On Mediated Equilibria of Cheap-Talk Games*

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Abstract

In the Crawford-Sobel (uniform, quadratic utility) cheap-talk model, we consider mediation in which the informed agent reports one possible element of a partition to a mediator (a communication device) and then the mediator suggests an action to the uninformed decision-maker according to the probability distribution of the device. We show that a mediated equilibrium involving exactly N elements to report and N actions to choose from, cannot improve upon the unmediated N-partition Crawford-Sobel equilibrium when the preference divergence parameter is small.

Keywords: Cheap Talk, Mediated Equilibrium.

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1 INTRODUCTION

The strategic interaction and information transmission between an uninformed decision maker and an informed agent was studied in the seminal paper by Crawford and Sobel (1982) (hereafter referred to as the CS model). CS have proved that any (Bayesian-Nash) equilibrium in this cheap-talk game is equivalent to a partition equilibrium where the informed agent reveals one of the finitely many elements (the number of which depends on the value of the preference divergence parameter) of the partition in which the true state of the nature lies.

In the CS model (as well as in the related literature[1]) the informed agent directly "talks" to the decision-maker. The question that naturally arises is what would happen if a mediator, who would receive inputs from the informed agent and send outputs to the decision maker, could be used?[2] Can such a mediated communication improve upon the payoffs of the players involved in the cheap-talk game?[3]

This particular issue had been completely overlooked in this otherwise healthy literature until the recent work by Krishna and Morgan (2004), and Myerson (website) who constructed examples to show that indeed mediated communication can lead to equilibria that are distinct and Pareto superior to the CS equilibria. The values of the preference divergence parameter considered in the examples are such that the only equilibrium in the CS model is the babbling equilibrium and thus the constructed mediated equilibria simply improve upon the babbling equilibrium. No general analytical result comparing a mediated equilibrium with the CS equilibrium has yet been established.[4]

[1] The CS model has been indeed extended in many directions. Recent publications in this literature include, among others, Aumann and Hart 2003, Battaglini 2002, Dessein 2002, Krishna and Morgan 2004.

[2] General communication mechanisms for games with incomplete information have been exhaustively analysed (See, among others, Forges 1986, 1990; Myerson 1982, 1986)

[3] This is somewhat similar to the issue of whether there exist correlated equilibria (Aumann 1974, 1987) of a normal form game that can improve upon the Nash equilibrium payoffs, the answer to which is in the work by Moulin and Vial (1978).

[4] Dessein (2002) considered delegation to an intermediary in the CS framework. However,

In this paper, following the example constructed by Krishna and Morgan (2004), we consider mediation in which the informed agent reports one possible element of a partition to a mediator (a communication device) and then the mediator suggests an action to the uninformed decision-maker according to the probability distribution of the device. Our aim however is to identify "simple" mediated equilibria that can improve upon CS partition equilibria for any specific value of the preference divergence parameter in the cheap talk game.

As mentioned, from the examples constructed by Krishna and Morgan (2004) and Myerson (website), we already know that there are mediated equilibria that indeed can improve upon a particular CS partition equilibrium. However, we first note that the number of elements of the partition (and the actions) required to construct such mediated equilibria are higher than those in the corresponding CS equilibrium. Indeed, we formalise this feature of the construction used in the literature and prove (Proposition 2) that any N-partition CS equilibrium can be improved upon by a mediated equilibrium involving $N+1$ elements and $N+2$ actions.

We then concentrate on a specific form of mediated equilibria that we call simple mediated equilibria in which the mediator is restricted to use the same number of inputs and outputs as the number of elements of the partition in the CS equilibrium. We observe (Proposition 1) that a CS partition equilibrium is equivalent to a simple mediated equilibrium and can be identified as a "corner point" of the set of all such equilibria. Then, we show that the optimum of this set is attained at this corner point when the value of the parameter is small enough. The main result of this paper (Theorem 1) is that the CS N-partition equilibrium cannot be improved upon by the corresponding simple mediated equilibrium when the preference divergence parameter is less than $\frac{1}{2N^2}$. We illustrate all our results with constructive examples involving different values of the preference divergence parameter.

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the role of his "intermediary" is different from that of "mediation" here.

2 THE MODEL

2.1 Crawford-Sobel Game

Our set-up is identical to the uniform-quadratic utility CS Model, as presented in the literature (see for instance, Krishna and Morgan 2004). Informed readers may wish to skip this subsection.

There are 2 agents. The informed agent, called the sender (S), precisely knows the state of the world, θ, where $\theta \sim U[0,1]$, and can send a possibly noisy signal at no cost, based on his private information, to the other agent, called the receiver (R). The receiver, however doesn't know θ but must choose some decision y based on the information contained in the signal. The receiver's payoff is $U^R(y,\theta) = -(y-\theta)^2$, and the sender's payoff is $U^S(y,\theta,b) = -(y-(\theta+b))^2$, where $b > 0$ is a parameter that measures the degree of congruence in their preferences.

CS have shown that any equilibrium of this game is essentially equivalent to a partition equilibrium where only a finite number of actions are chosen in equilibrium and each action corresponds to an element of the partition. For $b < \frac{1}{2N(N-1)}$, where $N \geq 2$ is an integer,[5] there is an equilibrium in which the state space is partitioned into N elements, characterised by $0 = a_0 < a_1 < a_2 < ...... < a_{N-1} < a_N = 1$, where $a_k = \frac{k}{N} + 2bk(k-N)$, in which S sends a message for each element $[a_{k-1}, a_k)$, and given this message, R takes the optimal action $y_k = \frac{a_{k-1}+a_k}{2}$. We call this the N-partition CS equilibrium. For $\frac{1}{2N(N+1)} \leq b < \frac{1}{2N(N-1)}$, the "best" equilibrium (the one that maximises EU^R) is the N-partition CS equilibrium[6]. For such an equilibrium, the receiver's expected payoff is $EU^R = -\frac{1}{12N^2} - \frac{b^2(N^2-1)}{3}$ while the sender's expected payoff is $EU^S = EU^R - b^2$.

[5] For $\frac{1}{4} \leq b \leq 1$, babbling is the only equilibrium.

[6] Note that for any $b < \frac{1}{2N(N+1)}$, an N-partition CS equilibrium does exist. However, it is not the "best".

2.2 Krishna-Morgan Game

Krishna-Morgan (hereafter referred to as KM) extend the CS analysis by modifying their game to include more than one round of communication between the receiver and the sender. The first round of communication involves a face-to-face meeting where the sender reveals whether $\theta \in [0, c)$ or $\theta \in [c, 1]$. Furthermore, in this round, the sender sends a message A_1 and the receiver sends a message A_2 where A_1 and A_2 belong to an appropriately chosen message set A. These messages, A_1 and A_2, are used to conduct a suitable joint lottery which gives rise to an outcome "success" with a certain probability p and "failure" with probability $1 - p$. Play in the second round of communication depends on the messages sent in the first round. If the sender reveals that $\theta \in [0, c)$, then a partition equilibrium in the interval $[0, c]$ is played regardless of other messages. If the sender reveals that $\theta \in [c, 1]$, and the outcome of the joint lottery is a success, then, in the second round, the sender further reveals whether $\theta \in [c, z)$ or $\theta \in [z, 1]$ for some appropriately chosen z. If the sender reveals that $\theta \in [c, 1]$, and the outcome of the joint lottery is a failure, then, in the second round, no additional information is revealed.

KM have shown that for $\frac{1}{2(N+1)^2} < b < \frac{1}{2N^2}$, where $N \geq 2$ is an integer, the above strategies constitute an equilibrium for any c satisfying $\frac{N-1}{N+1} - 4b(N-1) < c < a_{N-1}$. If the sender reveals that $\theta \in [0, c)$, then a partition equilibrium of size $N - 1$ will be played, which is characterised by $0 = z_0 < z_1 < z_2 < .... < z_{N-2} < z_{N-1} = c$, where $z_j = \frac{j}{N-1}c + 2bj(j - (N-1))$. Also, z and p are chosen such that $z = -2b + \frac{1}{2}c + \frac{1}{2}$ and $p = \frac{4}{3}\frac{(N-1)^2[4N(N-2)b^2+4b-(1-c)^2]+c^2}{(N-1)^2[16b^2-(1-c)^2]}$. Among this class of equilibria, the "best" equilibrium, i.e., the one that maximizes the receiver's expected payoff and corresponds to choosing $c = 2b(N-1)^2$, is Pareto superior to the N-partition CS equilibrium[7]. We will refer to this as the N-KM equilibrium.

[7] Note that this KM equilibrium does not strictly improve upon the corresponding CS equilibrium at points $b = \frac{1}{2N^2}$, where $N \geq 2$ is an integer.

2.3 Mediated Equilibrium

Within the CS framework, we now consider mediation, a possible structure of which could be as follows: S sends a message based on his private information to the mediator; the mediator then chooses an action according to a commonly-known probability distribution and recommends it to R. We here consider a specific form of mediation (mechanism) as formally defined below.

Definition 1 *An $N \times M$ mediated talk is $(\{x_k\}_{k=0}^{N}, \{y_j\}_{j=1}^{M}, \{p_{kj}\}_{k=1,...N;j=1,..M})$ where $0 = x_0 < x_1 < x_2 < ...... < x_{N-1} < x_N = 1$, each $y_j \in [0,1]$ for $j \in \{1, 2, ...M\}$, each $p_{kj} \in [0,1]$ for $k \in \{1, 2, ...N\}$, $j \in \{1, 2, ...M\}$ with $\sum_j p_{kj} = 1$.*

In an $N \times M$ mediated talk, S reports one of N possible elements, $[x_{k-1}, x_k)$, in which the true state θ may lie, to the mediator, and given the report $\theta \in [x_{k-1}, x_k)$, the mediator then recommends to R one action, y_j, out of M possible actions, with probability p_{kj}.

Such a mechanism[8] is said to be in equilibrium if it is incentive compatible for both players, that is, if (i) S has the incentive to be truthful to the mediator given the probabilities p_{kj}, and (ii) R has the incentive to obey the mediator's recommendation y_j, given the posterior probabilities on the state of nature. Formally,

Definition 2 *For any specific value of b, an $N \times M$ mediated equilibrium is an $N \times M$ mediated talk that satisfies incentive compatibility*

(i) for S: $f_k'(\theta) > 0$ and $f_k(x_k) = 0$ for all $k \in \{1, 2, ..., N-1\}$, where, $f_k(\theta) = \sum_{j=1}^{M} (p_{kj} - p_{k+1j}) [y_j - (\theta + b)]^2$, and

(ii) for R: $y_j = \underset{y}{argmax} - \sum_{k=1}^{N} q_{kj} \frac{1}{(x_k - x_{k-1})} \int_{x_{k-1}}^{x_k} (y - \theta)^2 d\theta$ for all $j \in \{1, 2, ..., M\}$, where, q_{kj} is the posterior probability that $\theta \in [x_{k-1}, x_k)$ and is given by $q_{kj} = \frac{(x_k - x_{k-1})p_{kj}}{\sum_{k=1}^{N}(x_k - x_{k-1})p_{kj}}$.

[8] This is the type of mechanism Krishna and Morgan (2004) considered to construct their example. Myerson (website) however has used a “discrete” version of such a mechanism.

An $N \times M$ mediated equilibrium can be characterised easily. Incentive compatibility for R requires

$$\sum_{k=1}^{N} p_{kj} \left[(y_j - x_{k-1})^2 - (y_j - x_k)^2 \right] = 0; \text{ for all } j = 1....M, \tag{1}$$

which implies

$$y_j = \frac{1}{2} \left[\frac{(1 - \sum_{k \neq j} p_{jk})(x_j^2 - x_{j-1}^2) + \sum_{k \neq j} p_{kj}(x_k^2 - x_{k-1}^2)}{(1 - \sum_{k \neq j} p_{jk})(x_j - x_{j-1}) + \sum_{k \neq j} p_{kj}(x_k - x_{k-1})} \right]. \tag{2}$$

Incentive Compatibility for S requires

$$\sum_{j=1}^{M} (p_{kj} - p_{k+1j}) \left[y_j - (x_k + b) \right]^2 = 0; \text{ for all } k = 1....N-1, \tag{3}$$

and

$$\sum_{j=1}^{M-1} (p_{k+1j} - p_{kj}) (y_j - y_M) > 0; \text{ for all } k = 1....N-1. \tag{4}$$

Using (3), we get

$$2(x_k + b) = \frac{\sum_{j=1}^{M-1} (p_{kj} - p_{k+1j}) \left(y_j^2 - y_M^2 \right)}{\sum_{j=1}^{M-1} (p_{kj} - p_{k+1j}) (y_j - y_M)}$$

$$= \frac{\begin{array}{c} \left(1 - \sum_{j \neq k} p_{kj} - p_{k+1k}\right) \left(y_k^2 - y_M^2\right) + \left(p_{kk+1} - 1 + \sum_{j \neq k+1} p_{k+1j}\right) (y_{k+1}^2 \\ - y_M^2) + \sum_{j \neq k, k+1} (p_{kj} - p_{k+1j}) \left(y_j^2 - y_M^2\right) \end{array}}{\begin{array}{c} \left(1 - \sum_{j \neq k} p_{kj} - p_{k+1k}\right) (y_k - y_M) + \left(p_{kk+1} - 1 + \sum_{j \neq k+1} p_{k+1j}\right) (y_{k+1} \\ - y_M) + \sum_{j \neq k, k+1} (p_{kj} - p_{k+1j}) (y_j - y_M) \end{array}}$$

$$\text{for all } k = 1....N-1. \tag{5}$$

Thus an $N \times M$ mediated equilibrium is characterised by (5) where the y_j's are given by (2) with the constraints that the inequalities in (4) are satisfied.

Definition 3 *An N-simple mediated equilibrium is an $N \times N$ mediated equilibrium.*

2.4 Examples and Illustrations

Example 1 *Take $b = \frac{1}{10}$. The best CS equilibrium involves a partition with two elements.*[9] *It is easy to check that the 2-partition CS equilibrium is characterised by $y_1 = \frac{3}{20}$, $y_2 = \frac{13}{20}$ and*[10] $a = \frac{3}{10}$. *Here, $EU^R = -\frac{37}{1200} \simeq -0.031$ and $EU^S = -\frac{49}{1200} \simeq -0.041$.*

Example 2 *Take $b = \frac{1}{10}$ as in Example 1. The 2-KM equilibrium corresponds to $c = \frac{2}{10}$, $z = \frac{4}{10}$, $p = \frac{5}{9}$. Here, $EU^R = -\frac{36}{1200} = -0.03$ and $EU^S = -\frac{48}{1200} = -0.04$. Note that this 2-KM equilibrium improves upon the corresponding 2-partition CS equilibrium.*

The following examples illustrate a few mediated equilibria. We characterise a 2-simple mediated equilibrium which is given by[11] $(x, y_1, y_2, p_{11}, p_{12}, p_{21}, p_{22})$, where, x, y_1, $y_2 \in (0, 1)$, $p_{11}, p_{12}, p_{21}, p_{22} \in [0, 1]$ and $p_{11}+p_{12} = 1$, $p_{21}+p_{22} = 1$. The incentive compatibility for S requires that $(p_{21} - p_{11})(y_1 - y_2) > 0$ and $\frac{y_1+y_2}{2} - x = b$. Note that the inequality is automatically satisfied as long as $y_1 \neq y_2$. The incentive compatibility constraints for R can be written as:

$$y_1 = \frac{(1 - p_{12})x^2 + p_{21}(1 - x^2)}{2[(1 - p_{12})x + p_{21}(1 - x)]} \tag{6}$$

$$y_2 = \frac{p_{12}x^2 + (1 - p_{21})(1 - x^2)}{2[p_{12}x + (1 - p_{21})(1 - x)]} \tag{7}$$

We can combine all the incentive constraints into the following equation:

$$\frac{(1 - p_{12})x^2 + p_{21}(1 - x^2)}{4[(1 - p_{12})x + p_{21}(1 - x)]} + \frac{p_{12}x^2 + (1 - p_{21})(1 - x^2)}{4[p_{12}x + (1 - p_{21})(1 - x)]} - x = b \tag{8}$$

[9] Recall that a 2-partition CS equilibrium exists for $b < \frac{1}{4}$ and for $\frac{1}{12} \leq b < \frac{1}{4}$, the best CS partition equilibrium involves two elements.

[10] We drop the subscript in a_1 in the 2-partition CS equilibrium for presentational simplicity in the rest of the paper.

[11] See footnote 10.

Thus, a 2-simple mediated equilibrium in this set-up can be characterised by three variables (p_{12}, p_{21}, x), where, $x \in (0,1)$ and $p_{12}, p_{21} \in [0,1]$ satisfying (8).

Example 3 *Take $b = \frac{1}{10}$ as in Example 1. It is easy to check that $y_1 = \frac{19}{50}$, $y_2 = \frac{31}{50}$, and $x = \frac{4}{10}$ with $p_{11} = \frac{4}{5}$, $p_{12} = \frac{1}{5}$, $p_{21} = \frac{3}{10}$, $p_{22} = \frac{7}{10}$ constitute a 2-simple mediated equilibrium. The utilities are: $EU^R = -\frac{517}{7500}$ and $EU^S = -\frac{592}{7500}$. Note that this 2-simple mediated equilibrium does not improve upon the corresponding 2-partition CS equilibrium.*

Example 4 *Take $b = \frac{1}{6}$. Here, the 2-partition CS equilibrium is characterised by $\{a = \frac{1}{6}, y_1 = \frac{1}{12}, y_2 = \frac{7}{12}\}$ with utilities $EU^R = -\frac{7}{144} = -0.04861111$ and $EU^S = -\frac{11}{144} = -0.07638888$. It is easy to check that $\{x = 0.2245201023, y_1 = 0.1745967377, y_2 = 0.6077768002, p_{11} = 0.97, p_{21} = 0.04\}$ constitute a 2-simple mediated equilibrium. The utilities in this equilibrium are, respectively, $EU^R = -0.04826241093$ and $EU^S = -0.076040188707$. Hence this 2-simple mediated equilibrium does improve upon the corresponding 2-partition CS equilibrium.*

Example 5 *Reconsider Example 2. Here, $b = \frac{1}{10}$. The 2-KM equilibrium in Example 2 is equivalent to the following 3×4 mediated equilibrium: $\{x_1 = \frac{2}{10}, x_2 = \frac{4}{10}$, $y_1 = \frac{1}{10}$, $y_2 = \frac{3}{10}$, $y_3 = \frac{7}{10}$, $y_4 = \frac{6}{10}$, $p_{11} = 1$, $p_{22} = \frac{5}{9}$, $p_{24} = \frac{4}{9}$, $p_{33} = \frac{5}{9}$, $p_{34} = \frac{4}{9}\}$. Clearly, this mediated equilibrium improves upon the corresponding 2-partition CS equilibrium.*

3 RESULTS

To state and prove our results, we take $b < \frac{1}{2N(N-1)}$, for which the N-partition CS equilibrium exists. We first show that a CS partition equilibrium is equivalent to a simple mediated equilibrium.

Proposition 1 *The N-partition CS equilibrium is equivalent to an N-simple mediated equilibrium.*

Proof. Consider the N-partition CS equilibrium as described in Section 2.1. Using these equilibrium values, construct an $N \times N$ mediated talk with $x_k = a_k = \frac{k}{N} + 2bk(k-N)$ for all $k \in \{1, ...N\}$; $y_j = \frac{a_{j-1}+a_j}{2}$ for all $j \in \{1, ...N\}$; $p_{kj} = 0$ for all k, $j \in \{1, ...N\}$, $k \neq j$. To be an N-simple mediated equilibrium, this $N \times N$ mediated talk must satisfy the incentive compatibility condition for R, given by (2), and the incentive compatibility condition for S, given by (5). Clearly, the first requires $y_j = \frac{a_{j-1}+a_j}{2}$, which is satisfied by definition. Note that (5) becomes $2(x_k + b) = y_k + y_{k+1} = \frac{x_{k-1}+2x_k+x_{k+1}}{2}$, which can be easily verified to be true. Hence the proof. ■

Example 5 suggests that one can improve upon a CS equilibrium by a mediated equilibrium suitably constructed using the corresponding KM equilibrium that exists under $\frac{1}{2(N+1)^2} < b < \frac{1}{2N^2}$. We formalise this feature and prove the following general result.

Proposition 2 *The N-KM equilibrium is equivalent to an $(N+1) \times (N+2)$ mediated equilibrium.*

Proof. Consider the KM equilibrium as described in Section 2.2. Using these equilibrium values, construct an $(N+1) \times (N+2)$ mediated talk given by $x_k = 2bk^2$, $k = 1, ...N-1$ and $x_N = \frac{1}{2} + b(N-1)^2 - 2b$; $y_j = b(2j^2 + 1 - 2j)$, $j = 1, ...N-1$ and $y_N = \frac{3b(N-1)^2}{2} + \frac{1}{4} - b$, $y_{N+1} = \frac{3}{4} + \frac{b(N-1)^2}{2} - b$, $y_{N+2} = \frac{1}{2} + b(N-1)^2$; $p_{kk} = 1$ and $p_{kj} = 0$ for all $j \neq k$, $k = 1, ...N-1$; $p_{Nj} = 0$ for all $j \neq N$, $N+2$ and $p_{NN} = p$, $p_{NN+2} = 1-p$; $p_{N+1j} = 0$ for all $j \neq N+1$, $N+2$ and $p_{N+1N+1} = p$, $p_{N+1N+2} = 1-p$ where

$$p = \frac{4}{3}\frac{4N(N-2)b^2 + 4b - 1 - 4(N-1)^4b^2 + 4(N-1)^2b + 4(N-1)^2b^2}{16b^2 - 1 - 4(N-1)^4b^2 + 4(N-1)^2b} \tag{9}$$

To be a mediated equilibrium, it must satisfy the incentive compatibility conditions for R and S. The incentive compatibility condition for R, given by (2), requires

$$y_j = \frac{x_j + x_{j-1}}{2} \text{ for all } j = 1, ...N+1 \text{ and } y_{N+2} = \frac{1 + x_{N-1}}{2} \tag{10}$$

which hold by definition. The incentive compatibility condition for S, given by (5), becomes $2(x_k + b) = y_k + y_{k+1}$ for all $k \in \{1, ..., N-3, N-2, N\}$ which can easily be checked to hold.

Also, (5) requires that $2(x_{N-1} + b) = \frac{y_{N-1}^2 - py_N^2 - (1-p)y_{N+2}^2}{y_{N-1} - py_N - (1-p)y_{N+2}}$ which is true because p is defined such that

$$[y_{N-1} - (x_{N-1} + b)]^2 = p[y_N - (x_{N-1} + b)]^2 + (1-p)[y_{N+2} - (x_{N-1} + b)]^2.$$

Hence the proof. ■

The main result of this paper compares an N-simple mediated equilibrium with the N-partition CS equilibrium.

Theorem 1 *For $b < \frac{1}{2N^2}$, an N-simple mediated equilibrium cannot improve upon the N-partition CS equilibrium.*

Proof. From Proposition 1, it suffices to show that, for $b < \frac{1}{2N^2}$, the N-simple mediated equilibrium which maximizes the expected utility of R corresponds to $p_{kj} = 0$ for all $k \neq j$; k, $j \in \{1, ...N\}$.

Now, note that

$$EU^R = -\sum_{k=1}^{N} \left[(1 - \sum_{j \neq k} p_{kj}) \int_{x_{k-1}}^{x_k} (y_k - \theta)^2 d\theta + \sum_{\substack{j=1 \\ j \neq k}}^{N} p_{kj} \int_{x_{k-1}}^{x_k} (y_j - \theta)^2 d\theta \right] \tag{11}$$

$$3EU^R = -\sum_{k=1}^{N} \left[\begin{array}{c} (1 - \sum_{j \neq k} p_{kj}) \left[(y_k - x_{k-1})^3 - (y_k - x_k)^3 \right] \\ + \sum_{\substack{j=1 \\ j \neq k}}^{N} p_{kj} \left[(y_j - x_{k-1})^3 - (y_j - x_k)^3 \right] \end{array} \right] \tag{12}$$

We now consider the constrained maximization problem:

$\underset{\{x_k\}_1^{N-1}, \{p_{kj}\}_{j \neq k}}{\text{Maximize}}$ (12) subject to (5) where the y_j's in (5) are given by (2).

We then prove that, for $b < \frac{1}{2N^2}$, the solution of the above problem is achieved at a "corner", namely, $p_{kj} = 0$ for all $k \neq j$; k, $j \in \{1, ...N\}$ and $x_k = x_k^{CS} = \frac{k}{N} + 2bk(k - N)$ for all $k \in \{1, ..., N-1\}$.[12]

[12] We here drop the constraint that (4) be satisfied and find the optimal solution of this modified constrained maximization problem with a larger "feasible set". However, notice that the (proposed) optimal solution, namely, the CS N-partition equilibrium, does satisfy the constraint (4) and hence will be the solution of the desired maximization problem.

Consider the Lagrangian:

$$
\begin{aligned}
L &= -\sum_{k=1}^{N}\left[\begin{array}{c} \left(1-\sum_{j\neq k}p_{kj}\right)\left[(y_k-x_{k-1})^3-(y_k-x_k)^3\right] \\ +\sum_{\substack{j=1\\ j\neq k}}^{N}p_{kj}\left[(y_j-x_{k-1})^3-(y_j-x_k)^3\right] \end{array}\right] \\
&\quad+\sum_{k=1}^{N-1}\lambda_k\left[2b-\frac{\begin{array}{c}\left(1-\sum_{j\neq k}p_{kj}-p_{k+1k}\right)\left(y_k^2-y_N^2\right) \\ +(p_{kk+1}-1+\sum_{j\neq k+1}p_{k+1j})(y_{k+1}^2-y_N^2) \\ +\sum_{j\neq k,k+1}\left(p_{kj}-p_{k+1j}\right)\left(y_j^2-y_N^2\right)\end{array}}{\begin{array}{c}\left(1-\sum_{j\neq k}p_{kj}-p_{k+1k}\right)\left(y_k-y_N\right) \\ +(p_{kk+1}-1+\sum_{j\neq k+1}p_{k+1j})(y_{k+1}-y_N) \\ +\sum_{j\neq k,k+1}\left(p_{kj}-p_{k+1j}\right)\left(y_j-y_N\right)\end{array}}+2x_k\right]
\end{aligned}
$$

To complete the proof, we just need to show that at the proposed solution, there exist $\{\lambda_k\}_{k=1}^{N-1}$ such that $\frac{\partial L}{\partial x_k}=0$ for all $k=1,....N-1$ and $\frac{\partial L}{\partial p_{kj}}<0$ for all $j\neq k$, when $b<\frac{1}{2N^2}$. The rest of the proof is in the appendix. ■

The above theorem for $N=2$ has been illustrated in the previous section. Example 3 indicates that the 2-partition CS equilibrium cannot be improved upon by a 2-simple mediated equilibrium when b is small ($b<\frac{1}{8}$). Example 4 on the other hand shows that a 2-simple mediated equilibrium can improve upon the 2-partition CS equilibrium when b is large enough ($\frac{1}{8}\leq b<\frac{1}{4}$).

It is worth pointing out that for a large N (corresponding to a small b), it is harder to improve upon an N-partition CS equilibrium by an N-simple mediated equilibrium, as the required range of b ($\frac{1}{2N^2}\leq b<\frac{1}{2N(N-1)}$) becomes small. Also, as noted in Section 2.2, the critical value "$\frac{1}{2N^2}$" has a significance in the analysis of Krishna and Morgan (2004).

4 CONCLUSION

In this note, we have just taken a step to directly compare a CS equilibrium with a specific mediated equilibrium. One obvious explanation of the restricted mechanism we use here appeals to the amount of complexity that a mediator can handle; if there are bounds on the information processing capacity of a mediator, then it is natural to ask if hiring a mediator, instead of relying on direct communication, is worthwhile.

We are aware that our results point at a potentially huge research agenda with a number of broad open questions: such as, can one fully characterise the set of mediated equilibria in this set-up and compare it with Krishna-Morgan's equilibria with *face-to-face* communication? Also, do our results generalize to a more general specification of utility functions and state distributions? However, we feel that the analytical (optimization) technique used to prove Theorem 1 in this note is of a specific nature and may not be useful to answer the above open questions.

5 APPENDIX

We complete the proof of our result here. Continuing the proof in section 3, now we are going to show that at the proposed solution, there exist $\{\lambda_k\}_{k=1}^{N-1}$ such that $\frac{\partial L}{\partial x_k} = 0$ for all $k = 1, \ldots. N-1$ and $\frac{\partial L}{\partial p_{kj}} < 0$ for all $j \neq k$, when $b < \frac{1}{2N^2}$.

First, at $p_{kj} = 0$ for all $k \neq j$; $k,\ j \in \{1, \ldots N\}$, it is easy to check that $\frac{\partial y_k}{\partial x_k} = \frac{\partial y_k}{\partial x_{k-1}} = \frac{1}{2}$, and $\frac{\partial y_k}{\partial x_j} = 0$ for all $j \neq k,\ k-1$. Also, $\frac{\partial y_j}{\partial p_{kj}} = \frac{(x_k - x_{k-1})(x_k + x_{k-1} - x_j - x_{j-1})}{2(x_j - x_{j-1})}$ for all $k \neq j$ and $\frac{\partial y_j}{\partial p_{kl}} = 0$ for all $l \neq j$, for all k.

Subsequently, it can be shown that $\frac{\partial L}{\partial x_k} = -3\left[(y_k - x_k)^2 - (y_{k+1} - x_k)^2\right] + \lambda_k - \frac{\lambda_{k+1}}{2} - \frac{\lambda_{k-1}}{2}$ for all $k = 1, \ldots. N-1$ (since λ_0 and λ_N are not defined, define them to be equal to zero).

Now $\frac{\partial L}{\partial x_k} = 0$ implies $12b(y_{k+1} - y_k) + 2\lambda_k - \lambda_{k+1} - \lambda_{k-1} = 0$ for all $k = 1, \ldots. N-1$.

This gives us a system of $N-1$ equations in $N-1$ variables, $\lambda_1, \ldots.. \lambda_{N-1}$, which can be succinctly written in matrix form as

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 & 0 & . & . & 0 \\ -1 & 2 & -1 & 0 & . & . & . \\ 0 & -1 & 2 & -1 & 0 & . & . \\ 0 & 0 & -1 & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ . & . & . & 0 & -1 & 2 & -1 \\ 0 & . & . & 0 & 0 & -1 & 2 \end{bmatrix}}_{(N-1)\times(N-1)} \underbrace{\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ . \\ . \\ . \\ \lambda_{N-1} \end{bmatrix}}_{(N-1)\times 1} = \underbrace{\begin{bmatrix} 12b(y_1 - y_2) \\ 12b(y_2 - y_3) \\ 12b(y_3 - y_4) \\ . \\ . \\ . \\ 12b(y_{N-1} - y_N) \end{bmatrix}}_{(N-1)\times 1}$$

The $(N-1)\times(N-1)$ matrix above is symmetric and tridiagonal, the ij-th element of the inverse of which is given by $\frac{1}{4N}(i+j- \mid j-i \mid)(2N- \mid j-i \mid -i-j)$ (using results by Hu and O'Connell 1996 and Yamani and Abdelmonem, 1997).

Thus, solving the equations, we get, $\lambda_k = -\frac{2bk(N-k)}{N}[3 - 2bN^2 + 4bkN]$ (which is < 0 for all $k = 1, ...N-1$).

We are now ready to show that, when $b < \frac{1}{2N^2}$, $\frac{\partial L}{\partial p_{kj}} < 0$ for all $j \neq k$, at the proposed solution (the CS equilibrium values of x_k's and y_k's) and with the above values λ_k for all $k = 1, ...N-1$.

For all $j \neq k$, we have

$$\begin{aligned} \frac{\partial L}{\partial p_{kj}} &= [(y_k - x_{k-1})^3 - (y_k - x_k)^3] - [(y_j - x_{k-1})^3 - (y_j - x_k)^3] \\ &\quad -\lambda_k \left[\frac{(y_j - y_k)(y_j - y_{k+1})}{(y_k - y_{k+1})}\right] - \lambda_{k-1}\left[\frac{(y_k - y_j)(y_j - y_{k-1})}{(y_{k-1} - y_k)}\right] \\ &\quad -(\lambda_j + \lambda_{j-1})\left[\frac{\partial y_j}{\partial p_{kj}}\right] \end{aligned}$$

We first prove $\frac{\partial L}{\partial p_{kj}} < 0$ for all $k \neq 1$, when $b < \frac{1}{2N^2}$. Substituting the values for the x_k's, y_k's and λ_k's, we have,

$$\frac{\partial L}{\partial p_{kj}} = \frac{\begin{array}{c}(k-j)^2(2bN^2-1)(2bN^2+1)(2bN^2+2bN-2bjN-2bkN-1)\\(12b^2N^4-36b^2kN^3-12jb^2N^3+24b^2N^3+32b^2k^2N^2\\+8jb^2N^2k+4b^2N^2-36b^2N^2k-12b^2jN^2+8b^2j^2N^2\\-12bN^2+18bkN+6bjN-12bN+3)\end{array}}{\begin{array}{c}N^3(2bN^2+2bN-4bjN-1)(2bN^2+4bN-4bkN-1)\\(2bN^2-4bkN-1)\end{array}}$$

As $b < \frac{1}{2N(N-1)}$ (implying $2bN^2 - 2bN \ -1 < 0$), we have $2bN^2 - 4bkN - 1 = 2bN^2 - 2bN \ -1 + 2bN(1-2k) < 0$.

Similarly, $2bN^2 + 2bN - 4bjN - 1 = 2bN^2 - 2bN \ -1 + 4bN(1-j) < 0$, and $2bN^2 + 2bN - 2bjN - 2bkN - 1 = 2bN^2 - 2bN \ -1 + 2bN(2-j-k) < 0$.

Also, $2bN^2 + 4bN - 4bkN - 1 = 2bN^2 - 2bN \ -1 + 2bN(3-2k) < 0$ as $k \geq 2$.

Finally, note that the factor,

$$\begin{aligned}
& 12b^2N^4-36b^2kN^3-12jb^2N^3+24b^2N^3+32b^2k^2N^2\\
& +8jb^2N^2k+4b^2N^2-36b^2N^2k-12b^2jN^2+8b^2j^2N^2\\
& -12bN^2+18bkN+6bjN-12bN+3\\
= \ & 12N^4b^2-12N^3b^2(3k+j-2)+4N^2b^2[8k^2+2jk+1-9k-3j+2j^2]\\
& -12N^2b+6Nb(3k+j-2)+3\\
= \ & [12N^4b^2-12N^2b+3]\\
& +(3k+j-2)[6Nb-12N^3b^2]+12N^2b^2(3k+j-2)\\
& +4N^2b^2[8k^2+2jk+1-9k-3j+2j^2]-12N^2b^2(3k+j-2)\\
= \ & 3(2bN^2-1)^2+6bN(3k+j-2)\left[1+2bN-2bN^2\right]\\
& +4N^2b^2[(k+j-3)^2+7k(k-2)+2(k-1)+j^2]
\end{aligned}$$

which is > 0 as $1 + 2bN - 2bN^2 > 0$ (as $b < \frac{1}{2N(N-1)}$) and $k \geq 2$. Hence, $\frac{\partial L}{\partial p_{kj}} < 0$ for all $j \neq k$, and for all $k > 1$, when, $(2bN^2 - 1) < 0$, i.e., when, $b < \frac{1}{2N^2}$.

Now we show that $\frac{\partial L}{\partial p_{kj}} < 0$ for $k = 1$, when $b < \frac{1}{2N^2}$. Here,

$$\frac{\partial L}{\partial p_{1j}} = \frac{\begin{array}{c}(1-j)^2(2bN^2+1)(2bN^2-2bjN-1)(12b^2N^4-12b^2N^3\\ -12jb^2N^3-4b^2jN^2+8b^2j^2N^2-12bN^2+6bN+6bjN+3)\end{array}}{N^3(2bN^2+2bN-4bjN-1)(2bN^2-4bN-1)}$$

As before, as $b < \frac{1}{2N(N-1)}$, it is easy to check that $2bN^2 - 2bjN - 1 < 0$, $2bN^2 + 2bN - 4bjN - 1 < 0$ and $2bN^2 - 4bN - 1 < 0$.

Finally, the factor,

$12b^2N^4 - 12b^2N^3 - 12jb^2N^3 - 4b^2jN^2 + 8b^2j^2N^2 - 12bN^2 + 6bN + 6bjN + 3$

$= 3(2N^2b-1)^2 + 6Nb(1+j)(2Nb+1-2N^2b) + 4N^2b^2[(j-2)^2 - 7 + j^2]$

which we need to show is > 0 for all $j \geq 2$. Clearly, it is so for all $j \geq 3$. For $j = 2$, the factor is equal to $3(2N^2b-1)^2 + 18Nb(2Nb+1-2N^2b) - 12N^2b^2$ which can be shown to be > 0 whenever $b < \frac{1}{2N^2 - \frac{4N}{3}}$. Since $\frac{1}{2N^2 - \frac{4N}{3}} > \frac{1}{2N^2}$, we have that the factor is for > 0 for all $j \geq 2$ when $b < \frac{1}{2N^2}$. Hence, $\frac{\partial L}{\partial p_{1j}} < 0$ for all $j \geq 2$, when $b < \frac{1}{2N^2}$. The proof of the theorem is thus completed.

6 REFERENCES

1. Aumann, R. J. (1974) "Subjectivity and correlation in randomized strategies," *Journal of Mathematical Economics,* 1, 67-96.

2. Aumann, R. J. (1987) "Correlated equilibrium as an expression of Bayesian rationality," *Econometrica,* 55, 1-18.

3. Aumann, R. J. and S. Hart (2003) "Long cheap talk," *Econometrica,* 71, 1619-1660.

4. Battaglini, M. (2002) "Multiple referrals and multidimensional cheap talk," *Econometrica,* 70, 1379-1401.

5. Crawford, V. P. and J. Sobel (1982) "Strategic information transmission," *Econometrica,* 50, 1431-1451.

6. Dessein, W. (2002) "Authority and communication in organizations," *Review of Economic Studies,* 69, 811-838.

7. Forges, F. (1986) "An approach to communication equilibria," *Econometrica,* 54, 1375-1385.

8. Forges, F. (1990) "Universal mechanisms," *Econometrica,* 58, 1341-1364.

9. Hu, G. Y. and R. F. O'Connell (1996) "Analytical inversion of symmetric tridiagonal matrices," *Journal of Physics A: Mathematical and General,* 29, 1511-1513.

10. Krishna, V. and J. Morgan (2004) "The art of conversation: eliciting information from experts through multi-stage communication," *Journal of Economic Theory,* 117, 147-179.

11. Moulin, H. and J. P. Vial (1978) "Strategically zero-sum games: the class of games whose completely mixed equilibria cannot be improved upon," *International Journal of Game Theory,* 7, 201-221.

12. Myerson, R. B. (1982) "Optimal coordination mechanisms in generalized principal-agent problems," *Journal of Mathematical Economics,* 10, 67-81.

13. Myerson, R. B. (1986) "Multistage games with communication," *Econometrica,* 54, 323-358.

14. Myerson, R. B. (website) "http://home.uchicago.edu/~rmyerson/research/crawfsob.xls"

15. Yamani, H. A. and M. S. Abdelmonem (1997) "The analytic inversion of any finite symmetric tridiagonal matrix," *Journal of Physics A: Mathematical and General,* 30, 2889-2893.