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Alexander Matros and Alex Possajennikov
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Suzanne Robey
Centre for Decision Research and Experimental Economics
School of Economics
University of Nottingham
University Park
Nottingham
NG7 2RD
Tel: +44 (0)115 95 14763
Fax: +44 (0) 115 95 14159
suzanne.robey@nottingham.ac.uk

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Common Value Allocation Mechanisms with Private Information: Lotteries or Auctions?*

Alexander Matros†
University of South Carolina and
Lancaster University

Alex Possajennikov‡
University of Nottingham

June 10, 2014

Abstract

We consider mechanisms for allocating a common-value prize between two players in an incomplete information setting. In this setting, each player receives an independent private signal about the prize value. The signals are from a discrete distribution and the value is increasing in both signals. First, we characterize symmetric equilibria in four mechanisms: a lottery; and first-price, second-price, and all-pay auctions. Second, we establish revenue equivalence of these auction mechanisms in this setting. Third, we describe conditions under which the expected revenue is higher in the lottery than in any of the auctions. Finally, we identify an optimal mechanism and its implementation by means of reserve prices in lottery and auction mechanisms.

Keywords: common value; contests; auctions

JEL Codes: D82; D44

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†Darla Moore School of Business, University of South Carolina, USA and Lancaster University Management School, Lancaster, UK. Email: alexander.matros@gmail.com

‡School of Economics, University of Nottingham, Nottingham, UK. Email: alex.possajennikov@nottingham.ac.uk

1 Introduction

Suppose that a resource, such as an oil tract, is to be given off for development. Firms interested in the resource can explore the ground and get a signal about the value of the resource. This value is common for all firms but the firms can get different signals about it. The firms bid for the right to develop the resource having only partial private information about the resource value. The resource is then allocated according to some mechanism based on the firms' bids.

Allocation mechanisms attracted a lot of attention in the economics literature. Typical mechanisms are auctions, where the prize (the resource in the setting above) is allocated deterministically to the highest bidder; and lotteries (contests), where the prize is allocated stochastically and the probability of getting the prize increases with the bid. For an overview of auction mechanisms, see, for example, Milgrom and Weber (1982) and Krishna (2009); lottery mechanisms are surveyed in Nitzan (1994), Corchón (2007), Congleton et al. (2008) and Konrad (2009). Common-value settings with incomplete information were considered mostly under auction mechanisms; see, for example, Wilson (1977), Milgrom (1979), Wang (1991), Klemperer (1998), Parreiras (2006), Malueg and Orzach (2012), Siegel (2014). For lotteries, starting with the original Tullock (1980) paper, the working assumption is that values (private or common) are known to all participants. To the best of our knowledge, only Wärneryd (2003, 2012) and Einy et al. (2013) consider settings where bidders have incomplete information in a common-value-prize contest. A comparison between different mechanisms in the incomplete information setting is missing in the literature, because explicit solutions for the lottery mechanism are difficult to derive.

In this paper we analyze and compare the lottery and the standard auction mechanisms (first-price, second-price and all-pay) in a common-value symmetric setting with a discrete distribution of signals. In order to make the analysis tractable, we restrict attention to two-player two-signal situations. We assume that each player receives independently either a low or a high private signal about the prize value. The value is an increasing function of both signals, thus each player is only partially informed about the common value. In this setting, we are able to find equilibria for all of the above mechanisms, compare their revenues, and identify situations in which a lottery is better for the seller.

The literature on incomplete-information contests considers discrete distributions of signals, because explicit solutions are not feasible even for simplest private-value continuous

distribution settings.[1] However, even with two-value discrete distributions, additional assumptions are needed to derive an equilibrium explicitly. Malueg and Yates (2004) consider two-bidder two-signal private-value contests with distributions characterized by one parameter: the correlation between bidders' signals. Münster (2009) derives an equilibrium in a lottery setting where one of the two private values that a bidder can have is zero but allowing for more general asymmetric distributions. Lim and Matros (2010) describe an equilibrium for n-player game in a similar setting.

In the present paper, we first analyze the lottery mechanism in our common-value setting. We characterize a unique symmetric equilibrium and find the ex-ante expected revenue in this equilibrium. Wärneryd (2003, 2012) and Einy et al. (2013) consider imperfectly discriminating contests with a common value and private information. However, they assume that some bidders know the common value with certainty while others only know the distribution of possible values. In contrast, we have symmetric partially informed bidders.

Second, we derive unique symmetric equilibria for the first-price, second-price, and all-pay auction mechanisms. Most of the auction literature focuses on continuous distributions of signals and pure-strategy equilibria. Wilson (1977) derives conditions for a symmetric equilibrium of first-price common-value auctions, while Milgrom (1979) generalizes these conditions. Milgrom (1981) characterizes a symmetric equilibrium of the second-price common-value auction. These common-value settings have a given distribution of common values and conditional distributions of signals. Klemperer (1998) suggests an alternative common-value setting (a "wallet game") in which signals are independently distributed and the common value is a deterministic function of the signals. The "wallet game" is a particular case in our model.

Milgrom (1981, footnote 8) indicates that equilibria with discrete distributions can be derived using mixed strategies. Maskin and Riley (1985) consider a discrete distribution setting, albeit with private valuations of the prize. They explicitly construct a mixed-strategy equilibrium for the first-price auction with two possible values. For the common-value first-price auction, Wang (1991) constructs an equilibrium for a finite number of signals (see also a two-value example in Bolton and Dewatripont, 2004, Ch. 7). A distinctive property of the equilibrium is that bidders with different signals mix over adjacent non-overlapping intervals, a feature that appears common in auction mechanisms with discrete distributions. For example, Konrad (2004) derives such an equilibrium in a private-value

[1] On the existence of equilibria in a continuous-distribution setting see Wasser (2013).

all-pay auction for a two-signal distribution and Siegel (2014) shows that a common-value all-pay auction with discrete signals also has such an equilibrium in a general setting. Our symmetric mixed-strategy equilibria for first-price and all-pay auctions are qualitatively similar.

A revenue comparison is a central topic in the auction analysis. While in a private-value setting the revenue equivalence often holds for all types of auctions, in a common-value setting seller's revenue can differ across auction mechanisms. Milgrom and Weber (1982) compare expected prices in affiliated-value auctions, showing that the price is (weakly) higher in the second-price auction than in the first-price auction. Generally, there does not appear to be a clear ranking of auction mechanisms for a common-value setting with discrete signals. Malueg and Orzach (2009, 2012), in a setting with differential information represented by partitions, show that either the first-price or the second-price auction can have a higher revenue. In our two-signal common-value model, however, the revenue equivalence result holds for the first-price, second-price, and all-pay auctions.

This revenue equivalence result helps us to compare the ex-ante expected revenue in the lottery with the one in auctions. There appears to be a certain belief that auctions are better than imperfectly discriminating contests for the revenue. They certainly are for the situations with a known common value. Moreover, considering situations with asymmetric information, Wärneryd (2012, p. 278) writes that "[f]rom the standpoint of a seller offering a good in an auction a perfectly discriminating mechanism is optimal ...". Although we find that for most of the parameter values auctions are indeed better, there are situations in which a lottery generates a higher ex-ante expected revenue than an auction does.

Most of the literature compares lotteries and auctions in complete information settings. Fang (2002) shows that if known private valuations are sufficiently asymmetric, a lottery can generate a higher revenue because the weaker player spends more in the lottery. Epstein et al. (2013), Franke et al. (2014), Mealem and Nitzan (2014a, b) consider various ways to optimally bias a lottery or an all-pay auction, based on players' asymmetries, and find that, depending on the bias, lotteries or auctions can generate higher revenue. Mealem and Nitzan (2012) provide a survey of these results. Einy et al. (2013) compare lotteries and auctions in asymmetric information setting and show that a lottery can have higher revenue than an auction. In contrast, we analyze an ex-ante symmetric situation. Nevertheless, the intuition is similar: the low-signal player bids more aggressively in a lottery than in an auction which might lead to a higher revenue in the lottery.

Finally, we identify an optimal mechanism and discuss its implementation by means of reserve prices in the lottery and auction mechanisms. If signals are correlated and informative, Cremer and McLean (1985) show that the seller can extract all the surplus from the bidders, also for interdependent (e.g. common) values (Bolton and Dewatripont, 2004, Ch. 7, illustrate this in a two-value example). Other methods to increase seller's revenue for continuous distributions include a reserve price (for common-value auctions see, for example, Milgrom and Weber, 1982) and entry fees (see Harstad, 1990). Generally, second-price auctions with the appropriate reserve price or the entry fee are better than first-price auctions with the corresponding price or fee. We find that although in our setting signals are independent, the seller can extract all the surplus from bidders. Moreover, all mechanisms can achieve this result with an appropriately chosen reserve price.

The rest of the paper is organized as follows. In Section 2 we present our common-value setting. The lottery is analyzed in Section 3 and auctions are studied in Section 4. Section 5 presents a revenue comparison between the lottery and auctions. Optimal mechanisms are considered in Section 6. Section 7 concludes. Proofs are given in Appendices.

2 Setting: Common Value and Private Signals

Consider the following situation. There is a valuable object for sale and there are two risk-neutral players. Each of the two players gets a private signal about the value of the object: Player 1 receives signal s_1 and Player 2 gets signal s_2. The value v of the object is an increasing function of two private signals, $v = g(s_1, s_2)$. The value of the object is common to the players. This setting is a generalization of the "wallet game" in Klemperer (1998), where the value is the sum of two signals.

Suppose that the signals have the following structure: each signal is either H (high) with probability $p \in [0,1]$, or L (low) with probability $(1-p) \in [0,1]$, independently of the other signal. That is,

$$s_i = \begin{cases} H, & \text{with probability } p, \\ L, & \text{with probability } 1-p. \end{cases}$$

Thus, the value of the object may be $g(L,L)$, $g(L,H)$, $g(H,L)$, or $g(H,H)$, with probabilities $(1-p)^2$, $(1-p)p$, $p(1-p)$, and p^2 respectively. We assume that $g(L,L) = 0$, $g(L,H) = g(H,L) = V > 0$, and $g(H,H) = (1+\alpha)V$, for $\alpha \geq 0$.[2] The parameter α

[2] Our results will hold if $0 < g(L,L) < V$. Our assumption, $g(L,L) = 0$, simplifies the exposition.

captures possible nonlinearity or complementarity in the signals. For example, for a fixed V, if $\alpha \to \infty$ then the object is essentially valuable only if both players get high signals, while if $\alpha \to 0$, then one high signal already makes the object sufficiently valuable while the second high signal does not add much extra value.

Two players spend some resources in order to obtain the object. The object will be allocated to players according to some mechanism. We first consider the standard mechanisms: a lottery and auctions (all-pay, first-price, and second-price). Subsequently, we consider an optimal mechanism and demonstrate how it can be implemented as a modification of each of the standard mechanisms.

3 Lottery

In this section we assume that the object is allocated to the players according to a lottery. If bids of two players are x_i and x_j, then player i wins the object with probability $\frac{x_i}{x_i+x_j}$, $j \neq i$, if $x_i > 0$ and zero otherwise. The bids are sunk, thus both the winner and the loser of a lottery pay their bids.

A pure strategy of player i consists of two bids, one if his private signal is L and the other if his signal is H. Let us call the first bid x_L^i and the second x_H^i. Thus a pure strategy is $x^i = (x_L^i, x_H^i)$. The expected payoff of player i, conditional on the received signal is

$$u_i(x_L^i, x^j | s_i = L) = (1-p)\frac{x_L^i}{x_L^i + x_L^j}0 + p\frac{x_L^i}{x_L^i + x_H^j}V - x_L^i; \tag{1}$$

$$u_i(x_H^i, x^j | s_i = H) = (1-p)\frac{x_H^i}{x_H^i + x_L^j}V + p\frac{x_H^i}{x_H^i + x_H^j}(1+\alpha)V - x_H^i. \tag{2}$$

We denote this game by $\mathcal{L}$.

3.1 Symmetric Equilibrium

A unique pure-strategy equilibrium of the lottery game is characterized in the following proposition:

Proposition 1 *In the unique pure-strategy equilibrium of the lottery game $\mathcal{L}$, equilibrium bids $x_L^i = x_L^j = x_L$ and $x_H^i = x_H^j = x_H$ are*

$$x_L = \begin{cases} \frac{1}{4}pV\left(1-p+D(p,\alpha)\right)\left(1+p-D(p,\alpha)\right), & \text{if } 0 \leq \alpha \leq 3, \\ 0, & \text{if } \alpha > 3, \end{cases} \tag{3}$$

and

$$x_H = \begin{cases} \frac{1}{4}pV\left(1-p+D(p,\alpha)\right)^2, & \textit{if } 0 \le \alpha \le 3, \\ \frac{1}{4}p(1+\alpha)V, & \textit{if } \alpha > 3, \end{cases} \tag{4}$$

where

$$D(p,\alpha) = \sqrt{1-p+p^2+\alpha p}.$$

Proof. See Appendix A.1.

Equilibrium bids have a piece-wise structure because for $\alpha \ge 3$ the payoff maximization problem of the low-signal player has a corner solution. The following example illustrates the equilibrium bidding as functions of α, for a given value of p.

Example 1. Suppose that $V = 1$ and $p = 1/4$. Then

$$x_L = \begin{cases} \frac{1}{16}\left(\frac{3}{4}+\frac{1}{4}\sqrt{13+4\alpha}\right)\left(\frac{5}{4}-\frac{1}{4}\sqrt{13+4\alpha}\right), & \text{if } 0 \le \alpha \le 3, \\ 0, & \text{if } \alpha > 3, \end{cases}$$

and

$$x_H = \begin{cases} \frac{1}{16}\left(\frac{3}{4}+\frac{1}{4}\sqrt{13+4\alpha}\right)^2, & \text{if } 0 \le \alpha \le 3, \\ \frac{1}{16}(1+\alpha), & \text{if } \alpha > 3. \end{cases}$$

These bids are drawn in Figure 1. As it can be seen from the figure, equilibrium bid x_H increases with α, while x_L decreases with α.

Some equilibrium properties also hold more generally. Specifically, x_H is greater than x_L, equilibrium bids are monotonic in both p and α, and the difference $x_H - x_L$ increases with α.

Proposition 2 *For the equilibrium bids x_L and x_H in the lottery game $\mathcal{L}$,*

- *$x_H \ge x_L$, with equality only for $p = 0$ or for $\alpha = 0$ and $p = 1$;*
- *x_L is decreasing in α and increasing in p, i.e. $\frac{\partial x_L}{\partial \alpha} \le 0$ and $\frac{\partial x_L}{\partial p} \ge 0$;*
- *x_H is increasing in both α and p, i.e. $\frac{\partial x_H}{\partial \alpha} \ge 0$ and $\frac{\partial x_H}{\partial p} \ge 0$;*
- *$\frac{\partial}{\partial \alpha}(x_H - x_L) \ge 0$, with equality only for $p = 0$.*

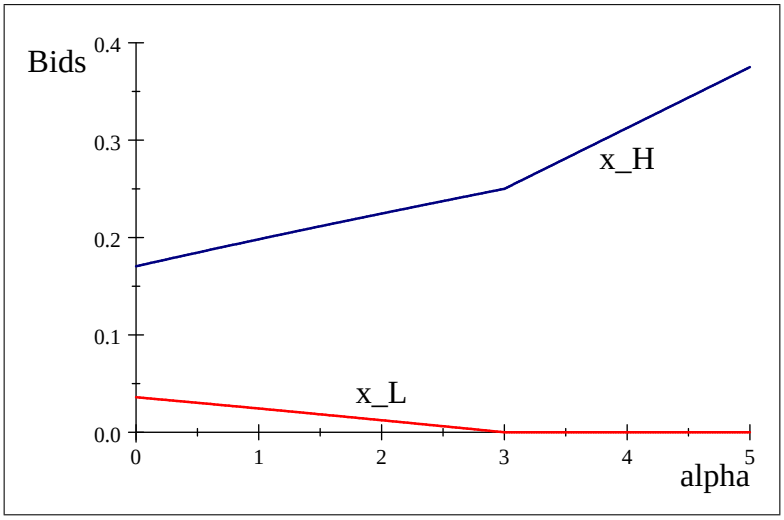

Figure 1: Lottery equilibrium bids as functions of α for $V = 1$ and $p = 1/4$.

Proof. See Appendix A.1.

Figures 1-4 illustrate the proposition. They show equilibrium bids as functions of p and α, for $V = 1$. Both equilibrium bids are increasing in p, i.e. the higher is the probability that the opponent has a high signal (and thus the prize is higher), the higher are the bids. The rate of change in α is different though: it is the highest for x_H and the lowest for x_L for higher α. It is rather surprising that equilibrium bid x_L decreases with α. For any p, equilibrium bid x_H is increasing in α. A player with a high signal is willing to spend more for a higher prize, and, as a result, at $\alpha = 3$, this spending is so high that a low-signal player drops out of the contest altogether.

A player with the high signal spends more, and the difference is increasing in α. Figure 4 plots $(x_H - x_L)$ as a function of p for various values of α, for $V = 1$. Note that the difference $(x_H - x_L)$ is always non-negative. If $\alpha = 0$, then this difference is smallest and non-monotonic in p. For the other considered values of α this difference is increasing in p.

Both x_H and x_L increase in p but behave differently with respect to α. This opens the possibility that the expected sum of bids, which can be interpreted as seller's revenue, may be non-monotonic in α, although α directly relates to the prize value. The next section considers this question.

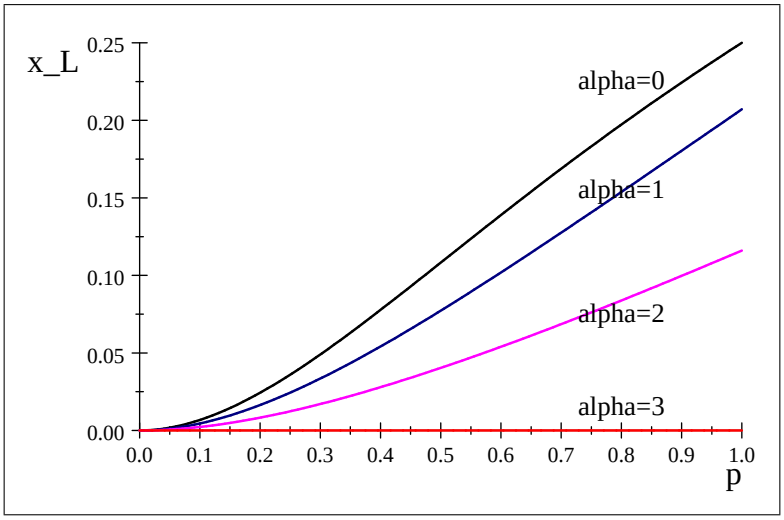

Figure 2: Lottery equilibrium bids x_L as functions of p for $V = 1$ and various α

3.2 Ex-ante Expected Revenue

The ex-ante expected revenue in the equilibrium of the lottery game is

$$\pi^{\mathcal{L}}(p,\alpha) = (1-p)^2(2x_L) + 2(1-p)p(x_L + x_H) + p^2(2x_H) = 2((1-p)x_L + px_H). \quad (5)$$

Since the equilibrium bids are given by equations (4) and (3), we get the following result.

Proposition 3 *In the equilibrium of the lottery game $\mathcal{L}$, the ex-ante expected revenue is*

$$\pi^{\mathcal{L}}(p,\alpha) = \begin{cases} \frac{1}{2}pV(1-p+D(p,\alpha))\left(1+p-2p^2+(2p-1)D(p,\alpha)\right), & \text{if } 0 \le \alpha \le 3, \\ \frac{1}{2}p^2(1+\alpha)V, & \text{if } \alpha > 3. \end{cases} \quad (6)$$

The following example illustrates the possibility that the ex-ante expected revenue is non-monotonic in α.

Example 1 continued. Suppose that $V = 1$ and $p = 1/4$. Then, the ex-ante expected revenue is

$$\pi^{\mathcal{L}}\left(\frac{1}{4},\alpha\right) = \begin{cases} \frac{1}{8}\left(\frac{3}{4}+\frac{1}{4}\sqrt{13+4\alpha}\right)\left(\frac{9}{8}-\frac{1}{8}\sqrt{13+4\alpha}\right), & \text{if } 0 \le \alpha \le 3, \\ \frac{1}{32}(1+\alpha), & \text{if } \alpha > 3. \end{cases}$$

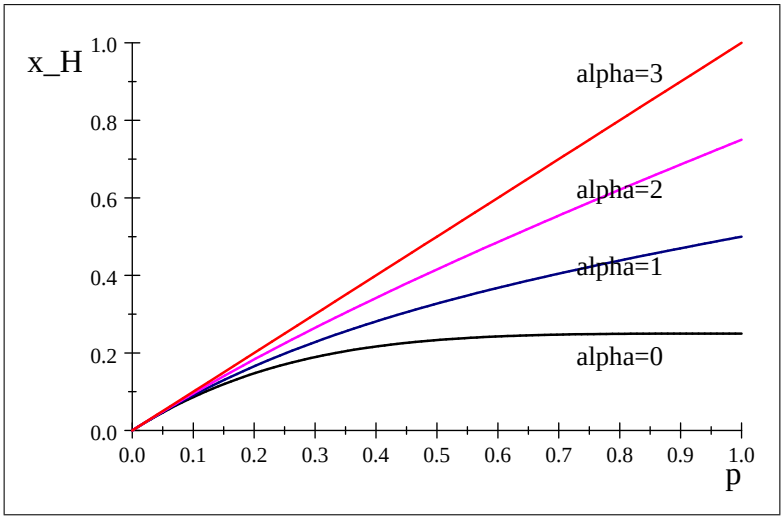

Figure 3: Lottery equilibrium bids x_H as functions of p for $V = 1$ and various α

Figure 5 illustrates this example. Note that the revenue decreases in α for $0 \leq \alpha \leq 3$, but increases for larger values of α. The revenue is non-monotonic in α for a particular value $p = 1/4$ in this example. In general, the ex-ante expected revenue increases in p but has a more intricate dependence on parameter α as the following proposition shows.

Proposition 4 *In the equilibrium of the lottery game $\mathcal{L}$:*

- *The ex-ante expected revenue is increasing in p, i.e. $\frac{\partial}{\partial p}\pi^{\mathcal{L}}(p,\alpha) \geq 0$, for any α.*
- *Suppose that $\alpha > 3$. Then, the ex-ante expected revenue is increasing in α, i.e. $\frac{\partial}{\partial \alpha}\pi^{\mathcal{L}}(p,\alpha) \geq 0$, for any p.*
- *Suppose that $0 \leq \alpha \leq 3$. Then,*

  i) the ex-ante expected revenue is decreasing in α, i.e. $\frac{\partial}{\partial \alpha}\pi^{\mathcal{L}}(p,\alpha) \leq 0$, for any α if $p \leq \frac{1}{2} - \frac{1}{10}\sqrt{5} \approx 0.27639$;

  ii) the ex-ante expected revenue is increasing in α, i.e. $\frac{\partial}{\partial \alpha}\pi^{\mathcal{L}}(p,\alpha) \geq 0$, for any α if $p \geq \frac{1}{3} \approx 0.33333$.

Proof. See Appendix A.1.

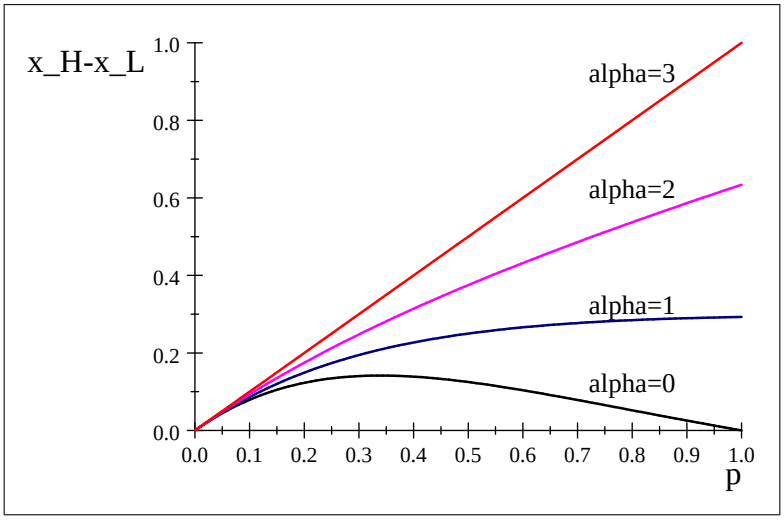

Figure 4: Difference $x_H - x_L$ in lottery equilibrium bids as function of p for $V = 1$ and various α

Figure 6 illustrates the proposition. The ex-ante expected revenue is increasing in p for all values of α, since both x_L and x_H are increasing with p. However, for low p, somewhat counter-intuitively, the ex-ante expected revenue for higher α is lower than that for lower α. The reason is that as α increases, although a high-signal player increases his bid, a low-signal player decreases his. A low value of p means that it is more likely that players will get a low signal thus in expectations the total bid goes down.

3.3 Overdissipation

We define overdissipation as a situation in which total spending is greater than the ex-post prize value, or

$$TS\,(p, \alpha) > g(s_1, s_2).$$

The total spending is

$$TS\,(p, \alpha) = \begin{cases} 2x_L, & \text{if } g(s_1, s_2) = 0, \\ x_L + x_H, & \text{if } g(s_1, s_2) = V, \\ 2x_H, & \text{if } g(s_1, s_2) = (1+\alpha)V. \end{cases}$$

Overdissipation can occur ex-post because players are not sure about the value of the prize ex ante. Namely,

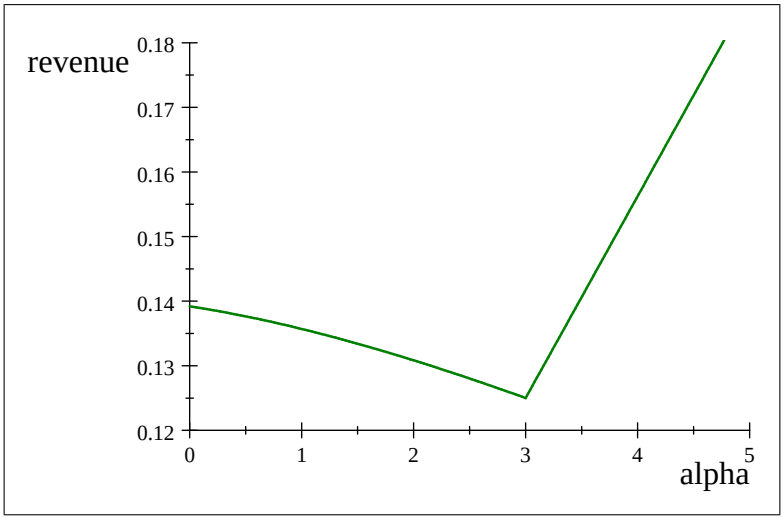

Figure 5: Lottery's expected revenue as a function of α for $V = 1$ and $p = 1/4$

Proposition 5 *Overdissipation occurs:*

- *For $\alpha < 3$ (and $p > 0$), if the value of the prize is 0;*
- *For $\alpha > 3$ and $p > \frac{4}{1+\alpha}$, if the value of the prize is V.*

Proof. See Appendix A.1.

If $\alpha < 3$, the low-signal player bids a positive amount hoping that the other player has a high signal and thus the value is V. If the prize value is 0, then each player bids $x_L > 0$. Not only overdissipation takes place but the lottery winner has a negative payoff, suffering a "winner's curse". If $\alpha > 3$, the high-signal player hopes that the prize is very large, $(1 + \alpha) V$, and bids so much that his expenditure is higher than V. If the prize value is V, the player wins the prize but again suffers a "winner's curse" (the loser, who must have received a low signal, bids 0 and gets 0 payoff).

4 Auctions

We consider all-pay, first-price, and second-price auctions in this section. In these auctions, both players submit bids b^i and b^j and the winner is determined by the highest bid: if

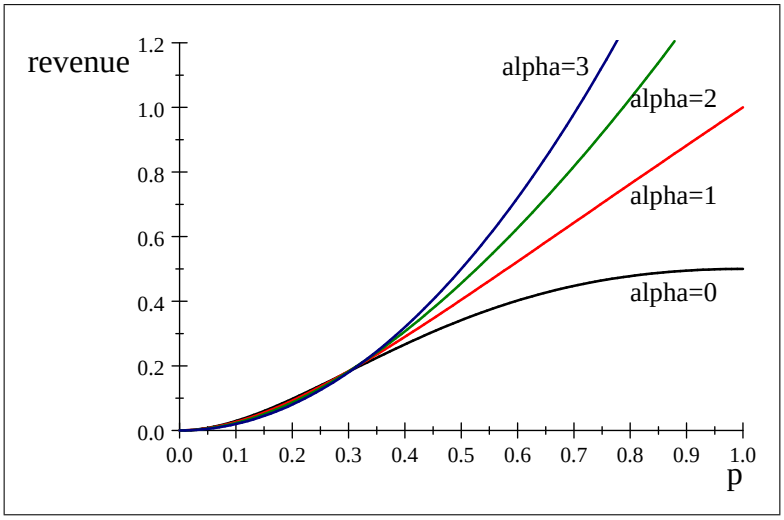

Figure 6: Lottery's expected revenue as a function of p for $V = 1$ and various α

$b^i > b^j$, then player i wins the object with probability 1 (in the case of a tie, $b^i = b^j$, each player gets the object with probability 1/2). While the allocation rule is the same in all three types of auctions, the payments differ as specified below. We denote by $\mathcal{A}$ any game with an auction allocation rule.

As in the lottery game, player i's auction strategy consists of two bids (or distributions of bids) $\left(b_L^i, b_H^i\right)$, depending on the signal player i gets. The unique equilibria of the all-pay and of the first-price auctions are symmetric; for the second-price auction there are multiple equilibria but a unique symmetric one. In a symmetric equilibrium both players use the same strategy thus we denote by b_L, b_H the symmetric equilibrium bids (or distributions of bids) and use the superscript to refer to the auction type being considered.

4.1 All-Pay Auction

In an all-pay auction, the highest bidder wins the auction and both bidders pay their bids. The expected payoffs of player i are

$$u_i(b_L^i, b^j|s_i = L) = \begin{cases} (1-p)0 + pV - b_L^i, & \text{if } b_L^i > b^j, \\ ((1-p)0 + pV)/2 - b_L^i, & \text{if } b_L^i = b^j, \\ -b_L^i, & \text{if } b_L^i < b^j, \end{cases}$$

$$u_i(b_H^i, b^j|s_i = H) = \begin{cases} (1-p)V + p(1+\alpha)V - b_H^i, & \text{if } b_H^i > b^j, \\ ((1-p)V + p(1+\alpha)V)/2 - b_H^i, & \text{if } b_H^i = b^j, \\ -b_H^i, & \text{if } b_H^i < b^j. \end{cases}$$

We denote the game described by the all-pay auction by $\mathcal{A}^{All}$. This auction is similar to the lottery in the sense that bids are sunk: both the winner and the loser of the all-pay auction have to pay their bids.

Proposition 6 *Bids $b_L^{All} = 0$ and b_H^{All} distributed on the interval $[0, p(1+\alpha)V]$ according to the cumulative distribution function $F_{All}(x) = \frac{x}{p(1+\alpha)V}$ constitute a unique equilibrium of the all-pay auction $\mathcal{A}^{All}$.*

Proof: See Appendix A.2.

Note that in this equilibrium, the low-signal player does not make a positive bid, while the high-signal player uniformly randomizes over an interval. This equilibrium is similar to the one derived in Siegel (2014) in a common-value all-pay auction setting with several discrete signals.

4.2 First-Price Auction

In the first-price auction, the highest bidder wins the auction and is the only one to pay his bid. The payoffs of player i are

$$u_i(b_L^i, b^j|s_i = L) = \begin{cases} (1-p)0 + pV - b_L^i, & \text{if } b_L^i > b^j, \\ \left((1-p)0 + pV - b_L^i\right)/2, & \text{if } b_L^i = b^j, \\ 0, & \text{if } b_L^i < b^j, \end{cases}$$

$$u_i(b_H^i, b^j|s_i = H) = \begin{cases} (1-p)V + p(1+\alpha)V - b_H^i, & \text{if } b_H^i > b^j, \\ \left((1-p)V + p(1+\alpha)V - b_H^i\right)/2, & \text{if } b_H^i = b^j, \\ 0, & \text{if } b_H^i < b^j. \end{cases}$$

The game described by the first-price auction is denoted by $\mathcal{A}^F$.

Proposition 7 *Bids $b_L^F = 0$ and b_H^F distributed on the interval $[0, p(1+\alpha)V]$ according to the cumulative distribution function $F_F(x) = \frac{(1-p)x}{p[(1+\alpha)V-x]}$ constitute a unique equilibrium of the first-price auction $\mathcal{A}^F$.*

Proof: See Appendix A.2.

In this equilibrium of the first-price auction, as in the all-pay auction, the low-signal player does not make a positive bid, while the high-signal player mixes over the same interval as in the all-pay auction, although with different probabilities. The high-signal player bids more aggressively in the first-price auction than in the all-pay auction because he does not lose his bid in the case of losing the auction. This kind of equilibrium is also derived in Wang (1991) for a different common-value first-price auction setting with an arbitrary finite number of signals.

4.3 Second-Price Auction

In the second-price auction, the highest bidder wins the auction and pays the lowest bid. The payoffs of player i are

$$u_i(b_L^i, b^j | s_i = L) = \begin{cases} (1-p)0 + pV - b^j, & \text{if } b_L^i > b^j, \\ \left((1-p)0 + pV - b_L^i\right)/2, & \text{if } b_L^i = b^j, \\ 0, & \text{if } b_L^i < b^j, \end{cases}$$

$$u_i(b_H^i, b^j | s_i = H) = \begin{cases} (1-p)V + p(1+\alpha)V - b^j, & \text{if } b_H^i > b^j, \\ \left((1-p)V + p(1+\alpha)V - b_H^i\right)/2, & \text{if } b_H^i = b^j, \\ 0, & \text{if } b_H^i < b^j. \end{cases}$$

We denote the game described by the second-price auction by $\mathcal{A}^S$.

Proposition 8 *Bids $b_L^S = 0$ and $b_H^S = (1+\alpha)V$ constitute a unique symmetric equilibrium of the second-price auction $\mathcal{A}^S$.*

Proof: See Appendix A.2.

Note that in this symmetric equilibrium bids are independent from p. The low-signal player bids 0 because he can win only against another low-signal player, in which case the

value of the object is 0. Because of this, the high-signal player knows that the outcome does not depend on his bid if the other player has a low signal; but the possibility of meeting another high-signal player pushes his bid up to the maximum value $(1+\alpha)V$. This kind of equilibrium is similar to the ones considered in Klemperer (1998) and Bolton and Dewatripont (2004, Ch. 7) for different signal distributions. There are also asymmetric equilibria (e.g. $b_L^i = b_H^i = 0$; $b_L^j = b_H^j = (1+\alpha)V$) but they are much less appealing in our symmetric setting.

4.4 Expected Revenue

In this section we compare expected revenues in auctions. Surprisingly, it turns out that in our setting all three auction formats are revenue equivalent. We calculate the ex-ante expected revenue in three auctions and demonstrate the revenue equivalence result in the next proposition.

Proposition 9 *The ex-ante expected revenues in symmetric equilibria in the all-pay, first-price, and second-price auctions are the same and equal to* $\pi^{\mathcal{A}}(p,\alpha) = p^2(1+\alpha)V$.

Proof. See Appendix A.2.

Although in general common-value auctions the revenue equivalence does not hold (see Milgrom and Weber, 1982), in our two-player two-signal setting it holds (see also Bolton and Dewatripont, 2004, Ch. 7, for a different two-signal distribution example). This means that we can consider one expression $\pi^{\mathcal{A}}(p,\alpha)$ for the expected revenue from any of the three auctions.

It is straightforward to see that the ex-ante expected revenue in auctions is monotonic in p and α. The following proposition is stated without proof:

Proposition 10 *For the ex-ante expected revenue in any of the auctions* $\mathcal{A}$*:*

- *Revenue is increasing in* p*, i.e.* $\frac{\partial}{\partial p}\pi^{\mathcal{A}}(p,\alpha) \geq 0$*, for any* α*.*
- *Revenue is increasing in* α*, i.e.* $\frac{\partial}{\partial \alpha}\pi^{\mathcal{A}}(p,\alpha) \geq 0$*, for any* p*.*

Unlike in the lottery game, the expected revenue always increases with p and α. In all three auctions, low-signal players drop out of bidding and high-signal players bid aggressively in the equilibria for any p and α.

5 Revenue Comparison

In this section we present our main result: the revenue comparison between the lottery and the auctions. It turns out that there is no clear ranking: for some values of p and α, the expected revenue is higher in the auctions but for others it is higher in the lottery.

As we have already seen in (6), the ex-ante expected revenue in the lottery is

$$\pi^{\mathcal{L}}(p,\alpha) = \begin{cases} \frac{1}{2}pV\left(1-p+D(p,\alpha)\right)\left(1+p-2p^2+(2p-1)D(p,\alpha)\right), & \text{if } 0 \leq \alpha \leq 3, \\ \frac{1}{2}p^2(1+\alpha)V, & \text{if } \alpha > 3. \end{cases}$$

Proposition 9 demonstrates the revenue equivalence result for the three auctions, showing that the ex-ante expected revenue in the auctions is

$$\pi^{\mathcal{A}}(p,\alpha) = p^2(1+\alpha)V.$$

Note that if $\alpha > 3$, then $\pi^{\mathcal{A}}(p,\alpha) > \pi^{\mathcal{L}}(p,\alpha)$ for any $p > 0$. If $0 \leq \alpha \leq 3$, then $\pi^{\mathcal{A}}(p,\alpha) < \pi^{\mathcal{L}}(p,\alpha)$ if and only if

$$p(1+\alpha) < \frac{1}{2}\left(1-p+D(p,\alpha)\right)\left(1+p-2p^2+(2p-1)D(p,\alpha)\right),$$

where $D(p,\alpha) = \sqrt{1-p+p^2+\alpha p}$. This gives a condition for the ex-ante expected revenue to be higher in the lottery than in an auction. The main result of this section is the following revenue comparison:

Theorem 1 *If $p \in [\widehat{p}, 1]$, then*

$$\pi^{\mathcal{A}}(p,\alpha) > \pi^{\mathcal{L}}(p,\alpha) \text{ for any } \alpha \geq 0.$$

If $p \in (0, \widehat{p})$, then

$$\pi^{\mathcal{A}}(p,\alpha) > \pi^{\mathcal{L}}(p,\alpha), \quad \text{for any } \alpha > \alpha(p),$$
$$\pi^{\mathcal{A}}(p,\alpha) < \pi^{\mathcal{L}}(p,\alpha), \quad \text{for any } \alpha < \alpha(p),$$

where $\widehat{p} = 0.1\left(9-\sqrt{6}\right) \approx 0.65505$ and $\alpha(p) = \frac{20p^2-36p+15}{4p^2-12p+9}$.

Proof. See Appendix A.3.

Theorem 1 describes the cases in which the ex-ante auction revenue is higher than the ex-ante lottery revenue and the cases in which the opposite holds. Figure 7 illustrates values of p and α discussed in Theorem 1.

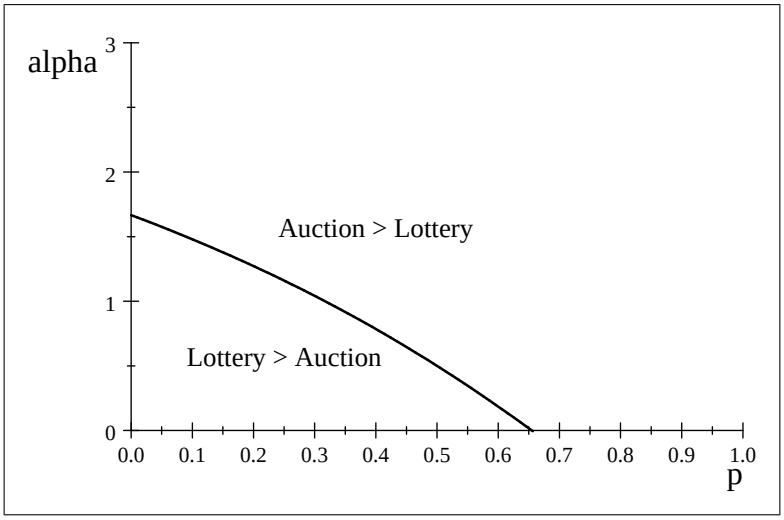

Figure 7: Revenue comparison between the lottery and auctions

For small values of p and α, the ex-ante expected lottery revenue is higher than the ex-ante expected auction revenue. This result is startling because an auction is typically considered as a mechanism leading to the highest expected revenue (see, for example, Wärneryd, 2012, p. 278). The reason for the higher expected revenue in the lottery is that the low-signal player drops out of the bidding for any value of p and α in the common-value auctions equilibria. In the lottery, however, such a player has a positive chance of winning by submitting a small positive bid. Although the high-signal player bids less in the lottery than in an auction, if the probability p that a player has a high signal and the complementarity α between the signals are small, then the ex-ante expected revenue from low-signal players outweights the losses from high-signal players in the lottery. On the other hand, if p and/or α are large enough, then the ex-ante expected revenue is higher in an auction.

Example 1 continued. Suppose that $V = 1$ and $p = 1/4$. Then

$$\pi^{\mathcal{L}}\left(\frac{1}{4}, \alpha\right) = \begin{cases} \frac{1}{8}\left(\frac{3}{4} + \frac{1}{4}\sqrt{13+4\alpha}\right)\left(\frac{9}{8} - \frac{1}{8}\sqrt{13+4\alpha}\right), & \text{if } 0 \leq \alpha \leq 3, \\ \frac{1}{32}(1+\alpha), & \text{if } \alpha > 3, \end{cases}$$

and

$$\pi^{\mathcal{A}}\left(\frac{1}{4}, \alpha\right) = \frac{1}{16}(1+\alpha).$$

Figure 8 illustrates the expected revenues in the lottery and auctions. It shows that if

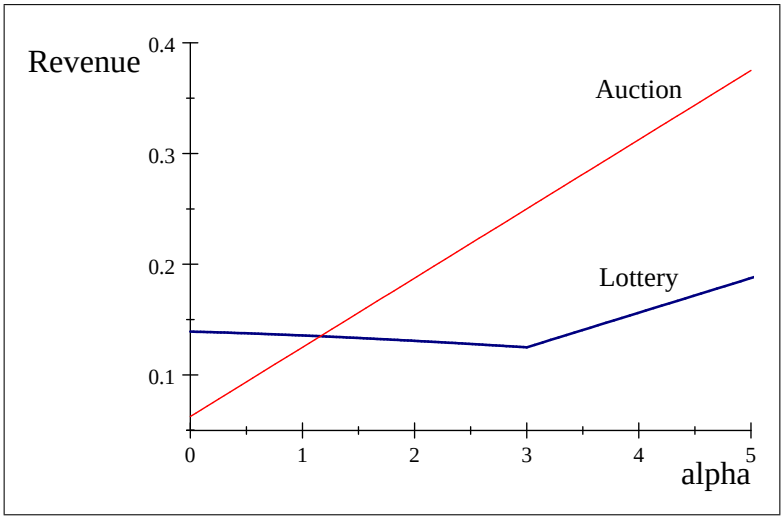

Figure 8: Expected revenues in lottery and auctions for $V = 1$ and $p = 1/4$

high signals are "substitutes" (the second high signal does not add much to the value, $\alpha < 1$), then the lottery generates a higher expected revenue. However, if high signals are "complements" (the second high signal reinforces the first, $\alpha > 1$), then an auction gives a higher expected revenue. If α is close to 1 (the second high signal is as good as the first) then both the lottery and an auction give approximately the same revenue.

6 Optimal Mechanism and Reserve Prices

As we have seen above, either the lottery or an auction can have higher ex-ante expected revenue. In this section we identify the maximum ex-ante expected revenue that the seller can obtain for given parameter values, p and α. We show how this revenue can be achieved both with the lottery and auction mechanisms using appropriate reserve prices.

6.1 Seller's Revenue Maximization

Recall that in our setting each player gets an independent signal (H with probability p and L with probability $1-p$) and the value is determined by the signals in the following way:

$$v = \begin{cases} 0, & \text{if signals are } (L,L), \\ V, & \text{if signals are } (L,H) \text{ or } (H,L), \\ (1+\alpha)V, & \text{if signals are } (H,H). \end{cases}$$

The signals are uncorrelated across bidders thus the result of Cremer and McLean (1985) about full surplus extraction does not apply. Nevertheless, in our setting the full surplus extraction by the seller is possible, as we show below.

If the seller learns the players' signals and, therefore, the value of the object, then the seller could extract this value from the players. This ex-ante value is

$$\pi^{FB} = p^2(1+\alpha)V + 2p(1-p)V + (1-p)^2 0 = pV(p(1+\alpha) + 2(1-p)).$$

This first-best ex-ante expected revenue is the maximum expected revenue that the seller can hope for in any mechanism. If the seller gets this revenue, there is no surplus left for the bidders. The next proposition shows that the seller can indeed achieve this revenue.

Proposition 11 *There exists a (direct symmetric) mechanism in which the seller's ex-ante expected revenue is equal to π^{FB}.*

Proof. See Appendix A.4.

The proof shows that if a direct symmetric mechanism is described by $T_{\hat{s}_i\hat{s}_j}$ and $q_{\hat{s}_i\hat{s}_j}$, where $(\hat{s}_i, \hat{s}_j)$ are the reported signals from the set $\{(L,L),(L,H),(H,L),(H,H)\}$, $T_{\hat{s}_i\hat{s}_j}$ is the payment of player i conditional on this player reporting $\hat{s}_i$ and the other player reporting $\hat{s}_j$, and $q_{\hat{s}_i\hat{s}_j}$ is the probability that player i gets the object conditional on the reports, then an optimal mechanism involves

$$q_{HH} = \frac{1}{2},\ q_{HL} = 1,\ q_{LH} = 0,\ q_{LL} = 0$$

and

$$T_{LL} = T_{LH} = 0,\ T_{HL} = V,\ T_{HH} = \frac{1}{2}(1+\alpha)V.$$

This is an intuitive mechanism: the object is given to the buyer with the highest (reported) signal who pays the object's value (according to the reported signals); in the case of a

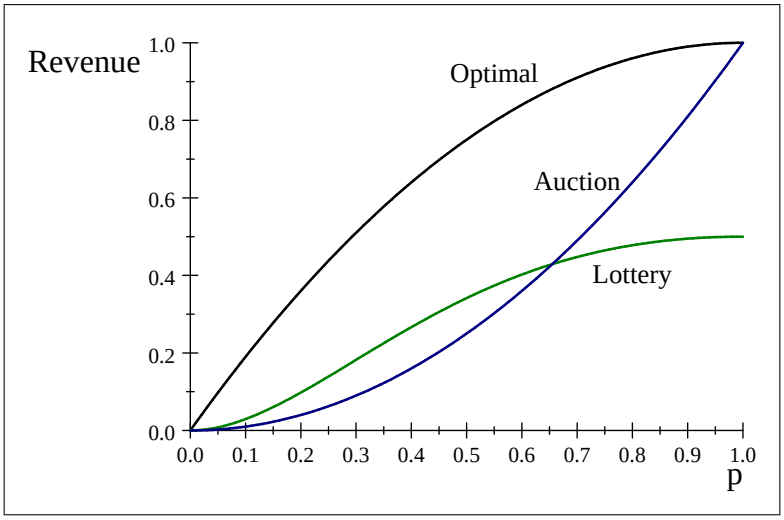

Figure 9: Expected revenues for $\alpha = 0$.

tie with high signals, each buyer pays half of the value and gets the object with equal probability. The important part of the mechanism is that if both signals are low, the seller does not sell the object. We will see below that the reserve price works in a similar way. Note that for the equality of the first-best revenue and this optimal mechanism's revenue it is important that the value is 0 with the two low signals thus the seller can retain the object in this case.

Figures 9 and 10 present the ex-ante expected revenues in the optimal mechanism, in the lottery, and in an auction as functions of p for a small and a large value of α (taking $V = 1$). The figures confirm that if α is small, the ex-ante expected revenue in the lottery can be higher than in an auction although this revenue is still below the optimal revenue. For larger α, the ex-ante expected revenue is higher in an auction. Note also that as $p \to 1$, the ex-ante auction revenue approaches the optimal revenue, since in an auction it is high-signal bidders who compete strongly for the object.

6.2 Reserve Prices

The optimal mechanism in the previous subsection is a direct mechanism in which buyers are asked to report their signals. This subsection demonstrates how the optimal mechanism can be implemented as an auction of any type (or the lottery) with an appropriate reserve

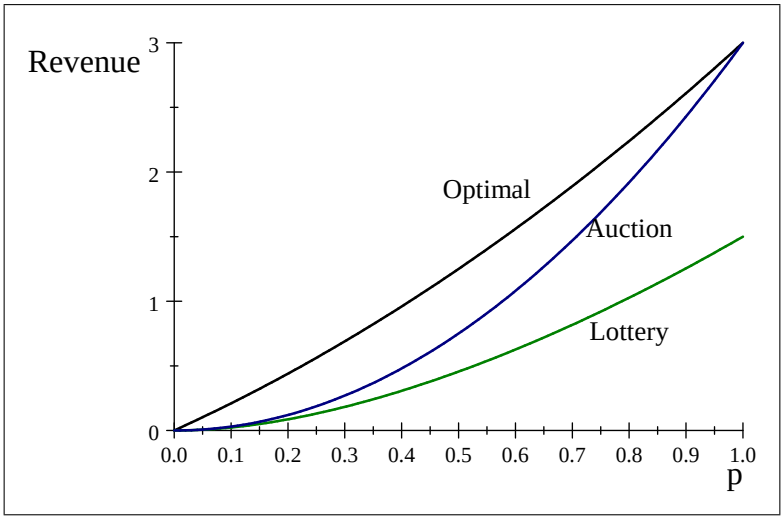

Figure 10: Expected revenues for $\alpha = 2$

price.

We assume that a reserve price r means that bids $x_i < r$ are not accepted, i.e. they do not enter the lottery or an auction. In the second-price auction, the reserve price also implies that if $x_i \geq r > x_j$, then the payment by bidder i is equal to r, i.e. the reserve price also plays the role of the second-highest bid in such a case. The following proposition states the main implementation result:

Proposition 12 *For any auction type (first-price, second-price, all-pay) or for the lottery, there exists a reserve price such that there exists an equilibrium in which the seller's ex-ante expected revenue is equal to π^{FB}.*

Proof. See Appendix A.4.

It follows from Proposition 12 that all four considered mechanisms allow the seller to achieve the full surplus extractions with an appropriate reserve price. The proof is constructive: for each of the four mechanisms, a reserve price and the corresponding symmetric equilibrium is found such that the seller's ex-ante expected revenue is equal to π^{FB}. The reserve prices are different across mechanisms: for the lottery the optimal reserve price is $r_{Lot} = p\frac{1}{2}(1+\alpha)V + (1-p)V$, for the second- and first-price auctions it is $r_S = r_F = V$, and for the all-pay auction it is $r_{All} = (1-p)V$. Note that the reserve prices are chosen in such

a way as to make the surplus of the high-signal bidder equal to 0 and force the low-signal bidder to submit zero bid.

7 Conclusion

Lotteries and auctions are typical allocation mechanisms. However, they are usually considered in different informational situations. On the one hand, lotteries are typically analyzed in the common-value complete information context. On the other hand, auctions are often considered in the private information environment. In this paper, we analyze both lottery and auctions mechanisms in the same informational setting where players have private information about a common-value prize. First, we construct unique symmetric equilibria for the lottery and the auctions. Second, we show that all standard auctions have the same ex-ante expected revenue. Third, we compare ex-ante expected revenues across the mechanisms. It turns out that if individual signals are "substitutes" – the second high signal does not increase the common value by much – and if the probability of high signal is small, then the ex-ante expected revenue in the lottery is higher than in an auction. Otherwise, the ex-ante expected revenue in an auction is higher than in the lottery. Thus our setting provides a possible rationale for using "beauty contests" or lotteries as allocation mechanisms of objects with common although not perfectly known value in some situations. Finally, we consider how the seller can modify the mechanisms using the reserve price in order to get higher revenue. We find that the seller could extract full surplus from the buyers in both the lottery and auction mechanisms using an appropriate reserve price to exclude the low-signal player and push the high-signal player's bid sufficiently high.

A Appendix

A.1 Proofs for the Lottery

Proof of Proposition 1. The first order conditions of player i's problems to maximize payoffs $(1)-(2)$ are

$$\begin{aligned} p\frac{x_H^j}{(x_L^i+x_H^j)^2}V-1 &= 0; \\ (1-p)\frac{x_L^j}{(x_H^i+x_L^j)^2}V+p\frac{x_H^j}{(x_H^i+x_H^j)^2}(1+\alpha)V-1 &= 0. \end{aligned}$$

The second order conditions are clearly satisfied as the left-hand sides of the above expressions are decreasing in x_L^i and x_H^i respectively.

If $x_H^i \neq x_H^j$, then it is not possible to satisfy the two equations for each of two players simultaneously. Therefore, the equilibrium is symmetric, with $x_L^j = x_L^i = x_L$ and $x_H^j = x_H^i = x_H$

$$p\frac{x_H}{(x_L+x_H)^2}V-1 = 0; \tag{7}$$

$$(1-p)\frac{x_L}{(x_L+x_H)^2}V+p\frac{1}{4x_H}(1+\alpha)V-1 = 0. \tag{8}$$

From equation (7) $x_L = \sqrt{pVx_H} - x_H$. Equation (8) becomes

$$-4x_H + 4(1-p)\sqrt{pV}\sqrt{x_H} + (1+\alpha)p^2V = 0.$$

A unique positive solution of this quadratic equation is

$$x_H = \frac{1}{4}pV(1-p+D(p,\alpha))^2,$$

where

$$D(p,\alpha) = \sqrt{1-p+p^2+\alpha p}. \tag{9}$$

Then,

$$x_L = \frac{1}{4}pV\left(1-p+D(p,\alpha)\right)\left(1+p-D(p,\alpha)\right).$$

The last expression becomes 0 for $\alpha = 3$ and negative for $\alpha > 3$. Thus, the previous derivations hold for $\alpha \leq 3$. For $\alpha > 3$, $x_L^i = x_L^j = x_L = 0$. Then, $x_H^i = x_H^j = x_H =$

$\frac{1}{4}p(1+\alpha)V$. If $x_H = \frac{1}{4}p(1+\alpha)V$, then the solution of the payoff maximization problem for the low-signal player is indeed the corner solution, $x_L = 0$.

Note also that there is no equilibrium in mixed strategies because the payoff functions are strictly concave. A player's best response is unique for any strategy (including mixed ones) of the other player, and, therefore, any equilibrium has to be in pure strategies. ■

For the subsequent proofs, it is useful to make the following observations.

Lemma 1 *Suppose* $0 \leq p \leq 1$ *and* $0 \leq \alpha \leq 3$. *Then*

- $D(p,\alpha) - p \geq 0$, *with equality only for* $\alpha = 0$ *and* $p = 1$;
- $1 - p + D(p,\alpha) > 0$;
- $1 + p - D(p,\alpha) \geq 0$, *with equality only for* $\alpha = 3$;
- $2\left(D(p,\alpha) - p\right) + (1-\alpha) \geq 0$, *with equality only for* $\alpha = 3$.

Proof. Observe that

$$D(p,\alpha) = \sqrt{(1 - p + p^2 + \alpha p)} \geq p,$$

because $1 - p + p^2 + \alpha p \geq p^2 \Leftrightarrow 1 - p + \alpha p \geq 0$, which holds for $\alpha \geq 0$ and $p \leq 1$. The equality holds only for $\alpha = 0$ and $p = 1$. Therefore $1 - p + D(p,\alpha) \geq 1 > 0$. Also,

$$1 + p \geq D(p,\alpha) = \sqrt{1 - p + p^2 + \alpha p}$$

if and only if $1 + 2p + p^2 \geq 1 - p + p^2 + \alpha p \Leftrightarrow 3p \geq \alpha p$, which holds for $\alpha \leq 3$. The equality holds only for $\alpha = 3$.

Since $D(p,\alpha) \geq p$, $2\left(D(p,\alpha) - p\right) + (1-\alpha) \geq 0$ if $0 \leq \alpha \leq 1$. Suppose that $1 < \alpha \leq 3$. Then,

$$D(p,\alpha) \geq p + \frac{(\alpha - 1)}{2},$$

if and only if

$$1 - p + p^2 + \alpha p \geq p^2 + (\alpha - 1)\,p + \frac{(\alpha-1)^2}{4} \Leftrightarrow \alpha \leq 3,$$

with equality only if $\alpha = 3$. ■

Proof of Proposition 2. Consider the difference

$$(x_H - x_L) = \begin{cases} \frac{1}{2}pV\left(1-p+D(p,\alpha)\right)\left(D(p,\alpha)-p\right), & \text{if } 0 \leq \alpha \leq 3, \\ \frac{1}{4}p(1+\alpha)V, & \text{if } \alpha \geq 3. \end{cases}$$

From Lemma 1, $(x_H - x_L) \geq 0$ with equality only for $p = 0$ or if $\alpha = 0$ and $p = 1$.

If $\alpha > 3$, then $x_L = 0$ and $\frac{\partial x_L}{\partial \alpha} = 0$. Consider $0 \leq \alpha < 3$. Note that

$$\frac{\partial D(p,\alpha)}{\partial \alpha} = \frac{p}{2D(p,\alpha)}$$

and

$$\frac{\partial D(p,\alpha)}{\partial p} = \frac{-1+2p+\alpha}{2D(p,\alpha)}.$$

Then

$$\frac{\partial x_L}{\partial \alpha} = \frac{p^2 V}{4}\frac{1}{D(p,\alpha)}\left(p - D(p,\alpha)\right).$$

Therefore, $\frac{\partial x_L}{\partial \alpha} \leq 0$ from Lemma 1. Note that $\frac{\partial x_L}{\partial \alpha} = 0$ only for $p = 0$ or for $\alpha = 0$ and $p = 1$.

Obviously, $\frac{\partial x_L}{\partial p} = 0$ for $\alpha \geq 3$. For $0 \leq \alpha \leq 3$,

$$\frac{\partial x_L}{\partial p} = \frac{1}{4}V\left[\left(1-p+D(p,\alpha)\right)\left(1+p-D(p,\alpha)\right) + p\frac{2\left(D(p,\alpha)-p\right)+(1-\alpha)}{D(p,\alpha)}\left(D(p,\alpha)-p\right)\right].$$

Therefore, from Lemma 1, $\frac{\partial x_L}{\partial p} \geq 0$. Note that $\frac{\partial x_L}{\partial p} = 0$ only for $\alpha = 3$.

Consider now $\frac{\partial x_H}{\partial \alpha}$. For $\alpha > 3$, $\frac{\partial x_H}{\partial \alpha} = \frac{1}{4}pV \geq 0$. Note that $\frac{\partial x_H}{\partial \alpha} = 0$ only if $p = 0$. Suppose that $0 \leq \alpha < 3$. Then,

$$\frac{\partial x_H}{\partial \alpha} = \frac{1}{4}pV\left(1-p+D(p,\alpha)\right)\frac{p}{D(p,\alpha)}.$$

From Lemma 1, $\frac{\partial x_H}{\partial \alpha} \geq 0$. Note that $\frac{\partial x_H}{\partial \alpha} = 0$ only if $p = 0$.

Consider $\frac{\partial x_H}{\partial p}$. If $\alpha \geq 3$, then $\frac{\partial x_H}{\partial p} = \frac{1}{4}(1+\alpha)V > 0$. Suppose that $0 \leq \alpha \leq 3$. Then,

$$\frac{\partial x_H}{\partial p} = \frac{1}{4}V\left(1-p+D(p,\alpha)\right)\left(1-p+D(p,\alpha) - p\frac{2\left(D(p,\alpha)-p\right)+(1-\alpha)}{D(p,\alpha)}\right).$$

Consider

$$\left(1-p+D(p,\alpha) - p\frac{2\left(D(p,\alpha)-p\right)+(1-\alpha)}{D(p,\alpha)}\right) =$$

$$\frac{D(p,\alpha) - 3pD(p,\alpha) + 1 - 2p + 3p^2 + 2\alpha p}{D(p,\alpha)}.$$

Suppose that $0 \leq p \leq 1/3$. Then $(1-3p)\,D(p,\alpha) \geq 0$ and $1 - 2p + 3p^2 + 2\alpha p > 0$, thus $(1-3p)\,D(p,\alpha) + 1 - 2p + 3p^2 + 2\alpha p > 0$. Suppose that $1/3 < p \leq 1$. Then, $(1-3p)\,D(p,\alpha) + 1 - 2p + 3p^2 + 2\alpha p \geq 0$ if and only if

$$\frac{1 - 2p + 3p^2 + 2\alpha p}{3p-1} \geq D(p,\alpha).$$

Since both sides of this inequality are positive, we get

$$\left(\frac{1 - 2p + 3p^2 + 2\alpha p}{3p-1}\right)^2 \geq \left(1 - p + p^2 + \alpha p\right),$$

or $p\,(\alpha+1)\left(4p\alpha + 3(1-p)^2\right) \geq 0$. Note that $p\,(\alpha+1)\left(4p\alpha + 3(1-p)^2\right) = 0$ if and only if $p = 0$ or if $p = 1$ and $\alpha = 0$.

Since $\frac{\partial x_H}{\partial \alpha} \geq 0$ and $\frac{\partial x_L}{\partial \alpha} \leq 0$, $\frac{\partial}{\partial \alpha}(x_H - x_L) \geq 0$, with equality only for $p = 0$. ■

Proof of Proposition 4. Note that

$$\frac{\partial}{\partial p}\pi^{\mathcal{L}}(p,\alpha) = 2\,(x_H - x_L) + 2\,(1-p)\left(\frac{\partial x_L}{\partial p}\right) + 2p\left(\frac{\partial x_H}{\partial p}\right).$$

From Proposition 2, $x_H \geq x_L$, $\frac{\partial x_L}{\partial p} \geq 0$, and $\frac{\partial x_H}{\partial p} \geq 0$. Hence, $\frac{\partial}{\partial p}\pi^{\mathcal{L}}(p,\alpha) \geq 0$.

Suppose that $\alpha > 3$. Then, $\pi^{\mathcal{L}}(p,\alpha) = \frac{1}{2}p^2(1+\alpha)V$. Therefore, $\frac{\partial}{\partial \alpha}\pi^{\mathcal{L}}(p,\alpha) = \frac{1}{2}p^2 V \geq 0$.

Suppose that $0 \leq \alpha \leq 3$. Then,

$$\pi^{\mathcal{L}}(p,\alpha) = \frac{1}{2}pV\left(1 - p + D(p,\alpha)\right)\left(1 + p - 2p^2 + (2p-1)D(p,\alpha)\right).$$

Since

$$\frac{\partial D(p,\alpha)}{\partial \alpha} = \frac{p}{2D(p,\alpha)},$$

we get

$$\frac{2}{pV}\frac{D(p,\alpha)}{p}\frac{\partial}{\partial \alpha}\pi^{\mathcal{L}}(p,\alpha) = 2p(1-p) + (2p-1)D(p,\alpha).$$

Then $\frac{\partial}{\partial \alpha}\pi^{\mathcal{L}}(p,\alpha) = 0$ if

$$2p(1-p) + (2p-1)D(p,\alpha) = 0. \tag{10}$$

Note that if $p \geq \frac{1}{2}$, then both terms are positive and the equality cannot be satisfied. In this case $\frac{\partial}{\partial \alpha}\pi^{\mathcal{L}}(p,\alpha) > 0$. Rewrite equality (10) as

$$2p(1-p) = (1-2p)\sqrt{(1 - p + p^2 + \alpha p)}. \tag{11}$$

For $p < \frac{1}{2}$, we get

$$\alpha = \frac{-5p^2 + 5p - 1}{p(1-2p)^2}.$$

Given that $p < \frac{1}{2}$, note that $\alpha \geq 0$ if $-5p^2 + 5p - 1 \geq 0$, or if

$$\frac{1}{2} - \frac{1}{10}\sqrt{5} \leq p < \frac{1}{2}. \tag{12}$$

On the other hand,

$$\frac{-5p^2 + 5p - 1}{p(1-2p)^2} \leq 3$$

if $-5p^2 + 5p - 1 - 3p(1-2p)^2 \leq 0$. This inequality holds for $0 \leq p < \frac{1}{2}$ if

$$0 \leq p \leq \frac{1}{3}. \tag{13}$$

Therefore, expression $\frac{5p-5p^2-1}{p(1-2p)^2}$ is between 0 and 3 only for p between $\frac{1}{2} - \frac{1}{10}\sqrt{5}$ and $\frac{1}{3}$. Hence, for any $p \in \left(\frac{1}{2} - \frac{1}{10}\sqrt{5}, \frac{1}{3}\right)$,

$$\frac{\partial}{\partial \alpha}\pi^{\mathcal{L}}(p,\alpha) \begin{array}{ll} < 0, & \text{for } \alpha > \frac{-5p^2+5p-1}{p(1-2p)^2}, \\ > 0, & \text{for } \alpha < \frac{-5p^2+5p-1}{p(1-2p)^2}. \end{array}$$

Moreover, $\pi^{\mathcal{L}}(p,\alpha)$ reaches a local maximum at $\alpha = \frac{5p-5p^2-1}{p(1-2p)^2}$.

It is straightforward to check that for any $\alpha \in [0,3]$,

$$\frac{\partial}{\partial \alpha}\pi^{\mathcal{L}}(p,\alpha) \begin{array}{ll} < 0, & \text{for } p < \frac{1}{2} - \frac{1}{10}\sqrt{5}, \\ > 0, & \text{for } p > \frac{1}{3}. \end{array} \blacksquare$$

Proof of Proposition 5. Consider the three cases corresponding to the three possible values of the prize.

Suppose that $g(s_1, s_2) = 0$. Then, both players have low signals and

$$TS(p,\alpha) > g(s_1, s_2) = 0,$$

if $2x_L > 0$, or, from (3), if $0 \leq \alpha < 3$.

Suppose that $g(s_1, s_2) = V$. Then, one signal is low and the other one is high and

$$TS(p,\alpha) > g(s_1, s_2) = V,$$

if $x_L + x_H > V$, or, from (3) and (4) if

$$\begin{array}{ll} \frac{1}{4}pV\left(1-p+D(p,\alpha)\right)^2 + \frac{1}{4}pV\left(1-p+D(p,\alpha)\right)\left(1+p-D(p,\alpha)\right) > V, & \text{if } 0 \leq \alpha \leq 3, \\ \frac{1}{4}p(1+\alpha)V > V, & \text{if } \alpha > 3. \end{array}$$

Therefore, overdissipation occurs if

$$\begin{cases} p\left(1-p+D(p,\alpha)\right) > 2, & \text{if } 0 \leq \alpha \leq 3, \\ p(1+\alpha) > 4, & \text{if } \alpha > 3. \end{cases}$$

Note that $D(p,3) > D(p,\alpha)$ for any $0 \leq \alpha < 3$. Since $D(p,3) = 1+p$,

$$p\left(1-p+D(p,\alpha)\right) \leq 2p \leq 2,$$

for $0 \leq \alpha \leq 3$ and $0 \leq p \leq 1$. Thus, there is no overdissipation for $0 \leq \alpha \leq 3$. Therefore,

$$TS\left(p,\alpha\right) > g(s_1,s_2) = V,$$

only if $p(1+\alpha) > 4$ and $\alpha > 3$.

Suppose that $g(s_1,s_2) = (1+\alpha)V$. Then, both signals are high and

$$TS\left(p,\alpha\right) > g(s_1,s_2) = (1+\alpha)V,$$

if $2x_H > (1+\alpha)V$, or, from (4), if

$$\begin{cases} pV\left(1-p+D(p,\alpha)\right)^2 > 2(1+\alpha)V, & \text{if } 0 \leq \alpha \leq 3, \\ p(1+\alpha)V > 2(1+\alpha)V, & \text{if } \alpha > 3. \end{cases}$$

From Proposition 2, $\frac{\partial x_H}{\partial p} \geq 0$, therefore,

$$p\left(1-p+D(p,\alpha)\right)^2 \leq \left(D(1,\alpha)\right)^2 = 1+\alpha < 2(1+\alpha)$$

Hence if $g(s_1,s_2) = (1+\alpha)V$ overdissipation cannot occur. ■

A.2 Proofs for Auctions

Proof of Proposition 6. Standard methods (see, for example, Siegel, 2014) show that bidders have to randomize on continuous intervals in equilibrium and these distributions cannot contain atoms. Furthermore, there cannot be gaps between equilibrium bid distributions b_L^i and b_H^i. Moreover, the lower bound of b_H^i has to be higher than the upper

bound of b_L^i (otherwise player i with signal H would be better off making b_L^i bid). Thus the equilibrium distributions of b_L^i and b_H^i are on adjacent intervals. The lower bounds of b_L^i and b_L^j as well as the upper bounds of b_H^i and b_H^j should be the same (otherwise one of the players can lower his bid and increase his payoff).

In such a monotonic equilirbium a low-signal player can win only against another low-signal player. But the value is 0 in this case, thus the low-signal player has to bid 0 in equilibrium.

Consider a player with H signal. Suppose that his opponent bids on the interval $[0, t_1]$ according to a distribution function F_{All}. Since the support of the bid distribution b_H^i is $[0, t_{All}]$ and given zero bid from the low-signal player, it follows that

$$(1-p)\,V + pF_{All}\,(x)\,(1+\alpha)\,V - x = (1-p)\,V,$$

since in a mixed-strategy equilibrium, all expected payoffs in the support are equal. This means that

$$F_{All}\,(x) = \frac{x}{p(1+\alpha)V}. \tag{14}$$

Since

$$1 = F_{All}\,(t_{All}) = \frac{t_{All}}{p(1+\alpha)V},$$

then $t_{All} = p(1+\alpha)V$.

Since because $F_{All}\,(x)$ is uniquely determined, the equilibrium is unique. ■

Proof of Proposition 7. Similarly to the all-pay auction, the equilibrium distributions have to be atomless, without holes, and on adjacent intervals for signals L and H. Furthermore, bids of low-signal players are $\bar{b}_L^i = \bar{b}_L^j = 0$ in the equilibrium.

Consider a player with H signal and suppose that he bids on the interval $[0, t_F]$ according to a distribution function F_F. Then, given zero bid from the low-signal player, his expected payoff is

$$(1-p)\,[V - x] + pF_F\,(x)\,[(1+\alpha)V - x] = (1-p)\,V,$$

which means that

$$F_F\,(x) = \frac{(1-p)\,x}{p\,((1+\alpha)V - x)}.$$

Since it is a mixed-strategy equilibrium, all strategies on the support $[0, t_F]$ give the same payoff,

$$(1-p)\,V = (1-p)\,(V - t_F) + p\,((1+\alpha)V - t_F)\,,$$

or $t_F = p(1+\alpha)V$. This is the unique equilibrium because $F_F(x)$ is uniquely determined. ■

Proof of Proposition 8. Consider a player with H signal. His expected payoff is

$$(1-p)(V-0) + p\frac{1}{2}((1+\alpha)V - (1+\alpha)V) = (1-p)V.$$

Any bid $x > 0$ gives the same expected payoff. Bid $x = 0$ lowers the probability of winning if the other player has signal L, thus leading to a lower payoff.

For a low-signal player, the equilibrium expected payoff is 0. Any bid $x < (1+\alpha)V$ leads to the same expected payoff of 0 and bidding $x \geq (1+\alpha)V$ gives

$$(1-p)(0-0) + pq(V - (1+\alpha)V) = -pq\alpha V \leq 0,$$

where q is the probability of winning against a player with signal H.

If $x_L > 0$ or $x_H \neq (1+\alpha)V$, there is always a deviation to $x_L = 0$ or $x_H = (1+\alpha)V$ that increases the expected payoff, thus the symmetric equilibrium is unique. ■

Proof of Proposition 9. Let us calculate expected bids and expected ex-ante revenue in the three auctions.

All-pay auction. It is easy to see from Proposition 6 that the expected bid of a high-signal bidder is $E[b_H^{All}] = \frac{p(1+\alpha)V}{2}$. Hence, the ex-ante expected revenue is

$$\pi^{All}(p,\alpha) = (1-p)^2\left(2b_L^{All}\right) + 2(1-p)p\left(b_L^{All} + E[b_H^{All}]\right) + p^2\left(2E[b_H^{All}]\right) = p^2(1+\alpha)V.$$

Second-price auction. It is easy to calculate the ex-ante expected revenue in the symmetric equilibrium of Proposition 8 as

$$\pi^S(p,\alpha) = (1-p)^2(0) + 2p(1-p)(0) + p^2((1+\alpha)V) = p^2(1+\alpha)V.$$

First-price auction. Since the object is sold with probability one, the ex-ante expected revenue can be calculated as

$$\pi^F(p,\alpha) = E[V] - E[u_1] - E[u_2], \tag{15}$$

where u_1 and u_2 are player 1's and 2's equilibrium payoffs. In our case

$$E[V] = (1-p)^2 \cdot 0 + 2p(1-p)V + p^2(1+\alpha)V$$

and the equilibrium payoffs in the first-price auction are

$$E[u_i] = (1-p)\cdot 0 + p(1-p)V.$$

Therefore, the ex-ante expected revenue is

$$\pi^F(p,\alpha) = 2p(1-p)V + p^2(1+\alpha)V - 2p(1-p)V = p^2(1+\alpha)V. \blacksquare$$

A.3 Proof for Revenue Comparison

Proof of Theorem 1. Recall that

$$\pi^{\mathcal{A}}(p,\alpha) = p^2(1+\alpha)V$$

and

$$\pi^{\mathcal{L}}(p,\alpha) = \frac{1}{2}pV\left(1-p+\sqrt{1-p+p^2+\alpha p}\right)\left(1+p-2p^2+(2p-1)\sqrt{1-p+p^2+\alpha p}\right).$$

Let $p>0$. Then $\pi^{\mathcal{A}}(p,\alpha) > \pi^{\mathcal{L}}(p,\alpha)$ if and only if

$$3\alpha - 1 - 2p\alpha + 6p - 4p^2 > 4(1-p)\sqrt{1-p+p^2+\alpha p}. \tag{16}$$

The right-hand side of the inequality is always non-negative. The left hand side is negative if

$$\alpha < \frac{1-6p+4p^2}{3-2p}.$$

In this case $\pi^{\mathcal{A}}(p,\alpha) < \pi^{\mathcal{L}}(p,\alpha)$.

If the left-hand side of (16) is positive, then $\pi^{\mathcal{A}}(p,\alpha) > \pi^{\mathcal{L}}(p,\alpha)$ if and only if

$$\left(3\alpha - 1 - 2p\alpha + 6p - 4p^2\right)^2 - 16(1-p)^2\left(p\alpha - p + p^2 + 1\right) > 0.$$

The quadratic equation $\left(3\alpha - 1 - 2p\alpha + 6p - 4p^2\right)^2 - 16(1-p)^2\left(p\alpha - p + p^2 + 1\right) = 0$ has two solutions, $\alpha = -1$ and

$$\alpha = \frac{20p^2 - 36p + 15}{-12p + 4p^2 + 9} = \alpha(p). \tag{17}$$

It holds that $\left(3\alpha - 1 - 2p\alpha + 6p - 4p^2\right)^2 - 16(1-p)^2\left(p\alpha - p + p^2 + 1\right) > 0$ if $\alpha < -1$ and $\alpha > \alpha(p)$.Therefore $\pi^{\mathcal{A}}(p,\alpha) > \pi^{\mathcal{L}}(p,\alpha)$ if and only if $\alpha > \alpha(p)$ for non-negative α. Note

also that $\alpha(p) > \frac{1-6p+4p^2}{3-2p}$ for $p \in (0,1)$. Hence, $\pi^{\mathcal{A}}(p,\alpha) < \pi^{\mathcal{L}}(p,\alpha)$ is covered by $\alpha < \alpha(p)$ even if $\alpha < \frac{1-6p+4p^2}{3-2p}$.

If $20p^2 - 36p + 15 < 0$, then expression (17) is negative and thus $\pi^{\mathcal{A}}(p,\alpha) > \pi^{\mathcal{L}}(p,\alpha)$ for any non-negative α. Note that

$$20p^2 - 36p + 15 < 0 \text{ if } p \in (0.1(9-\sqrt{6}), 0.1(9+\sqrt{6})) \approx (0.65505, 1.1449).$$

Therefore, for $p > \hat{p} = 0.1(9-\sqrt{6})$ and $p \leq 1$ for any $\alpha \geq 0$ it holds that $\pi^{\mathcal{A}}(p,\alpha) > \pi^{\mathcal{L}}(p,\alpha)$. ■

A.4 Proofs for an Optimal Mechanism and Reserve Prices

Proof of Proposition 11. A general (direct symmetric) mechanism can be described by $\{T_i(\hat{s}_i,\hat{s}_j); q_i(\hat{s}_i,\hat{s}_j)\}_{i,j=1,2;j\neq i}$, where $(\hat{s}_i,\hat{s}_j)$ is the vector of reported signals from the set $\{(L,L),(L,H),(H,L),(H,H)\}$, $T_i(\hat{s}_i,\hat{s}_j)$ is the payment of player i conditional on him reporting $\hat{s}_i$ and the other player reporting $\hat{s}_j$, and $q_i(\hat{s}_i,\hat{s}_j)$ is the probability that player i gets the object conditional on the reports. It holds that $q_i(\hat{s}_i,\hat{s}_j) \geq 0$ and $q_i(\hat{s}_i,\hat{s}_j) + q_j(\hat{s}_j,\hat{s}_i) \leq 1$ for all report vectors.

Consider mechanisms with $q_i(H,H) = \frac{1}{2}$, $q_i(H,L) = 1$, $q_i(L,H) = 0$, $q_i(L,L) = 0$, i.e. if both players report H, the object is allocated to either of them with equal probability; if one player reports H and the other L, the object is allocated to the one who reported H; and if both players report L, the object is retained by the seller. To save space, denote $T_i(\hat{s}_i,\hat{s}_j)$ and $q_i(\hat{s}_i,\hat{s}_j)$ as $T_{\hat{s}_i\hat{s}_j}$ and $q_{\hat{s}_i\hat{s}_j}$, for example, $T_i(L,L)$ is written as T_{LL} etc.

The seller solves the following problem to find an optimal mechanism:

$$\max_{P_{LL},P_{LH},P_{HL},P_{HH}} (p^2 T_{HH} + 2p(1-p)(T_{HL}+T_{LH}) + (1-p)^2 T_{LL})$$

$$\begin{aligned}
\text{s.t.} \quad -pT_{LH} + (1-p)\left(\frac{1}{2}0 - T_{LL}\right) &\geq p\left(\frac{1}{2}V - T_{HH}\right) - (1-p)T_{HL} && (\text{IC}_L)\\
p\left(\frac{1}{2}(1+\alpha)V - T_{HH}\right) + (1-p)(V - T_{HL}) &\geq -pT_{LH} - (1-p)T_{LL} && (\text{IC}_H)\\
-pT_{LH} + (1-p)\left(\frac{1}{2}0 - T_{LL}\right) &\geq 0 && (\text{IR}_L)\\
p\left(\frac{1}{2}(1+\alpha)V - T_{HH}\right) + (1-p)(V - T_{HL}) &\geq 0, && (\text{IR}_H)
\end{aligned}$$

with two incentive compatibility and two individual rationality constraints. Suppose that constraints (IR_L) and (IC_H) are binding. From the binding (IR_L) constraint,

$$T_{LH} = -\frac{1-p}{p} T_{LL}. \tag{18}$$

Then, from the binding (IC_H) constraint and substituting equation (18),

$$T_{HH} = \frac{1}{2}(1+\alpha)V + \frac{1-p}{p}V - \frac{1-p}{p}T_{HL}. \tag{19}$$

After substitution of equations (18) and (19), the seller's objective function becomes

$$pV(p(1+\alpha) + 2(1-p)).$$

Thus, the expected revenue from an optimal mechanism of this kind is

$$\pi^{OM} = pV(p(1+\alpha) + 2(1-p)) = \pi^{FB}.$$

Since π^{FB} is the maximum revenue the seller can get in any mechanism, this kind of mechanism is sufficient to extract all surplus from the bidders. ■

Proof of Proposition 12. Consider the lottery and the auction settings. In each case, we find the corresponding optimal reserve price.

Lottery. Consider a reserve price $r_{Lot} > 0$ such that an H-signal bidder by bidding $x_H = r_{Lot}$ receives the expected payoff 0, if an L-signal bidder bids $x_L = 0$. The condition for such a reserve price is

$$p\frac{1}{2}(1+\alpha)V + (1-p)V - r_{Lot} = 0,$$

or

$$r_{Lot} = p\frac{1}{2}(1+\alpha)V + (1-p)V. \tag{20}$$

An L-signal bidder can participate in a lottery only if his bid y_L is at least r_{Lot}. It is easy to check that an L-signal bidder should not participate in the lottery by bidding $x_L < r_{Lot}$, for example, $x_L = 0$.

The expected payoff of an H-signal bidder is $u_H(y_H) = p\frac{y_H}{y_H + r_{Lot}}(1+\alpha)V + (1-p)V - y_H$. The derivative $u'_H(y_H) = p\frac{r_{Lot}}{(y_H + r_{Lot})^2}(1+\alpha)V - 1$ is decreasing in y_H. Note that $u'_H(r_{Lot}) = p\frac{1}{4r_{Lot}}(1+\alpha)V - 1$. From (20), $4r_{Lot} > p(1+\alpha)V$, $u'_H(r_{Lot}) < 0$, and thus $u'_H(y_H) < 0$

for any $y_H > r_{Lot}$. Since $u_H(r_{Lot}) = 0$ by construction, the expected payoff of an H-signal bidder is maximized at $x_H = r_{Lot}$. Thus $x_H = r_{Lot}$ and $x_L = 0$ is an equilibrium of the lottery with reserve price r_{Lot}. The ex-ante expected seller revenue is

$$p^2(2r_{Lot}) + 2p(1-p)(r_{Lot}) + (1-p)^2(0) = pV(p(1+\alpha) + 2(1-p)),$$

the same as the first-best revenue.

Second-price auction. Consider now the second-price auction with reserve price r_S. In this auction, the highest bidder i wins the auction if $b_i \geq r_S$ and the payment is equal to $\max\{b_j, r_S\}$. Consider the reserve price

$$r_S = V$$

and bids $b_L = 0$ and $b_H = (1+\alpha)V$.

The expected payoff of an L-signal bidder bidding $b_L = 0$ is 0 because he never wins the object. If he sets his bid b such that $r_S = V \leq b < (1+\alpha)V$, then he wins the object with probability $1-p$. However, the value of the object is 0 in this case and thus the expected payoff is negative. If an L-signal bidder bids $b = (1+\alpha)V$, then his expected payoff is $(1-p)(0-r_S) + p\frac{1}{2}(V - (1+\alpha)V) < 0$. Hence, setting $b_L = 0$ is optimal.

The expected payoff of an H-signal bidder bidding $b_H = (1+\alpha)V$ is $(1-p)(V-V) + p\frac{1}{2}((1+\alpha)V - (1+\alpha)V) = 0$. If he sets $b < r_S$, he never wins. If he bids b such that $r_S = V \leq b < (1+\alpha)V$, his expected payoff is $(1-p)(V-V) + p \cdot 0 = 0$. If an H-signal bidder sets $b > (1+\alpha)V$, his expected payoff is $(1-p)(V-V) + p((1+\alpha)V - (1+\alpha)V) = 0$. In all cases, setting $b_H = (1+\alpha)V$ is the best choice. The ex-ante seller's revenue in this equilibrium is

$$p^2(1+\alpha)V + 2p(1-p)r_S + (1-p)^2 \cdot 0 = p^2(1+\alpha)V + 2p(1-p)V,$$

the same as the first-best revenue.

First-price auction. Consider now the first-price auction with a reserve price r_F. Consider bids $b_L = 0$ and b_H distributed according to the distribution function F_F on interval $[r_F, \bar{b}_F]$. In an equilibrium, the expected payoff of an H-signal bidder should be the same for all bids in the interval $[r_F, \bar{b}_F]$, including $b = r_F$ and $b = \bar{b}_F$. Thus,

$$\begin{aligned}
(1-p)(V - r_F) &= (1-p)(V-x) + pF_F(x)((1+\alpha)V - x) \text{ and} \\
(1-p)(V - r_F) &= (1-p)(V - \bar{b}_F) + p((1+\alpha)V - \bar{b}_F).
\end{aligned}$$

From the first equality

$$F_F(x) = \frac{(1-p)(x-r_F)}{p((1+\alpha)V-x)}.$$

Note that $F(r_F) = 0$. From the second equality $\bar{b}_F = (1-p)r_F + p(1+\alpha)V$. Set $r_F = V$. Then $\bar{b}_F = V + p\alpha V$ and

$$F_F(x) = \frac{(1-p)(x-V)}{p((1+\alpha)V-x)}. \tag{21}$$

Given (21), the expected payoff of an H-signal bidder is 0 from any bid $b \in [r_F, \bar{b}_F]$. The expected payoff from bidding $b < r_F$ is also 0. The expected payoff from bidding $b > \bar{b}$ is $(1-p)V + p(1+\alpha)V - b < 0$. The expected payoffs of an L-signal bidder are 0 for any bid below r_F and negative for $b \geq r_F = V$ because $(1-p)(0-b) + pF(b)(V-b) < 0$.

Note that the bids of the high-signal bidder are above the reservation price with probability 1. Therefore the good is always sold if there is a high-signal bidder. If both bidders have low signal, then the good is not sold but the value is 0, meaning that there is no loss in overall surplus. Therefore the formula (15) can sitll be used:

$$\pi^F = E[V] - E[u_1^F] - E[u_2^F],$$

where u_1^F and u_2^F are bidders' equilibrium payoffs. Since these payoffs are 0 by construction, the seller's revenue is

$$\pi^F = E[V] = \pi^{FB}.$$

All-pay auction. Consider an all-pay auction with a reserve price r_{All}. Consider bids $b_L = 0$ and b_H distributed according to a distribution function F_{All} on interval $[r_{All}, \bar{b}_{All}]$. The payoff of an H-signal bidder should be the same for all bids in the interval $[r_{All}, \bar{b}_{All}]$, including $b = r_{All}$ and $b = \bar{b}_{All}$. Thus,

$$\begin{aligned}
(1-p)V - r_{All} &= (1-p)V + pF_{All}(x)(1+\alpha)V - x \\
(1-p)V - r_{All} &= (1-p)V + p(1+\alpha)V - \bar{b}_{All}.
\end{aligned}$$

From the first equality

$$F_{All}(x) = \frac{x - r_{All}}{p(1+\alpha)V}.$$

Note that $F_{All}(r_{All}) = 0$. From the second equality

$$r_{All} = \bar{b}_{All} - p(1+\alpha)V.$$

Set $\bar{b}_{All} = (1-p)V + p(1+\alpha)V$. Then $r_{All} = (1-p)V$ and

$$F_{All}(x) = \frac{x-(1-p)V}{p(1+\alpha)V}. \tag{22}$$

Given (22), the expected payoff of an H-signal bidder is 0 from any bid $b \in [r_{All}, \bar{b}_{All}]$. The expected payoff from bidding $b < r_{All}$ is $-b \leq 0$ (with equality only for $b = 0$). The expected payoff from bidding $b > \bar{b}_{All}$ is $(1-p)V + p(1+\alpha)V - b < 0$. The expected payoff of an L-signal bidder is 0 from $b_L = 0$ and negative for $b > 0$.

Similarly with the first-price auction, the bid of a high-signal bidder is higher than the reservation price for sure. Therefore there is no loss of surplus in object allocation and the expected revenue of the seller is

$$\pi^{All} = E[V] - E[u_1^{All}] - E[u_2^{All}],$$

where u_1^{All} and u_2^{All} are bidders' equilibrium payoffs. Again, since the payoffs are 0 by construction, the seller's revenue is

$$\pi^{All} = E[V] = \pi^{FB}. \blacksquare$$

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