Online Appendix to Accompany
Optimal Delegation with a Finite Number of States

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4th April 2011

In this online appendix, we provide some details of proofs which are not included in the main text of the paper.

Proof of Observation 2

Let $b \in \left(\frac{1}{2(T-1)}, \frac{1}{(T-1)}\right)$ and $\Delta \in \mathcal{D}_T^*(b)$. From Theorem 1, this implies that $|\Delta| = T$. To compute the moments of the decision taken by the agent when she is offered Δ, we must distinguish between two different cases:

- *Case 1: T is odd.* By Theorem 1(ii),

$$d_t(\Delta, b) = \begin{cases} \ell^e(b, T, T-1) + \frac{t}{T-1} & \text{if } t \text{ is even,} \\ 2b - \ell^e(b, T, T-1) + \frac{t-1}{T-1} & \text{if } t \text{ is odd,} \end{cases}$$

so that the expected value of the agent's decision is

$$\mathbb{E}^E(b) \equiv \frac{1}{T}\sum_{t=0}^{T-1} d_t(\Delta, b) = \frac{1}{T}\left[(T-1)b + \ell^e(b, T, T-1) + \frac{1}{T-1}\sum_{t=0}^{T-1} t\right] = \frac{(1+2b)(T-1)^2}{2T^2} \ .$$

By definition, the variance of the decision is given by

$$\mathbb{V}^E(b) \equiv \frac{1}{T}\sum_{t=0}^{T-1}\left[d_t(\Delta, b) - \mathbb{E}^E(b)\right]^2 \ ,$$

so that

$$\begin{aligned} T\frac{d\mathbb{V}^E(b)}{db} &= 2\sum_{t=0}^{T-1}\left[d_t(\Delta, b) - \mathbb{E}^E(b)\right] \cdot \frac{d}{db}\left[d_t(\Delta, b) - \mathbb{E}^E(b)\right] \\ &= 2\sum_{t=0}^{T-1}\left[d_t(\Delta, b) - \mathbb{E}^E(b)\right] \cdot \frac{d}{db}\left[d_t(\Delta, b) - \mathbb{E}^E(b)\right] \ , \end{aligned}$$

where

$$\frac{d}{db}\left[d_t(\Delta,b)-\mathbb{E}^E(b)\right]=\begin{cases}\frac{T-1}{T^2} & \text{if } t \text{ is even,}\\ \frac{3T-1}{T^2} & \text{if } t \text{ is odd.}\end{cases}$$

Tedious calculations then reveal that

$$\begin{aligned} T\frac{d\mathbb{V}^E(b)}{db} &= \frac{1}{T^3(T-1)}\left\{T\left(2T^3-4T^2+T-1\right)+2b\left(T^3+T-1\right)(T-1)+1\right\} \\ &\geq \frac{2T-1}{T}>0\ , \end{aligned}$$

where the first inequality comes from $b \geq \frac{1}{2(T-1)}$.

- *Case 2: T is even.* By Theorem 1(ii),

$$d_t(\Delta,b)=\begin{cases} b-\frac{T+1}{2T(T-1)}+\frac{t}{T-1} & \text{if } t \text{ is odd}\\ b-\frac{T+3}{2T(T-1)}+\frac{t}{T-1} & \text{if } t \text{ is even.}\end{cases}$$

As a consequence, the expected value of the agent's decision must be of the form

$$\mathbb{E}^O(b)\equiv\frac{1}{T}\sum_{t=0}^{T-1}d_t(\Delta,b)=b+\ \text{constant}\ .$$

This in turn implies that $\left[d_t(\Delta,b)-\mathbb{E}^O(b)\right]^2$ must be independent of b for each $t\in\mathbf{T}$, thus proving the variance of the decision does not depend on b.

Detailed proof of Lemma 2

As Δ is top-loaded, the agent takes decision $\delta_{t+\tau}$ in state $t+\tau$, for every $\tau<K$, and takes δ_{t+K} in states $K\leq t+\tau\leq s-1$.

For each $\tau\leq K$, let $\kappa_{t+\tau}\equiv\delta_{t+\tau}-\frac{t+\tau}{T-1}$. By optimality, $(\kappa_{t+\tau})_{\tau=0\ldots,K}$ must solve

$$\min_{\{\kappa_{t+\tau}\}_{\tau=t,\ldots,K}}\left\{\sum_{\tau=0}^{K-1}\kappa_{t+\tau}^2+\sum_{\tau=K}^{s-1}\left(\kappa_{t+K}-\frac{\tau-K}{T-1}\right)^2\right\}$$

subject to

$$\kappa_{t+\tau}+\kappa_{t+\tau+1}=2b-\frac{1}{T-1}\ ,\ \forall\tau=0,\ldots,K-1\ ; \tag{1}$$

where the equality in (1) must hold at Δ because it satisfies the chain property. It is easy to see that any solution to this problem is also a solution to

$$\min_{\{\kappa_{t+\tau}\}_{\tau=t,\ldots,K}}\left\{\sum_{\tau=0}^{K-1}\kappa_{t+\tau}^2+(s-K)\left(\kappa_{t+K}-\frac{s-K-1}{T-1}\right)\kappa_{t+K}\right\}\ , \tag{2}$$

subject to (1). From the incentive constraints (1), we obtain by recursion that

$$\kappa_{t+\tau}=\begin{cases}\kappa_t & \text{if } \tau \text{ is even}\\ 2b-\frac{1}{T-1}-\kappa_t & \text{if } \tau \text{ is odd}\end{cases}\ ,\ \forall\tau=0,\ldots,K.$$

Substituting into (2), simple convex optimization reveals that, at an optimum:

$$\kappa_t = \begin{cases} \ell^e(b,s,K) & \text{if } K \text{ is even,} \\ \ell^o(b,s,K) & \text{if } K \text{ is odd,} \end{cases}$$

where $\ell^e(b,s,K)$ and $\ell^o(b,s,K)$ are defined before Lemma 2 in Section 2.3.

We end the proof by showing that the principal can reduce his expected loss whenever $b \notin \left[b^{\min}(K,s), b^{\max}(K,s)\right]$ — a contradiction to Δ being optimal. More precisely, for every $m \in \mathbb{N}$, let $D^*(m) \equiv \left\{\delta_t^*(m), \ldots, \delta_{t+m-1}^*(m)\right\}$, where

$$\delta_{t+\tau}^*(m) = \begin{cases} \ell(b,s,m-1) + \frac{t+\tau}{T-1} & \text{if } \tau \text{ is even,} \\ 2b - \ell(b,s,m-1) + \frac{t+\tau-1}{T-1} & \text{if } \tau \text{ is odd.} \end{cases}$$

and ℓ is defined in the text before Lemma 2. We will show that if $b \notin \left[b^{\min}(K,s), b^{\max}(K,s)\right]$ then there is a $K' \neq K$ such that the principal can reduce his loss by offering $D^*(K'+1)$ to the agent.

Suppose that K is even. The change in the principal's loss following a deviation to $D^*(K+2)$ when $K < s-1$ can be decomposed as follows:

(i) In each state $t+\tau$, with τ even and no more than K, the change in the principal's loss is:

$$\ell^o(b,s,K+1)^2 - \ell^e(b,s,K)^2 = \varphi(b,s,K)\left[\varphi(b,s,K) + 2\ell^e(b,s,K)\right] \ ,$$

where

$$\varphi(b,s,K) \equiv \ell^o(b,s,K+1) - \ell^e(b,s,K) = \frac{s-K-1}{s}\left(2b - \frac{s-K}{T-1}\right) \ .$$

(ii) In each state $t+\tau$, with τ odd and no more than $K-1$, the change in the principal's loss is:

$$\left[2b - \frac{1}{T-1} - \ell^o(b,s,K+1)\right]^2 - \left[2b - \frac{1}{T-1} - \ell^e(b,s,K)\right]^2$$
$$= \varphi(b,s,K)\left[\varphi(b,s,K) - 2\left(2b - \frac{1}{T-1} - \ell^e(b,s,K)\right)\right] \ .$$

(iii) In each state $t+K+i$ for $i = 1, \ldots, s-K-1$, the change in the principal's loss is:

$$\left[2b - \ell^o(b,s,K+1) - \frac{i}{T-1}\right]^2 - \left[\ell^e(b,s,K) - \frac{i}{T-1}\right]^2$$
$$= \xi(b,s,K)\left\{\xi(b,s,K) + 2\left[\ell^e(b,s,K) - \frac{i}{T-1}\right]\right\} \ ,$$

where

$$\xi(b,s,K) \equiv 2b - \ell^o(b,s,K+1) - \ell^e(b,s,K) = \frac{1}{s}\left(2b + \frac{K}{T-1}\right) \ .$$

Note that

$$\sum_{i=1}^{s-K-1} \left\{ \xi(b,s,K) + 2\left[\ell^e(b,s,K) - \frac{i}{T-1}\right]\right\} = (s-K-1)\left[\xi(b,s,K) + 2\ell^e(b,s,K) - \frac{s-K}{T-1}\right]$$
$$= (K+1)\frac{s-K-1}{s}\left(2b - \frac{s-K}{T-1}\right)$$
$$= (K+1)\varphi(b,s,K) \ .$$

Thus, summing the changes in the principal's loss over all states, we obtain

$$(K+1)\varphi(b,s,K)\left[\varphi(b,s,K) + 2\ell^e(b,s,K) - \frac{K}{K+1}\left(2b - \frac{1}{T-1}\right) + \xi(b,s,K)\right]$$

$$= (K{+}1)\varphi(b,s,K)\left[\frac{1}{K+1}\left(2b + \frac{K}{T-1}\right)\right] = \frac{2(s-K-1)}{s}\left(2b + \frac{K}{T-1}\right)\left[b - b^{\min}(K,s)\right] \ .$$

This proves that the deviation to $D^*(K+2)$ is unprofitable only if $b \geq b^{\min}(K,s)$. The argument above also implies that the principal can profitably deviate from $D^*(K+2)$ to $D^*(K+1)$ unless $b \leq b^{\max}(K+1,s) = b^{\min}(K,s)$.

The change in the principal's expected loss if he deviates to $D^*(K)$ (assuming $K > 0$) can be decomposed as follows:

(i) In each state $t+\tau$, with τ even and strictly less than $K-1$, the change in the principal's loss is:

$$\ell^o(b,s,K-1)^2 - \ell^e(b,s,K)^2 = \phi(b,s,K)\left[\phi(b,s,K) + 2\ell^e(b,s,K)\right] \ ,$$

where

$$\phi(b,s,K) \equiv \ell^o(b,s,K-1) - \ell^e(b,s,K) = \frac{s-K}{s}\left(2b - \frac{s-K+1}{T-1}\right) \ .$$

(ii) In each state $t+\tau$, with τ odd and lower or equal to $K-1$, the change in the principal's loss is:

$$\left[2b - \frac{1}{T-1} - \ell^o(b,s,K-1)\right]^2 - \left[2b - \frac{1}{T-1} - \ell^e(b,s,K)\right]^2$$
$$= \phi(b,s,K)\left[\phi(b,s,K) - 2\left(2b - \frac{1}{T-1} - \ell^e(b,s,K)\right)\right] \ .$$

(iii) In each state $t+K+i$ for $i = 0,\ldots,s-K-1$, the change in the principal's loss is:

$$\left[2b - \ell^o(b,s,K-1) - \frac{i+2}{T-1}\right]^2 - \left[\ell^e(b,s,K) - \frac{i}{T-1}\right]^2$$
$$= \psi(b,s,K)\left\{\psi(b,s,K) + 2\left[\ell^e(b,s,K) - \frac{i}{T-1}\right]\right\} \ ,$$

where

$$\psi(b,s,K) \equiv 2\left(b - \frac{1}{T-1}\right) - \ell^o(b,s,K-1) - \ell^e(b,s,K) = -\frac{K}{s(T-1)} \ .$$

Note that

$$\begin{aligned}\sum_{i=0}^{s-K-1}\left\{\psi(b,s,K) + 2\left[\ell^e(b,s,K) - \frac{i}{T-1}\right]\right\} &= (s-K)\left[\psi(b,s,K) + 2\ell^e(b,s,K) - \frac{s-K-1}{T-1}\right] \\ &= K\frac{s-K}{s}\left(2b - \frac{s-K+1}{T-1}\right) \\ &= K\phi(b,s,K) \ .\end{aligned}$$

Thus, summing the principal's loss over all states, we obtain

$$\begin{aligned} & K\phi(b,s,K)\left[\phi(b,s,K) + 2\ell^e(b,s,K) - \left(2b - \frac{1}{T-1}\right) + \psi(b,s,K)\right] \\ &= K\phi(b,s,K)\left(-\frac{1}{T-1}\right) = -\frac{2K}{T-1}\frac{s-K}{s}\left[b - b^{\max}(K,s)\right] \ .\end{aligned}$$

This proves that the deviation to $D^*(K)$ is unprofitable only if $b \leq b^{\max}(K,s)$. The argument above also implies that the principal can profitably deviate from $D^*(K)$ to $D^*(K+1)$ unless $b \geq b^{\min}(K-1,s) = b^{\max}(K,s)$.

We have therefore proved that the principal can reduce his expected loss by choosing $D^*(K+2)$ instead of Δ when $b < b^{\min}(K,s)$, and by choosing $D^*(K)$ instead of Δ when $b > b^{\max}(K,s)$. [1] As $\Delta \in \mathcal{D}^*_{s,t}(b)$, this implies that $b \in \left[b^{\min}(K,s), b^{\max}(K,s)\right]$.

Computational details for the proof of Lemma 3

The detailed proof for $\Upsilon(b) \geq 0$ is as follows. We distinguish between four cases:

- **Case 1:** $\min\{K, K'\} > 0$.

In this case, we have both

$$b \in \left[\frac{\tau - K}{2(T-1)}, \frac{\tau - K}{2(T-1)}\right] \text{ and } b \in \left[\frac{s-\tau+1-K'}{2(T-1)}, \frac{s-\tau+2-K'}{2(T-1)}\right] \ .$$

As $\tau - K$ and $s - \tau + 1 - K'$ are integers, this in turn implies that

$$\tau - K = s - \tau + 1 - K' \ . \tag{3}$$

Now, let

$$\begin{aligned}\Upsilon(b) &\equiv 2\left(b + \frac{t+\tau-1}{T-1}\right) - \delta_{t+\tau-1} - \delta_{t+\tau} \\ &\geq 2\left(b + \frac{t+\tau-1}{T-1}\right) - \left[2b - \ell^o(b,\tau,K) + \ell^e\left(b, s-\tau+1, K'\right) + \frac{2t+\tau+K-1}{T-1}\right]\end{aligned}$$

[1] We have ignored the case in which $K = s-1$ and $b < b^{\min}(K,s)$ because we then have $b < 1/2(T-1)$.

First, note that

$$\frac{d}{db}\left[2b-\ell^{o}(b,\tau,K)\right]=\frac{d}{db}\left[\frac{K+1}{\tau}b+\frac{\tau-1}{2\tau(T-1)}+\frac{(\tau-K)^2}{2\tau(T-1)}\right]=\frac{K+1}{\tau}\leq 1$$

and

$$\frac{d}{db}\left[\ell^{e}\left(b,s-\tau+1,K'\right)\right]=\frac{d}{db}\left[\frac{K'}{s-\tau+1}b\right]=\frac{K'}{s-\tau+1}\leq 1\ .$$

As a consequence, $\Upsilon'(b)\geq 0$ and, from (3):

$$\Upsilon(b)\geq\Upsilon\left(\frac{\tau-K}{2(T-1)}\right)=\Upsilon\left(\frac{s-\tau+1-K'}{2(T-1)}\right)\ .$$

At $b=\frac{\tau-K}{2(T-1)}$, we have[2]

$$2\left(b+\frac{t+\tau-1}{T-1}\right)-\frac{2t+\tau+K-1}{T-1}=\underbrace{\frac{2(\tau-K)}{2(T-1)}}_{\hookrightarrow\Upsilon_1}+\underbrace{\frac{\tau-K-1}{T-1}}_{\hookrightarrow\Upsilon_2}\ ,$$

$$-\left[2b-\ell^{o}(b,\tau,K)\right]=\underbrace{-\frac{K+1}{\tau}\frac{\tau-K}{2(T-1)}}_{\hookrightarrow\Upsilon_3}\underbrace{-\frac{\tau-1}{2\tau(T-1)}}_{\hookrightarrow\Upsilon_3}\underbrace{-\frac{(\tau-K)^2}{2\tau(T-1)}}_{\hookrightarrow\Upsilon_1}$$

and

$$\begin{aligned}-\ell^{e}\left(b,s-\tau+1,K'\right)&=-\frac{K'}{s-\tau+1}\frac{\tau-K}{2(T-1)}+\frac{1}{2(T-1)}-\frac{(s-\tau+1-K'^2}{2(s-\tau+1)(T-1)}\\&=\underbrace{-\frac{K'}{s-\tau+1}\frac{\tau-K}{2(T-1)}}_{\equiv\Upsilon_4}+\underbrace{\frac{1}{2(T-1)}}_{\hookrightarrow\Upsilon_2}\underbrace{-\frac{(\tau-K)^2}{2(s-\tau+1)(T-1)}}_{\hookrightarrow\Upsilon_1}\ ,\end{aligned}$$

where the second equality follows from (3). Thus,

$$\begin{aligned}\Upsilon_1&\equiv\frac{2(\tau-K)}{2(T-1)}-\frac{(\tau-K)^2}{2\tau(T-1)}-\frac{(\tau-K)^2}{2(s-\tau+1)(T-1)}\\&=\frac{\tau-K}{2(T-1)}\left(2-\frac{\tau-K}{\tau}-\frac{\tau-K}{s-\tau+1}\right)\ ,\\\Upsilon_2&\equiv\frac{\tau-K-1}{T-1}+\frac{1}{2(T-1)}\ ,\\\Upsilon_3&\equiv-\frac{K+1}{\tau}\frac{\tau-K}{2(T-1)}-\frac{\tau-1}{2\tau(T-1)}\\&=-\frac{1}{2\tau(T-1)}\left[(K+1)(\tau-K)+\tau-1\right]\ ,\\\Upsilon_4&\equiv-\frac{K'}{s-\tau+1}\frac{\tau-K}{2(T-1)}\ .\end{aligned}$$

[2] We use "$\hookrightarrow\Upsilon_i$" to mean "is part of Υ_i".

Hence,

$$\begin{aligned}\Upsilon_1 + \Upsilon_4 &= \frac{\tau - K}{2(T-1)}\left(2 - \frac{\tau - K}{\tau} - \frac{\tau - K + K'}{s - \tau + 1}\right)\\ &= \frac{\tau - K}{2(T-1)}\left(2 - \frac{\tau - K}{\tau} - \frac{s - \tau + 1}{s - \tau + 1}\right)\\ &= \frac{\tau - K}{2(T-1)}\left(1 - \frac{\tau - K}{\tau}\right)\\ &= \frac{K(\tau - K)}{2\tau(T-1)} \equiv \Upsilon_5 \ .\end{aligned}$$

where the second inequality follows from (3). Moreover,

$$\begin{aligned}\Upsilon_5 + \Upsilon_3 &= \frac{1}{2\tau(T-1)}\left[K(\tau - K) - (K+1)(\tau - K) - \tau + 1\right]\\ &= \frac{K+1}{2\tau(T-1)} - \frac{1}{T-1} \equiv \Upsilon_6 \ ,\end{aligned}$$

and

$$\begin{aligned}\Upsilon_6 + \Upsilon_2 &= \frac{\tau - K - 1}{T-1} + \frac{K+1}{2\tau(T-1)} - \frac{1}{2(T-1)}\\ &= \frac{\tau - K - 1}{T-1}\left(1 - \frac{1}{2\tau}\right) \geq 0\end{aligned}$$

since $K \leq \tau - 1$.

- **Case 2:** $0 = K < K' \leq s - \tau$.
In this case, $\delta_{t+\tau-1} = \frac{2t+\tau-1}{2(T-1)}$, and $b \geq \frac{s-\tau+1-K'}{2(T-1)}$. If $s' = s - \tau + 1$ then

$$\begin{aligned}\Upsilon(b) &\geq 2\left(b + \frac{t+\tau-1}{T-1}\right) - \left[\frac{2t+\tau-1}{2(T-1)} + \ell^e\left(b, s', K'\right) + \frac{t+\tau}{T-1}\right]\\ &= 2b - \ell^e\left(b, s', K'\right) + \frac{\tau - 3}{2(T-1)}\\ &= \frac{2s' - K'}{s'}b - \frac{(s' - K'^2}{2s'(T-1)} + \frac{\tau - 2}{2(T-1)}\\ &\geq \frac{s' - K' + \tau - 2}{2(T-1)} = \frac{s - K' - 1}{2(T-1)} \geq 0\end{aligned}$$

since $K' \leq s - \tau \leq s - 1$.

- **Case 3:** $0 = K' < K \leq \tau - 1$.
In this case, $\delta_{t+\tau} = \frac{2t+s+\tau}{2(T-1)}$, and $b \geq \max\left\{\frac{\tau-K}{2(T-1)}, \frac{s-\tau+1}{2(T-1)}\right\}$; so

$$
\begin{aligned}
\Upsilon(b) &\geq 2\left(b+\frac{t+\tau-1}{T-1}\right)-\left[2b-\ell^o(b,\tau,K)+\frac{t+K}{T-1}+\frac{2t+s+\tau}{2(T-1)}\right] \\
&\geq b+\frac{t+\tau-1}{T-1}-\left[2b-\ell^o(b,\tau,K)+\frac{t+K}{T-1}\right]-\frac{1}{2(T-1)} \\
&= b-\left[\frac{K+1}{\tau}b-\frac{\tau-1}{2\tau(T-1)}-\frac{(\tau-K)^2}{2\tau(T-1)}\right]-\frac{1}{2(T-1)}+\underbrace{\frac{t+\tau-1}{T-1}-\frac{t+K}{T-1}}_{\geq 0} \\
&\geq \frac{\tau-K-1}{\tau}\frac{\tau-K}{2(T-1)}-\frac{1}{2\tau(T-1)}+\frac{(\tau-K)^2}{2\tau(T-1)} \\
&= \frac{\tau-K-1}{2\tau(T-1)}\left[2(\tau-K)+1\right]\geq 0\ .
\end{aligned}
$$

- **Case 4:** $K=K'=0$.

In this case, $\delta_{t+\tau-1}=\frac{2t+\tau-1}{2(T-1)}$, $\delta_{t+\tau}=\frac{2t+s+\tau}{2(T-1)}$, and $b\geq\max\left\{\frac{\tau}{2(T-1)},\frac{s-\tau+1}{2(T-1)}\right\}$; so

$$
\begin{aligned}
\Upsilon(b) &\geq 2\left(b+\frac{t+\tau-1}{T-1}\right)-\left[\frac{2t+\tau-1}{2(T-1)}+\frac{2t+s+\tau}{2(T-1)}\right] \\
&\geq \frac{2(\tau-1)}{T-1}-\frac{2(\tau-1)}{2(T-1)}=\frac{\tau-1}{T-1}\geq 0\ .
\end{aligned}
$$

Expertise

Observation 3. *If $b\in\left(\frac{1}{2(T-1)},\frac{1}{T-1}\right)$ then the expert outperforms an amateur who cannot distinguish between states $T-2$ and $T-1$, and takes the same decisions as the expert in all other states.*

Proof: Recall that the principal prefers to appoint this amateur and offer her δ_E if and only if[3]

$$
\left[\frac{2T-K}{T}b-\frac{(T-K)^2+3T}{2T(T-1)}\right]^2<\left[\frac{K}{T}b+\frac{(T-K)^2-T}{2T(T-1)}\right]^2 \tag{4}
$$

We need to treat cases in which T is odd and even separately, starting with the former: The left-hand side of (4) is negative because $T>1$ implies that

$$
b<\frac{1}{T-1}<\frac{3T+1}{2(T-1)(T+1)}\ .
$$

Consequently, the principal prefers not to appoint this amateur and offer her δ_E if $b<\frac{1}{T-1}$.

[3]See the proof of Proposition 3(ii) in the main text.

As $\delta_E > \delta_{K-2}$, the principal can only satisfy the amateur's incentive constraints by replacing δ_{K-1} and δ_{K-2} with a single decision δ if $\delta \geq \delta_E$. At best, the principal can induce her to take decision $\frac{2K-3}{2(T-1)}$ in E, yielding a loss of $\frac{1}{2(T-1)^2}$ in that event. If $b < \frac{1}{T-1}$ then $K = T-1$, and the principal would lose $\frac{2(T^2+1)}{T^2}\left[b - \frac{1}{2(T-1)}\right]^2$ in E, which exceeds $\frac{1}{2(T-1)^2}$ if and only if

$$4\left(T^2+1\right)(T-1)^2 b^2 - 4T\left(T^2+1\right)(T-1)b + 1 > 0:$$

that is, for

$$\frac{T^2+1+T\sqrt{T^2+1}}{2\left(T^2+1\right)(T-1)} < b < \frac{1}{T-1}.$$

Now $\delta_E > \frac{2T-5}{2(T-1)}$ if and only if $b > \frac{2T+1}{2(T^2-1)}$, where the latter term is less than $\frac{1}{T-1}$, in which case the principal cannot improve on the amateur's loss by raising the decision above δ_E. Comparing terms for $b \in \left(\frac{1}{2(T-1)}, \frac{1}{T-1}\right)$: an amateur who is offered a single decision outperforms the expert if and only if

$$\frac{T^2+1+T\sqrt{T^2+1}}{2\left(T^2+1\right)(T-1)} < b < \frac{2T+1}{2\left(T^2-1\right)}.$$

It is easy to confirm that the requisite lower bound exceeds the requisite upper bound on b; so this amateur cannot outperform the expert if $b < \frac{1}{T-1}$.

Now suppose that T is even. It is easy to confirm that $\delta_{K-2} < \frac{2T-5}{2(T-1)}$; so replacing δ_{T-3} and δ_{T-2} with $\frac{2T-5}{2(T-1)}$ does not violate the amateur's incentive constraints. Substituting $K = T-1$: the expert outperforms an amateur who takes $\frac{2T-5}{2(T-1)}$ in E (and therefore any such amateur) if and only if $b < \frac{1}{T-1}$; so the expert again outperforms an amateur who takes the same decisions in states outside E if $b \in \left(\frac{1}{2(T-1)}, \frac{1}{T-1}\right)$.

Other amateurs for $T = 4$

`We include the ODS and losses for the cases omitted from Appendix C.2 for the four-state case:`

`If` $E = \{2,3\}$ `then`

	Δ^{23}	λ^{23}	Λ^{23}
$b < \frac{1}{6}$	$0, \frac{1}{3}, \frac{5}{6}$	$\frac{1}{72}$	$b^2 + \frac{1}{72}$
$\frac{1}{6} < b < \frac{1}{3}$	$b - \frac{1}{6}, b + \frac{1}{6}, \frac{5}{6}$	$\frac{1}{2}b^2 - \frac{1}{6}b + \frac{1}{36}$	$\frac{1}{2}b^2 + \frac{1}{36}$
$\frac{1}{3} < b < \frac{1}{2}$	$\frac{1}{2}b, \frac{3}{2}b, \frac{1}{2}b + \frac{2}{3}$	$\frac{3}{4}b^2 - \frac{1}{3}b + \frac{1}{18}$	$\frac{1}{4}b^2 + \frac{1}{18}$
$\frac{1}{2} < b < \frac{2}{3}$	$\frac{3}{2}b - \frac{1}{2}, \frac{1}{2}b + \frac{1}{2}$	$\frac{3}{4}b^2 - \frac{1}{2}b + \frac{5}{36}$	$\frac{1}{4}b^2 + \frac{5}{36}$
$b > \frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E=\{0,1,2\}$ then

b	Δ^{012}	λ^{012}	Λ^{012}
$b<\frac{1}{3}$	$\frac{1}{3},1$	$\frac{1}{18}$	$b^2+\frac{1}{18}$
$\frac{1}{3}<b<\frac{2}{3}$	$\frac{1}{2}b+\frac{1}{6},\frac{3}{2}b+\frac{1}{2}$	$\frac{3}{4}b^2-\frac{1}{2}b+\frac{5}{36}$	$\frac{1}{4}b^2+\frac{5}{36}$
$b>\frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2+\frac{5}{36}$

The same details apply when $E=\{0,2\}$.

If $E=\{1,2,3\}$ then

b	Δ^{123}	λ^{123}	^{123}
$\frac{1}{6}<b<\frac{1}{3}$		$\frac{1}{18}$	$b^2+\frac{1}{18}$
$\frac{1}{3}<b<\frac{2}{3}$	$\frac{3}{2}b-\frac{1}{2},\frac{1}{2}b+\frac{1}{2}$	$\frac{3}{4}b^2-\frac{1}{2}b+\frac{5}{36}$	$\frac{1}{4}b^2+\frac{5}{36}$
$b>\frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2+\frac{5}{36}$

The same details apply when $E=\{1,3\}$.

If $E=\{0,3\}$ then

b	Δ^{03}	λ^{03}	Λ^{03}
$b<\frac{1}{6}$	$b+\frac{1}{4},b+\frac{5}{12},b+\frac{7}{12}$	$b^2-\frac{1}{6}b+\frac{19}{144}$	$\frac{19}{144}$
$\frac{1}{6}<b<\frac{2}{9}$		$\frac{3}{4}b^2-\frac{1}{6}b+\frac{5}{36}$	$\frac{1}{4}b^2+\frac{5}{36}$
$b>\frac{2}{9}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2+\frac{5}{36}$

If $E=\{0,1,3\}$ then

b	Δ^{013}	λ^{013}	Λ^{013}
$b<\frac{2}{9}$		$\frac{3}{4}b^2-\frac{1}{6}b+\frac{5}{36}$	$\frac{1}{4}b^2+\frac{5}{36}$
$b>\frac{2}{9}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2+\frac{5}{36}$

If $E=\{0,2,3\}$ then

b	Δ^{023}	λ^{023}	Λ^{023}
$b<\frac{2}{9}$	$\frac{3}{2}b+\frac{1}{6},\frac{1}{2}b+\frac{1}{2}$	$\frac{3}{4}b^2-\frac{1}{6}b+\frac{5}{36}$	$\frac{1}{4}b^2+\frac{5}{36}$
$b>\frac{2}{9}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2+\frac{5}{36}$

If $E = \{0,1\},\{2,3\}$ then

b	$\Delta^{01,23}$	$\lambda^{01,23}$	$\Lambda^{01,23}$
$b < \frac{1}{3}$	$\frac{1}{6}, \frac{5}{6}$	$\frac{1}{36}$	$b^2 + \frac{1}{36}$
$\frac{1}{3} < b < \frac{2}{3}$	$b - \frac{1}{6}, b + \frac{1}{2}$	$b^2 - \frac{2}{3}b + \frac{5}{36}$	$\frac{5}{36}$
$b > \frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{0,2\},\{1,3\}$ then

b	$\Delta^{02,13}$	$\lambda^{02,13}$	$\Lambda^{02,13}$
$b < \frac{1}{6}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{9}$	$b^2 + \frac{1}{9}$
$\frac{1}{6} < b < \frac{1}{3}$	$b - \frac{1}{6}, b + \frac{1}{2}$	$b^2 - \frac{1}{3}b + \frac{5}{36}$	$\frac{5}{36}$
$b > \frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{0,3\},\{1,2\}$ or $E = \{0,1,2,3\}$ then

b	Δ^E	λ^E	Λ^E
$b > 0$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

Other amateurs for $T = 5$

We include the ODS and losses for the cases omitted from Appendix C.3 for the five-state case:

If $E = \{0,1\},\{2,3,4\}$ then

b	$\Delta^{01,234}$	$\lambda^{01,234}$	$\Lambda^{01,234}$
$b < \frac{5}{16}$	$\frac{1}{8}, \frac{3}{4}$	$\frac{1}{32}$	$b^2 + \frac{1}{32}$
$\frac{5}{16} < b < \frac{5}{8}$	$\frac{6}{5}b - \frac{1}{4}, \frac{4}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{3}{25}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,1\},\{3,4\}$ then

b	$\Delta^{01,34}$	$\lambda^{01,34}$	$\Lambda^{01,34}$
$b < \frac{3}{16}$	$\frac{1}{8}, \frac{1}{2}, \frac{7}{8}$	$\frac{1}{80}$	$b^2 + \frac{1}{80}$
$\frac{3}{16} < b < \frac{3}{8}$	$\frac{2}{5}b + \frac{1}{20}, \frac{8}{5}b + \frac{1}{5}, \frac{2}{5}b + \frac{4}{5}$	$\frac{16}{25}b^2 - \frac{6}{25}b + \frac{7}{200}$	$\frac{9}{25}b^2 + \frac{7}{200}$
$\frac{3}{8} < b < \frac{5}{8}$	$\frac{6}{5}b - \frac{1}{4}, \frac{4}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{3}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,1,2,3\}$ or $E = \{0,3\},\{1,2\}$ then

b	Δ^E	λ^E	Λ^E
$b < \frac{5}{16}$	$\frac{3}{8}, 1$	$\frac{1}{16}$	$b^2 + \frac{1}{16}$
$\frac{5}{16} < b < \frac{5}{8}$	$\frac{2}{5}b + \frac{1}{4}, \frac{8}{5}b + \frac{1}{2}$	$\frac{16}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{1,2\},\{0,4\}$ then

b	$\Delta^{12,04}$	$\lambda^{12,04}$	$\Lambda^{12,04}$
$b < \frac{1}{16}$	$\frac{3}{8}, \frac{1}{2}, \frac{3}{4}$	$\frac{17}{160}$	$b^2 + \frac{17}{160}$
$\frac{1}{16} < b < \frac{\sqrt{10}-1}{16}$	$b + \frac{5}{16}, b + \frac{7}{16}, \frac{3}{4}$	$\frac{4}{5}b^2 - \frac{1}{10}b + \frac{7}{64}$	$\frac{1}{5}b^2 + \frac{7}{64}$
$\frac{\sqrt{10}-1}{16} < b < \frac{1}{4}$	$\frac{2}{5}b + \frac{2}{5}, \frac{8}{5}b + \frac{3}{5}$	$\frac{16}{25}b^2 - \frac{3}{25}b + \frac{23}{200}$	$\frac{9}{25}b^2 + \frac{23}{200}$
$b > \frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{1,2\},\{3,4\}$ then

b	$\Delta^{12,34}$	$\lambda^{12,34}$	$\Lambda^{12,34}$
$b < \frac{3}{16}$	$0, \frac{3}{8}, \frac{7}{8}$	$\frac{1}{80}$	$b^2 + \frac{1}{80}$
$\frac{3}{16} < b < \frac{9}{32}$	$\frac{4}{3}b + -\frac{1}{4}, \frac{2}{3}b + \frac{1}{4}, \frac{7}{8}$	$\frac{8}{15}b^2 - \frac{1}{5}b + \frac{1}{32}$	$\frac{7}{15}b^2 + \frac{1}{32}$
$\frac{\sqrt{10}-1}{16} < b < \frac{1}{4}$	$\frac{4}{5}b - \frac{1}{10}, \frac{6}{5}b + \frac{1}{10}, \frac{4}{5}b + \frac{13}{20}$	$\frac{24}{25}b^2 - \frac{11}{25}b + \frac{13}{200}$	$\frac{1}{25}b^2 + \frac{13}{200}$
$b > \frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{1,2\},\{0,3,4\}$ then

b	$\Delta^{12,034}$	$\lambda^{12,034}$	$\Lambda^{12,034}$
$b < \frac{5}{48}$	$\frac{3}{8}, \frac{7}{12}$	$\frac{11}{96}$	$b^2 + \frac{11}{96}$
$\frac{5}{48} < b < \frac{5}{24}$	$\frac{6}{5}b + \frac{1}{4}, \frac{4}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{1}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{24}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,4\},\{2,3\}$ then

b	$\Delta^{04,23}$	$\lambda^{04,23}$	$\Lambda^{04,23}$
$b < \frac{1}{16}$	$\frac{1}{4}, \frac{1}{2}, \frac{5}{8}$	$\frac{17}{160}$	$b^2 + \frac{17}{160}$
$\frac{1}{16} < b < \frac{5\left(\sqrt{2}-1\right)}{16}$	$\frac{1}{4}, b + \frac{7}{16}, b + \frac{9}{16}$	$\frac{4}{5}b^2 - \frac{1}{10}b + \frac{7}{64}$	$\frac{1}{5}b^2 + \frac{7}{64}$
$\frac{5\left(\sqrt{2}-1\right)}{16} < b < \frac{5}{16}$	$\left\{\frac{8}{5}b, \frac{2}{5}b + \frac{1}{2}\right\}$	$\frac{16}{25}b^2 - \frac{1}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{16}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,1,4\},\{2,3\}$ then

b	$\Delta^{014,23}$	$\lambda^{014,23}$	$\Lambda^{014,23}$
$b < \frac{5}{48}$	$\frac{5}{12}, \frac{5}{8}$	$\frac{11}{96}$	$b^2 + \frac{11}{96}$
$\frac{5}{48} < b < \frac{5}{24}$	$\frac{4}{5}b + \frac{1}{3}, \frac{6}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{1}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{24}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,1,2\},\{3,4\}$ then

b	$\Delta^{012,34}$	$\lambda^{012,34}$	$\Lambda^{012,34}$
$b < \frac{5}{16}$	$\frac{1}{4}, \frac{7}{8}$	$\frac{11}{32}$	$b^2 + \frac{1}{32}$
$\frac{5}{16} < b < \frac{5}{8}$	$\frac{4}{5}b, \frac{6}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{3}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,2\},\{1,3\}$ then

b	$\Delta^{02,13}$	$\lambda^{02,13}$	$\Lambda^{02,13}$
$b < \frac{1}{8}$	$\frac{1}{4}, \frac{1}{2}, 1$	$\frac{1}{20}$	$b^2 + \frac{1}{20}$
$\frac{1}{8} < b < \frac{5(\sqrt{2}-1)}{8}$	$b + \frac{1}{8}, b + \frac{3}{8}, 1$	$\frac{4}{5}b^2 - \frac{1}{5}b + \frac{1}{16}$	$\frac{1}{5}b^2 + \frac{1}{16}$
$\frac{5(\sqrt{2}-1)}{8} < b < \frac{5}{8}$	$\frac{2}{5}b + \frac{1}{4}, \frac{8}{5}b + \frac{1}{2}$	$\frac{16}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,3\},\{1,4\}$ then

b	$\Delta^{03,14}$	$\lambda^{03,14}$	$\Lambda^{03,14}$
$b < \frac{1}{16}$	$\frac{3}{8}, \frac{1}{2}, \frac{5}{8}$	$\frac{9}{80}$	$b^2 + \frac{9}{80}$
$\frac{1}{16} < b < \frac{1}{4}$	$\frac{1}{4}, \frac{2}{3}b + \frac{7}{12}, \frac{4}{3}b + \frac{2}{3}$	$\frac{8}{15}b^2 - \frac{1}{15}b + \frac{1}{12}$	$\frac{7}{15}b^2 + \frac{1}{12}$
$\frac{1}{4} < b < \frac{5}{8}$	$\frac{6}{5}b, \frac{4}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,3\},\{2,4\}$ then

b	$\Delta^{03,24}$	$\lambda^{03,24}$	$\Lambda^{03,24}$
$b < \frac{1}{16}$	$\frac{1}{4}, \frac{3}{8}, \frac{3}{4}$	$\frac{13}{160}$	$b^2 + \frac{13}{160}$
$\frac{1}{16} < b < \frac{1}{8}$	$\frac{4}{3}b + \frac{1}{6}, \frac{2}{3}b + \frac{1}{3}, \frac{3}{4}$	$\frac{8}{15}b^2 - \frac{b}{15} + \frac{1}{12}$	$\frac{7}{15}b^2 + \frac{1}{12}$
$\frac{1}{8} < b < \frac{1}{6}$	$\frac{1}{3}, \frac{3}{4}$	$\frac{1}{12}$	$b^2 + \frac{1}{12}$
$\frac{1}{6} < b < \frac{3}{8}$	$\frac{4}{5}b + \frac{1}{5}, \frac{6}{5}b + \frac{11}{20}$	$\frac{24}{25}b^2 - \frac{8}{25}b + \frac{11}{100}$	$\frac{1}{25}b^2 + \frac{11}{100}$
$b > \frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{1,3\},\{2,4\}$ then

b	$\Delta^{13,24}$	$\lambda^{13,24}$	$\Lambda^{13,24}$
$b < \frac{1}{8}$	$0, \frac{1}{2}, \frac{3}{4}$	$\frac{1}{20}$	$b^2 + \frac{1}{20}$
$\frac{1}{8} < b < \frac{1}{4}$	$0, b + \frac{3}{8}, b + \frac{5}{8}$	$\frac{4}{5}b^2 - \frac{b}{5} + \frac{1}{16}$	$\frac{1}{5}b^2 + \frac{1}{16}$
$\frac{1}{4} < b < \frac{5}{16}$	$0, \frac{5}{8}$	$\frac{1}{16}$	$b^2 + \frac{1}{16}$
$\frac{5}{16} < b < \frac{5}{8}$	$\frac{8}{5}b - \frac{1}{2}, \frac{2}{5}b + \frac{1}{2}$	$\frac{16}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,2,3\},\{1,4\}$ then

b	$\Delta^{023,14}$	$\lambda^{023,14}$	$\Lambda^{023,14}$
$b < \frac{5}{48}$	$\frac{5}{12}, \frac{5}{8}$	$\frac{11}{96}$	$b^2 + \frac{11}{96}$
$\frac{5}{48} < b < \frac{5}{24}$	$\frac{4}{5}b + \frac{1}{3}, \frac{6}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{1}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{24}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,1,3\},\{2,4\}$ then

b	$\Delta^{013,24}$	$\lambda^{013,24}$	$\Lambda^{013,24}$
$b < \frac{5}{24}$	$\frac{1}{3}, \frac{3}{4}$	$\frac{1}{12}$	$b^2 + \frac{1}{12}$
$\frac{5}{24} < b < \frac{5}{12}$	$\frac{4}{5}b + \frac{1}{6}, \frac{6}{5}b + \frac{1}{2}$	$\frac{24}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{1}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,1,3\}$ then

b	Δ^{013}	λ^{013}	Λ^{013}
$b < \frac{1}{12}$	$\frac{1}{3}, \frac{1}{2}, 1$	$\frac{7}{120}$	$b^2 + \frac{7}{120}$
$\frac{1}{12} < b < \frac{1}{6}$	$\frac{1}{2}b + \frac{7}{24}, \frac{3}{2}b + \frac{3}{8}, 1$	$\frac{3}{5}b^2 - \frac{1}{10}b + \frac{1}{16}$	$\frac{2}{5}b^2 + \frac{1}{16}$
$\frac{1}{6} < b < \frac{3}{16}$	$\frac{3}{8}, 1$	$\frac{1}{16}$	$b^2 + \frac{1}{16}$
$\frac{3}{16} < b < \frac{1}{2}$	$\frac{2}{5}b + \frac{11}{30}, \frac{8}{5}b + \frac{3}{10}, \frac{2}{5}b + \frac{7}{10}$	$\frac{16}{25}b^2 - \frac{4}{25}b + \frac{17}{200}$	$\frac{9}{25}b^2 + \frac{17}{200}$
$b > \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{0,1,4\}$ then

b	Δ^{014}	λ^{014}	Λ^{014}
$b < \frac{7+2\sqrt{10}}{48}$	$\frac{2}{5}b + \frac{2}{5}, \frac{8}{5}b + \frac{3}{5}$	$\frac{16}{25}b^2 - \frac{3}{25}b + \frac{23}{200}$	$\frac{9}{25}b^2 + \frac{23}{200}$
$\frac{7+2\sqrt{10}}{48} < b < \frac{3}{8}$	$\frac{b}{2} + \frac{19}{48}, \frac{3}{2}b + \frac{7}{16}, \frac{3}{4}$	$\frac{3}{5}b^2 - \frac{1}{20}b + \frac{7}{64}$	$\frac{2}{5}b^2 + \frac{7}{64}$
$\frac{3}{8} < b < \frac{5}{8}$	$\frac{2}{5}b + \frac{2}{5}, \frac{8}{5}b + \frac{3}{5}$	$\frac{16}{25}b^2 - \frac{3}{25}b + \frac{23}{200}$	$\frac{9}{25}b^2 + \frac{23}{200}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E = \{1,2,4\}$ then

b	Δ^{124}	λ^{124}	Λ^{124}
$b < \frac{1}{12}$	$0, \frac{7}{12}, \frac{3}{4}$	$\frac{7}{120}$	$b^2 + \frac{7}{120}$
$\frac{1}{12} < b < \frac{1}{6}$	$0, \frac{1}{2}b + \frac{13}{24}, \frac{3}{2}b + \frac{5}{8}$	$\frac{3}{5}b^2 - \frac{1}{10}b + \frac{1}{16}$	$\frac{2}{5}b^2 + \frac{1}{16}$
$\frac{1}{6} < b < \frac{5}{16}$	$0, \frac{5}{8}$	$\frac{1}{16}$	$b^2 + \frac{1}{16}$
$\frac{5}{16} < b < \frac{5}{8}$	$\frac{8}{5}b - \frac{1}{2}, \frac{2}{5}b + \frac{1}{2}$	$\frac{16}{25}b^2 - \frac{2}{5}b + \frac{1}{8}$	$\frac{9}{25}b^2 + \frac{1}{8}$
$b > \frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2 + \frac{1}{8}$

If $E=\{2,3,4\}$ then

b	Δ^{234}	λ^{234}	Λ^{234}
$b<\frac{1}{8}$	$0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{40}$	$b^2+\frac{1}{40}$
$\frac{1}{8}<b<\frac{3}{8}$	$b-\frac{1}{8},b+\frac{1}{8},\frac{3}{4}$	$\frac{2}{5}b^2-\frac{1}{10}b+\frac{1}{32}$	$\frac{2}{5}b^2+\frac{1}{32}$
$\frac{3}{8}<b<\frac{1}{2}$	$\frac{2}{5}b+\frac{1}{10},\frac{8}{5}b-\frac{1}{10},\frac{2}{5}b+\frac{3}{5}$	$\frac{16}{25}b^2-\frac{7}{25}b+\frac{13}{200}$	$b^2+\frac{13}{200}$
$\frac{1}{2}<b<\frac{5}{8}$	$\frac{8}{5}b-\frac{1}{2},\frac{2}{5}b+\frac{1}{2}$	$\frac{16}{25}b^2-\frac{2}{5}b+\frac{1}{8}$	$\frac{9}{25}b^2+\frac{1}{8}$
$b>\frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2+\frac{1}{8}$

If $E=\{1,2,3,4\}$ then

b	Δ^{1234}	λ^{1234}	Λ^{1234}
$b<\frac{1}{16}$	$0,\frac{5}{8}$	$\frac{1}{16}$	$b^2+\frac{1}{16}$
$\frac{1}{16}<b<\frac{5}{8}$	$\frac{8}{5}b-\frac{1}{2},\frac{2}{5}b+\frac{1}{2}$	$\frac{16}{25}b^2-\frac{2}{5}b+\frac{1}{8}$	$\frac{9}{25}b^2+\frac{1}{8}$
$b>\frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2+\frac{1}{8}$

If $E=\{0,2,3,4\}$ then

b	Δ^{0234}	λ^{0234}	Λ^{0234}
$b<\frac{5}{32}$	$\frac{1}{4},\frac{9}{16}$	$\frac{7}{64}$	$b^2+\frac{7}{64}$
$\frac{5}{32}<b<\frac{5}{16}$	$\frac{8}{5}b,\frac{2}{5}b+\frac{1}{2}$	$\frac{16}{25}b^2-\frac{1}{5}b+\frac{1}{8}$	$\frac{9}{25}b^2+\frac{1}{8}$
$b>\frac{5}{16}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2+\frac{1}{8}$

If $E=\{0,1,2,4\}$ then

b	Δ^{0124}	λ^{0124}	Λ^{0124}
$b<\frac{5}{32}$	$\frac{7}{16},\frac{3}{4}$	$\frac{7}{64}$	$b^2+\frac{7}{64}$
$\frac{5}{32}<b<\frac{5}{16}$	$\frac{2}{5}b+\frac{3}{8},\frac{8}{5}b+\frac{1}{2}$	$\frac{16}{25}b^2-\frac{1}{5}b+\frac{1}{8}$	$\frac{9}{25}b^2+\frac{1}{8}$
$b>\frac{5}{16}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2+\frac{1}{8}$

If $E=\{0,1,2,3\}$ then

b	Δ^{0123}	λ^{0123}	Λ^{0123}
$b<\frac{5}{16}$	$\frac{3}{8},1$	$\frac{1}{16}$	$b^2+\frac{1}{16}$
$\frac{5}{16}<b<\frac{5}{12}$	$\frac{2}{5}b+\frac{1}{4},\frac{8}{5}b+\frac{1}{2}$	$\frac{24}{25}b^2-\frac{2}{5}b+\frac{1}{8}$	$\frac{9}{25}b^2+\frac{1}{8}$
$b>\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2+\frac{1}{8}$

If $E=\{0,2\},\{1,3,4\}$ then

b	$\Delta^{02,134}$	$\lambda^{02,134}$	$\Lambda^{02,134}$
$b<\frac{5}{24}$	$\frac{1}{4},\frac{2}{3}$	$\frac{1}{12}$	$b^2+\frac{1}{12}$
$\frac{5}{24}<b<\frac{5}{12}$	$\frac{6}{5}b,\frac{4}{5}b+\frac{1}{2}$	$\frac{24}{25}b^2-\frac{2}{5}b+\frac{1}{8}$	$\frac{1}{25}b^2+\frac{1}{8}$
$b>\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2+\frac{1}{8}$

If $E=\{0,2,4\}$ then

b	Δ^{024}	λ^{024}	Λ^{024}
$b<\frac{1}{8}$	$\frac{1}{4},\frac{1}{2},\frac{3}{4}$	$\frac{1}{10}$	$b^2+\frac{1}{10}$
$\frac{1}{8}<b<\frac{1}{4}$	$\frac{6}{5}b+\frac{1}{10},\frac{4}{5}b+\frac{2}{5},\frac{6}{5}b+\frac{3}{5}$	$\frac{24}{25}b^2-\frac{6}{25}b+\frac{23}{200}$	$\frac{1}{25}b^2+\frac{23}{200}$
$\frac{1}{4}<b<\frac{5}{16}$	$\frac{8}{5}b,\frac{2}{5}b+\frac{1}{2}$	$\frac{16}{25}b^2-\frac{1}{5}b+\frac{1}{8}$	$\frac{9}{25}b^2+\frac{1}{8}$
$b>\frac{5}{16}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2+\frac{1}{8}$

If $E=\{0,3,4\}$ then

b	Δ^{024}	λ^{024}	Λ^{024}
$b<\frac{1}{24}$	$\frac{1}{4},\frac{1}{2},\frac{7}{12}$	$\frac{13}{120}$	$b^2+\frac{13}{120}$
$\frac{1}{24}<b<\frac{15-10\sqrt{2}}{8}$	$\frac{1}{4},\frac{3}{2}b+\frac{7}{16},\frac{1}{2}b+\frac{9}{16}$	$\frac{3}{5}b^2-\frac{1}{20}b+\frac{7}{64}$	$\frac{2}{5}b^2+\frac{7}{64}$
$\frac{15-10\sqrt{2}}{8}<b<\frac{5}{16}$	$\frac{8}{5}b,\frac{2}{5}b+\frac{1}{2}$	$\frac{16}{25}b^2-\frac{1}{5}b+\frac{1}{8}$	$\frac{9}{25}b^2+\frac{1}{8}$
$b>\frac{5}{16}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2+\frac{1}{8}$

If $E=\{0,2\},\{1,4\}$ then

b	$\Delta^{02,14}$	$\lambda^{02,14}$	$\Lambda^{02,14}$
$b<\frac{1}{16}$	$\frac{1}{4},\frac{5}{8},\frac{3}{4}$	$\frac{13}{160}$	$b^2+\frac{13}{160}$
$\frac{1}{16}<b<\frac{1}{8}$	$\frac{1}{4},\frac{2}{3}b+\frac{7}{12},\frac{4}{3}b+\frac{2}{3}$	$\frac{8}{15}b^2-\frac{1}{15}b+\frac{1}{12}$	$\frac{7}{15}b^2+\frac{1}{12}$
$\frac{1}{8}<b<\frac{5}{24}$	$\frac{1}{4},\frac{2}{3}$	$\frac{1}{12}$	$b^2+\frac{1}{12}$
$\frac{5}{24}<b<\frac{5}{12}$	$\frac{6}{5}b,\frac{4}{5}b+\frac{1}{2}$	$\frac{24}{25}b^2-\frac{2}{5}b+\frac{1}{8}$	$\frac{1}{25}b^2+\frac{1}{8}$
$b>\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2+\frac{1}{8}$

`If` $E=\{0,3\},\{1,2,4\}$ `then`

b	$\Delta^{03,124}$	$\lambda^{03,124}$	$\Lambda^{03,124}$
$b<\frac{5}{48}$	$\frac{3}{8},\frac{7}{12}$	$\frac{11}{96}$	$b^2+\frac{11}{96}$
$\frac{5}{48}<b<\frac{5}{24}$	$\frac{6}{5}b+\frac{1}{4},\frac{4}{5}b+\frac{1}{2}$	$\frac{24}{25}b^2-\frac{1}{5}b+\frac{1}{8}$	$\frac{1}{25}b^2+\frac{1}{8}$
$b>\frac{5}{24}$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2+\frac{1}{8}$

`If` $E=\{0,2,4\},\{1,3\}$ `or` $\{0,1,2,3,4\}$ `then`

b	Δ^E	λ^E	Λ^E
$b>0$	$\frac{1}{2}$	$\frac{1}{8}$	$b^2+\frac{1}{8}$