

Network Games

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Abstract

In a variety of contexts – ranging from public goods provision to information collection – a person’s well being depends on her own action as well as on the actions taken by her neighbors. We develop a framework to analyze such strategic interactions when neighborhood structure, modeled in terms of an underlying network of connections, affects payoffs. Our framework has two distinctive features: it permits a variety of payoff functions and applications, and it allows for variations in terms of how much players know about the overall network structure. We provide a number of results characterizing how the network structure, an individual’s position within the network, the nature of games (strategic substitutes versus complements and positive versus negative externalities), and the level of information (incomplete versus complete), shape individual behavior and payoffs.

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1 Introduction

In a diverse range of social and economic interactions - including public goods provision, job search, political alliances, trade, friendships, and information collection - an agent's well being depends on her own actions as well as on the actions taken by other agents in close proximity; i.e., her neighbors.¹ For example, the decision of an agent of whether or not to buy a new product, or to attend a meeting, might be influenced by the choices of her friends and acquaintances. The pattern of neighborhoods affecting each agent is often formalized in terms of a network of relations. Our goal in this paper is to examine how individual behavior varies with position within a network as well as how changes in the network structure - increasing the number of connections or redistributing connections - affect individual behavior and welfare.

The paper develops a general framework to address these questions. There are two distinctive features of this framework. First, we allow for a rich class of payoff structures. Second, we allow for variations in what agents know about the underlying network structure.

In a network game, individual incentives depend on the actions of her neighbors since her neighbors' choices of actions alter the marginal returns to her own actions. We focus on settings where a player's actions can be ordered, and so they can be thought of as doing more or less of the action (or in a special case, either taking an action or not taking it). There are two features of this neighborhood effect that play an important role in our analysis. The first is whether the underlying payoff structure, and hence the network game, exhibits strategic substitutes or complements. Indeed, strategic complementarities determine how a change in a neighbor's actions affects incentives for own actions. The second and equally important aspect of the neighborhood effect is the expectations concerning the neighbors' actions, which in turn depend on neighbors' expectations on their neighbors' actions, and so on. In other words, agents' information regarding the network structure can have a significant impact on individual choices. In reality, prevailing networks are complicated objects, so in many

¹For empirical work on network effects see Coleman (1966), Conley and Udry (2005), Granovetter (1994), Topa (2001), and Glaeser, Sacerdote, and Scheinkman (1996), among others.

cases an individual will only have imperfect knowledge of the details of their structure. This observation motivates a study of the role of incomplete information in network games.²

We study information issues in terms of each player's knowledge of the number of her own and others' connections in the network (i.e., their degrees). Suppose that $P(k)$ is the probability that a given player has degree k in a network. The analysis in this paper starts by focusing on the case of *Incomplete Information (II)* where each player knows only the degree distribution $P(k)$ and her own degree (i.e., how many neighbors she herself has). We contrast this with the traditional "benchmark case" of *Complete Information (CI)* where players possess complete knowledge of the prevailing network. Finally, in order to better understand the implications of information in a network context, we also discuss how matters are affected if players have intermediate levels of local information; that is, they know something about the degrees of all other players who are no more than some given distance away (in terms of minimum path length in the network).

In general, our approach to analyzing the strategic situation is based in Harsanyi's apparatus of Bayesian games and the corresponding notion of Bayesian Nash Equilibrium. The type of each player coincides with the "local" information she enjoys. Under II, her type is given by her own degree, while if her information radius is one it includes her own degree and those of her neighbors, and under CI it is given by the full network. In this latter case, Bayesian Nash equilibrium coincides with the standard concept of Nash Equilibrium.

Our *first* set of results, Propositions 2-3, relate to a comparison of equilibrium actions and payoffs of players who differ in their position in a given network, captured in terms of their degree. We show that, under incomplete information, in every (symmetric) equilibrium,³ actions are increasing in player degree if neighbors' actions are strategic complements, and decreasing in player degree if neighbors' actions are strategic substitutes. We also show that, in every equilibrium, expected payoffs are increasing in player degree if payoffs exhibit positive externalities, and decreasing in player degree if payoffs exhibit negative external-

²For empirical evidence on incomplete network knowledge see, e.g., Kumbasar, Romney and Batchelder (1994), Bondonio (1998), and Casciaro (1998).

³Existence of such equilibrium is established by Proposition 1.

ities. These results help us in understanding how the nature of the underlying strategic game determines whether well-connected players, say, work harder or free ride on their less connected cohort and if having more connections is good or bad for personal payoffs. We find, however, that such clear-cut monotonicity results crucially depend on the information enjoyed by players. As a first illustration of this, we provide examples showing that these degree-monotonicity properties can be violated under complete information.

The *second* set of results, Propositions 4-7, concern changes in network structure in terms of adding links. Under incomplete information, this is studied by comparing networks whose degree distributions can be related via First Order Stochastic Dominance (FOSD). The key insight here is that a more connected network induces uniformly higher equilibrium actions as a function of a player's degree. While we are able to prove this conclusion in full generality for games where actions are strategic complements, we restrict attention to certain classes of games (binary-action games⁴ and continuous-action games with quadratic payoffs) for the case of strategic substitutes. In games with strategic complements, a FOSD shift in the degree distribution increases equilibrium actions. In contrast, in games with strategic substitutes, the results can be qualitatively reversed. For instance, in a binary-action game, the overall percentage of players taking the action 1 decreases.

We also study the effects of FOSD shifts in degree distribution on equilibrium payoffs. In games with strategic complements and positive externalities, we show that payoffs of players of any degree as well as average payoffs increase. In games with strategic substitutes, the payoffs of players of any degree actually decrease while the impact on average payoffs depends on the details of the changes in the degree distribution.

The *third* set of results, Propositions 8-9, address the issue of characterizing the effect of a *general* redistribution of weight in network connectivity – in particular, one that might keep the number of links fixed (i.e., the average degree constant) and simply allocate them

⁴In binary-action games, a widely applicable class, players choose one of two actions (say 0 and 1, for instance buying a product or not, changing a behavior or not, etc.). Equilibria in such games have a particularly simple structure. There is a threshold degree (type) \hat{t} with all degrees $t < \hat{t}$ choosing action 1 (0) and all degrees $t > \hat{t}$ choosing action 0 (1) in strategic-substitute (-complement) games.

differently (e.g., in a more or less “polarized” fashion). Given the generality of the objective, we are only able to provide concrete results for binary-action games. In this context, both for games of strategic complements and substitutes, the change in the threshold defining the equilibrium is found to solely depend on how the total probability weight is shifted relative to those thresholds. This, in turn, has implications for welfare depending on the type of strategic interaction (complements or substitutes) and the nature of the externalities (positive or negative).

The above results are obtained in a setting where players know the degree distribution and their own degree. The monotonicity of actions with regard to degree in *all* symmetric equilibrium plays a central role in the derivation of these results. However, as we mentioned above, these monotonicity properties do not necessarily obtain under complete information. This finding leads us to examine monotonicity properties of equilibrium for intermediate levels of information between incomplete and complete information. We show that in a setting where players know the degree distribution, their own degree as well as the degree of all their neighbors, there always exists a (suitably defined) monotonic equilibrium, although there may also exist some non-monotonic ones. This result then suggests that we can recover some of the key insights obtained under incomplete information also in a setting where players have richer information about the network.⁵

We now place the paper in the context of the literature. The main contribution of the current paper lies in the development and analysis of a general framework to study the effects of social interactions on individual behavior. Three aspects of the framework: the rather general nature of payoffs, the general network structure, and the allowance for varying levels of information are worth emphasizing and contrasting with the existing literature in the field. In particular, almost all the existing work on network games to date – see, e.g., Ballester, Calvó-Armengol, and Zenou (2005), Bramoullé and Kranton (2005), Galeotti (2005), Goyal and Moraga-Gonzalez (2001) – has assumed complete information and worked with specific formulations both with regard to payoff functions and with regard to the network

⁵The interpretation of the model with incomplete information is discussed in greater detail in section 2.

structures.⁶ In order to further clarify the scope of our paper, we now discuss in greater detail its relationship with three recent papers: Bramoullé and Kranton (2005), Galeotti and Vega-Redondo (2005) and Sundararajan (2005).⁷

Bramoullé and Kranton (2005) consider a game where players search for valuable information and information is freely shared among neighbors. Players' utilities depend on a sum of their own efforts and efforts of neighbors. They assume that efforts of players are strategic substitutes and that each player has complete network information. They find that there is multiplicity of equilibria and that the comparative statics within and across networks are ambiguous. By contrast, Propositions 2-3 show that if information is incomplete then equilibria are monotone in actions and payoffs. Moreover, Proposition 5 shows that the effects of adding links (in a binary version of their game) are clear cut: every degree player chooses actions with greater probability, receives lower expected externalities from their neighbors, and earns lower payoffs. These results highlight the important role of network information in shaping behavior.

Galeotti and Vega-Redondo (2005) and Sundararajan (2005) analyze a network game with incomplete information. Sundararajan (2005) studies games with strategic complements under a specific information setting – incomplete information. The present paper develops and analyzes a general framework which allows for games with complements as well as substitutes, studies the effect of network changes (e.g., as it becomes more connected or more heterogenous), and allows for different levels of information. Galeotti and Vega-Redondo (2005) assume that payoffs are given by the product of neighbors' actions (i.e., are multiplicative complements) and also focus on the incomplete information case. Their payoff specification does not quite fit into our setting, as we will discuss below, as it violates a condition which is instrumental to our work in deducing how network changes influence

⁶In particular, regular networks (in which all players have the same degree) and core-periphery structures (the star network is a special case of such structures) have been extensively explored in the literature.

⁷It is worth noting that Lopez-Pintado and Watts (2005) study social influence games and their general motivation is similar to ours. However, they abstract from network structure altogether, the effects of which are the focus of the present paper.

behavior (this condition is satisfied in most applications that have been studied in the literature. For example, existing models of crime networks, local public goods, and collaboration among firms, all constitute special cases of our framework). As such, there is no overlap in our analyses or conclusions.

The paper also relates to a strand of papers in the computer science literature on *graphical games* (see, e.g., Kearns, Littman, and Singh, 2001, and Kakade, Kearns, Langford, and Ortiz, 2003). While the underlying model tackled in that literature is close to ours, the focus is completely different. The literature on graphical games is concerned with efficient algorithms for finding Nash equilibria, and is not concerned with what those equilibria look like or how they are influenced by the setting. Our results are complementary in that they provide a characterization of the equilibria that these algorithms ultimately reach, and focus on how the setting influences the structure of the equilibria.

The rest of the paper is organized as follows: Section 2 develops the theoretical model. Section 3 presents results on equilibrium behavior as a function of location within a network. Sections 4 and 5 examine the effects of changing networks on equilibrium behavior. Section 6 discusses levels of knowledge between incomplete and complete information, while Section 7 concludes. The proofs are relegated to an Appendix.

2 The General Model

This section presents the main elements of our theoretical framework: the network relations between players, the nature of strategies and payoffs, the information a player has about the network relations, and the equilibrium concepts.

Networks: The connections between a finite set of players $N = \{1, \dots, n\}$ are described by an undirected network. That network is represented by a symmetric matrix $g \in \{0, 1\}^{n \times n}$, with $g_{ij} = 1$ denoting that i and j are connected. We follow the convention of setting $g_{ii} = 0$ for all i . The set of all possible non-directed networks with n vertices is denoted by G .

Let $N_i(g) = \{j | g_{ij} = 1\}$ represent the set of direct neighbors of i . For any integer $k \geq 1$, $N_i^k(g)$ denotes the k -neighborhood of i in g ; that is, all the players that can be reached from i by paths of length no more than k . So, inductively $N_i^1 = N_i$ and $N_i^k = N_i^{k-1} \cup (\cup_{j \in N_i^{k-1}} N_j)$. The *degree*, $k_i(g)$, of player i is the number of i 's direct connections:

$$k_i(g) = |N_i(g)|.$$

We denote the degree distribution of the network by P , where $P(k)$ is the frequency of nodes with degree k .⁸

Strategies and Payoff Functions: Each player i takes an action x_i in X , where X is a compact subset of $[0, 1]$. Without loss of generality, we assume throughout that $0, 1 \in X$. We consider both discrete and connected action sets X . The payoff of player i when the profile of actions is $x = (x_1, \dots, x_n)$ is given by:

$$u_i(x, g) = v_{k_i(g)}(x_i, x_{N_i(g)})$$

where $x_{N_i(g)}$ is the vector of actions taken by the neighbors of i . Thus the payoff of a player depends on her own action, as well as on the actions that her direct neighbors take.

Note that the payoff function depends on the player's degree but not on her identity. Therefore, any two players who have the same degree have the same payoff function. We shall also assume that v_k depends on the vector $x_{N_i(g)}$ in an anonymous way, so that if x' is a permutation of x (both k -dimensional vectors) then $v_k(x_i, x) = v_k(x_i, x')$ for any x_i . If X is a connected action set then v_k is taken to be differentiable in all arguments and concave in own action.

The Nature of Payoff Interdependence: There are two different aspects regarding how other players' actions affect a given player's utilities that we need to keep track of. The first is whether increasing the actions by other players leads a player to want to increase or

⁸In this paper, we focus on undirected networks; in some applications such as learning from others' actions, it is possible that player i observes j but the converse is not true. Directed networks are more appropriate for such applications. Many of the arguments that we develop in this paper will carry over to the setting of directed networks.

decrease her actions. The second is whether increasing the actions of other players increases or decreases a player's utility.

A game exhibits *strategic complements* if it satisfies increasing differences. That is, for all k , $x_i > x'_i$, and $x \geq x'$: $v_k(x_i, x) - v_k(x'_i, x) \geq v_k(x_i, x') - v_k(x'_i, x')$. Similarly, the game is said to exhibit *strategic substitutes* if it satisfies decreasing differences. That is, for all k , $x_i > x'_i$, and $x \geq x'$: $v_k(x_i, x) - v_k(x'_i, x) \leq v_k(x_i, x') - v_k(x'_i, x')$. These notions are said to apply strictly if the payoff inequalities are strict whenever $x \neq x'$.

A game exhibits *positive externalities* if for each v_k , and for all $x \geq x'$, $v_k(x_i, x) \geq v_k(x_i, x')$, whereas it displays *negative externalities* if $v_k(x_i, x) \leq v_k(x_i, x')$. Correspondingly, the game exhibits *strict externalities* (positive or negative) if the former payoff inequalities are strict whenever $x \neq x'$.

In order to make comparisons of actions and payoffs across players of different degrees, we make the following assumption.

Assumption A $v_{k+1}(x_i, (x, 0)) = v_k(x_i, x)$ for any $(x_i, x) \in X^{k+1}$.

Thus adding a link to a neighbor who chooses action 0 is payoff equivalent to not having an additional neighbor.

Assumption A is critical to a number of our results and is important in drawing welfare conclusions. It is restrictive, as it implies that adding new neighbors is akin to increasing the action that a player perceives being played by neighbors. This assumption is appropriate in situations such as information sharing, where it is the total information gathered by neighbors and shared with a player that is important. It applies to local competition among firms, where a firm cares about the total activity (e.g., production or advertising) by its neighbors. It also applies to local collective action problems (e.g. a local version of the model studied by Chwe (2000)) when the payoff to a player depends on the aggregate effort of her neighbors.

Assumption A is violated if a player cares about the average action of her neighbors (rather than the absolute levels) or the fraction of individuals choosing a particular action

(rather than the total number). This is true in the coordination game studied by Morris (2000). Assumption A is also violated if payoffs are a product of the actions of neighbors, as in Galeotti and Vega-Redondo (2005).⁹

There are two important remarks that we make here.

First, in situations where the number of neighbors taking different actions matters, network structure has important implications, as having more or fewer neighbors affects the nature of payoffs across players in the network. Thus, position in a network can matter in interesting ways. In settings where one only cares about the average number of neighbors taking an action, network structure plays a much more limited role. In that case, having more or fewer neighbors might have no impact if the fraction of them taking an action is the same.¹⁰

The second comment is that there is no right or wrong assumption about how payoffs depend on neighbors' actions. Assumption A captures many applications of interest, and there are others where it fails. The following example illustrates the scope of Assumption A as well as clarifies different kinds of possibilities that arise.

EXAMPLE 1 *Payoffs Depend on the Sum of Actions*

Player i 's payoff function when she chooses x_i and her k neighbors choose the profile (x_1, \dots, x_k) is:

$$v_k(x_i, x_1, \dots, x_k) = f(x_i + \lambda \sum_{j=1}^k x_j) - c(x_i), \quad (1)$$

where $f(\cdot)$ is non-decreasing and $c(\cdot)$ is a "cost" function associated with own effort. The parameter $\lambda \in \Re$ determines the nature of the externality (positive or negative, if λ is correspondingly positive or negative). This general example yields (strict) strategic substitutes or complements when (assuming differentiability) $\lambda f''$ is negative or positive, respectively.

⁹They use the payoff function: $u_i(x_i, x) = x_i \prod_{j \in N_i(g)} x_j - \alpha \frac{x_i^2}{2}$, where $x_j \in \Re_+$ for all j .

¹⁰Network structure still plays some role in overall behavior. For instance, in Morris (2000), it influences whether more than one action can survive in an equilibrium in a coordination game. But it plays a much less direct role than it does in our analysis.

More specifically, the case where f is concave, $\lambda = 1$, and $c(\cdot)$ is increasing and linear corresponds to the case of information sharing as a local public good studied by Bramoullé and Kranton (2005), where actions are strategic substitutes. In contrast, if $\lambda = 1$, but f is convex (with $c'' > f'' > 0$), then we obtain a model with strategic complementarities of the sort proposed by Goyal and Moraga-Gonzalez (2001) to study collaboration among local monopolies. In fact, the formulation in (1) is general enough to accommodate a good number of further examples in the literature such as human capital investment (Calvo-Armengol and Jackson 2004, 2005), crime networks (Ballester, Calvo-Armengol, and Zenou, 2005), some coordination problems (Ellison 1993), and the onset of social action (Chwe, 2000).

The following two specializations of Example 1 are also useful to keep in mind.

EXAMPLE 2 *Quadratic Payoff Functions*

Here $X = [0, 1]$ and we specialize Example 1 to a case where $f(y) = \gamma y + \alpha y^2$ and $c(x_i) = \beta x_i^2$ for some $\gamma, \alpha, \beta > 0$.

EXAMPLE 3 *“Best-Shot” Public Goods Games*

$X = \{0, 1\}$ and we may interpret 1 as acquiring information (or providing any local and discrete public good) and 0 as not acquiring it. We posit that $f(0) = 0$, $f(x) = 1$ for all $x \geq 1$, so that acquiring one piece of information suffices. Costs, on the other hand, are assumed to satisfy $0 = c(0) < c(1) < 1$ so that no individual can find it optimal to dispense with the information but prefers one of her neighbors to gather it.¹¹

Information Structures: What a player knows about the network in which she resides can certainly make a difference regarding her behavior. At one extreme we can consider a situation where a player knows the entire network, including her own degree, the degrees of

¹¹The Best-Shot game is a good metaphor for many situations in which there are significant spillovers between players' actions. For instance, consumers learn from relatives and friends (Feick and Price, 1987), in research and development, innovations often get transmitted between firms, and similarly in agriculture, experimentation is often shared amongst farmers (Foster and Rosenzweig, 1995, Conley and Udry, 2005). For a discussion of best shot games within the context of public good games, see Hirshleifer (1983).

her neighbors, and the degrees of the neighbors of her neighbors and so on. We refer to this as *complete information*. Near the other extreme is a case where a player knows nothing more than her own degree and the degree distribution of the network. We refer to this as *incomplete information*.

In Section 6, we shall return to discuss some other cases.

Independently of the depth of their network knowledge, players are always assumed to be informed of the degree distribution $P(\cdot)$, which is taken to be common knowledge. More specifically, we suppose that players believe that the prevailing network g has been drawn stochastically from a family of networks G so that the following two properties are satisfied:

- (a) The probability of any given player having k links in g is $P(k)$.
- (b) The degrees $k_i(g)$ and $k_j(g)$ displayed by *any* two players i and j (even if they happen to be neighbors) are stochastically independent.

Under (a) and (b), for any given player, the probability that a random neighbor of hers is of degree k is given by:

$$\tilde{P}(k) = \frac{kP(k)}{\langle k \rangle}, \quad (2)$$

where $\langle k \rangle = E_P[k]$ is the average degree in the network. This is the standard formalization of the idea that a randomly chosen link is likely to point to a node of a certain degree in proportion to that node's degree.

In order for a mechanism to exist that guarantees (a) and (b), we must take $n \rightarrow \infty$.¹² While we assume independence across degrees of players in the network, it is worth noting that many of our results hold under certain directions of degree correlation (e.g., see the remark following Proposition 2).

¹²An example of such a mechanism is the configuration model in the theory of random graphs, e.g. Bender and Canfield (1978) or Bollobás (1980). But this mechanism only guarantees (a) and (b) in the limit, as the set of nodes grows unboundedly. In this sense, therefore, all of our results must be interpreted either as requiring some bounds on rationality of the players' beliefs, or as holding approximately for a large enough population.

The strategic implications of different information structures can be analyzed within the usual Harsanyi framework of Bayesian games by a suitable specification of the *type spaces* of players, \mathcal{T}_i . That is, given the type $t_i \in \mathcal{T}_i$ revealed to any given player i , her beliefs are simply obtained as the posterior induced by such t_i and the prior satisfying (a) and (b). More precisely, the two contexts we focus on are as follows.

Incomplete Information (II): $\mathcal{T}_i = \{0, 1, \dots, n - 1\}$ for all i , and the type $t_i(g)$ revealed to i when any given g prevails is $t_i(g) = k_i(g)$.

Complete Information (CI): $\mathcal{T}_i = G$ for all i , and the type $t_i(g)$ revealed to i when any given g prevails is $t_i(g) = g$.

Equilibrium A strategy of player i is a mapping $\sigma_i : \mathcal{T}_i \rightarrow \Delta(X)$, where $\Delta(X)$ is the set of distribution functions on X . A strategy profile is denoted by $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$. The expected payoff to player i , facing a strategy profile $\boldsymbol{\sigma} = \{\sigma_i, \boldsymbol{\sigma}_{-i}\}$, is

$$U(\boldsymbol{\sigma}, t_i) = \sum_{g \in G} \left[\int_{(x_i, x_{N_i(g)}) \in X^{k_i(g)+1}} v_k(x_i, x_{N_i(g)}) d\boldsymbol{\sigma}(g) \right] P(g|t_i), \quad (3)$$

where $d\boldsymbol{\sigma}(g)$ is the measure on x induced when g is the realized network and players employ the profile of strategies $\boldsymbol{\sigma}$.

We now elaborate on different possible interpretations of the information structure. There are several reasons for examining these two information structures. First, there are different applications where one or the other might be more appropriate. Incomplete information is more relevant in situations where the network is more volatile and actions are not easily adjusted. For instance, if an individual might interact with some number of individuals over time, but detailed information about their identities, friends, and behaviors is not available, and the individual must choose an action which is not then easily changed, incomplete information is appropriate. This applies, for instance, to a context in which an agent chooses whether or not to learn a (human or computer) language.

In contrast, there are situations where actions are more easily coordinated or adjusted. Given that players only care about their neighbors' actions, if they can observe those actions and they choose an action that is a best response, then they must be at an equilibrium. For instance, consider a group of students choosing whether to attend some social event or not. Suppose each student is interested in attending the event if at least some number of their friends also attends. Each student can then discuss their intentions with their friends and update the decision to go out or not. The student's friends do the same, as do their friends, and so on. If the process of discussion and updating of action choice continues, it will ultimately reach a complete information equilibrium. This can happen even if the agents know nothing about the larger network structure. In fact, any process where actions are adjusted to best reply to neighbors' actions, will either cycle or eventually come to rest at a complete information equilibrium.¹³

Thus, both information settings apply, but possibly to different situations. Our formal analysis of these two information structures illustrates how the information structure affects behavior. Furthermore, the analysis teaches us which equilibrium features are robust to changes in available information and which ones are more sensitive.

DEFINITION 1 *An equilibrium is a profile of strategies σ such that for all i , $t_i \in \mathcal{T}_i$, and $x_i \in X_i$,*

$$U(\sigma, t_i) \geq U(x_i, \sigma_{-i}, t_i). \tag{4}$$

We say that a profile of strategies is *symmetric* if $\sigma_i = \sigma_j$ for all i and j , and an equilibrium is said to be symmetric if it is comprised of a symmetric strategy profile. Finally, we say that an equilibrium is in pure strategies if for every $i \in N$ and for every type $t_i \in \mathcal{T}_i$, $\sigma_i(t_i)$ places probability 1 on some element in X .

¹³See Jackson and Yariv (2006) for an analysis of dynamic processes exhibiting this flavor.

PROPOSITION 1 *There exists an equilibrium in the network game defined above under either information structure. Under incomplete information there always exists a symmetric equilibrium. If the game is of strategic complements, then it has an equilibrium in pure strategies (that can be chosen to be symmetric in the case of incomplete information).*

The proof of the first two claims is standard and omitted.¹⁴ The proof for the case of strategic complements follows from arguments in Propositions 4 and 6 below.

3 Comparing Choices Within a Network

We now study how position in a network affects behavior, by examining how equilibrium actions change with players' degrees.

Under incomplete information, we denote a symmetric strategy profile by $\sigma = \{\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}\}$ where $\sigma_k \equiv \sigma(k)$ specifies the 'action' chosen by every player with degree k .

We say that a strategy σ is *monotone increasing* whenever σ_k FOSD $\sigma_{k'}$, for each $k > k'$. A *monotone decreasing* strategy is defined analogously. For brevity, a symmetric equilibrium is called monotone (increasing or decreasing) when it is given by a monotone strategy.

PROPOSITION 2 *In a game of incomplete information with strict strategic complements (substitutes) every symmetric equilibrium is monotone increasing (decreasing).*

The strictness in complements or substitutes is important for this result. For instance, if players were completely indifferent between all actions, then non-monotone equilibria would clearly be possible.

The intuition of the proposition is as follows. Consider the strategic complements case. Consider a player with degree $k + 1$. Suppose all of her neighbors follow a symmetric equilibrium strategy, but her $(k + 1)$ 'th neighbor chooses the minimal 0 action. Assumption A implies that her best response would be identical to the equilibrium best response of a

¹⁴Existence of a symmetric equilibrium follows from standard arguments. For example, see remark (ii) following Theorem 2 in Jackson, Simon, Swinkels and Zame (2002), and note that the games here are a special case where communication is unnecessary as the outcome is single-valued.

degree k player. However, in any non-trivial equilibrium (where at least one player chooses an action different from 0), the $(k + 1)$ 'th neighbor would be choosing, on average, a positive action. Strict complementarities imply that our player best responds with (weakly) higher actions than her k degree peers. The opposite reasoning applies to the case of strategic substitutes.

This intuition underlying the above result suggests that it can be extended to settings with degree correlations. The following remark elaborates on this.

REMARK 1 *Positive association arises when the conditional degree distribution of any given neighbor of a player of degree k , denoted \tilde{P}_k , first order stochastically dominates the corresponding distribution $\tilde{P}_{k'}$ of a player of degree k' , where $k > k'$. The positive-monotonicity property holds in games of strategic complements if the degrees of neighbors are positively associated. Similarly, the negative-monotonicity result for strategic substitutes also holds if the degrees of neighbors are negatively associated.*

We now turn to the relation between degree and payoffs. The following result shows that equilibrium payoffs satisfy a monotonicity property in the incomplete information case.

PROPOSITION 3 *In a game of incomplete information with positive (negative) externalities every symmetric equilibrium has expected payoffs that are non-decreasing (non-increasing) in degree.*

We emphasize that under positive externalities, players with more neighbors earn higher payoffs irrespective of whether the game exhibits strategic complements or substitutes. Thus, even if the game displays strategic substitutes and higher degree players exert lower effort, they still earn a higher payoff. In every case, therefore, there is a clear advantage to being well connected in terms of having more connections. In the case of complements this comes from the fact that players with a higher degree expect higher overall actions by neighbors. In the case of substitutes, the results may be interpreted as saying that better connected players exploit network connections to free ride on those that are less-well connected.

The intuition behind Proposition 3 is as follows. Consider the case of positive externalities and look at a player with degree $k + 1$. Suppose, as before, that all of her neighbors follow the symmetric equilibrium strategy, but her $(k + 1)$ 'th neighbor chooses the minimal 0 action. Assumption A implies that our player would be able to replicate the expected payoff of a k degree player by simply using the strategy of the degree k player. However, if there is a positive probability that the $(k + 1)$ 'th neighbor chooses a positive action then positive externalities imply a higher expected payoff for our $k + 1$ degree player. Thus, the $(k + 1)$ degree player can assure herself an expected payoff which is at least as high as that of any k degree player.

Propositions 2 and 3 establish a clear-cut effect of a player's degree on her relative effort levels and payoffs for a broad class of games. These results are dependent on the incomplete information setting. For example, these results contrast with results obtained by Bramoullé and Kranton (2005) for a public-good context where the payoff function is as specified in (1), with a concave f , a linear cost function $c(\cdot)$, and $\lambda = 1$. Under the implicit assumption of complete information, they find that there exist Nash equilibria in which higher degree individuals choose higher actions and earn lower payoffs as compared to lower degree players.¹⁵ To provide a concrete illustration, consider the particular case given by the best-shot game from Example 3 and a star network with $n \geq 3$ players. In this network, there is a Nash equilibrium in which the center (of degree $n - 1$) chooses 0 and the other players choose 1. There is also, however, an equilibrium in which the center chooses 1 and the other players choose 0. This second equilibrium violates the monotonicity both in action and payoff.

The best-shot game satisfies strategic substitutes and positive externalities. Therefore, our results establish that the actions displayed at an equilibrium under incomplete information must fall in degree while payoffs must correspondingly increase. That is, an outcome such as that induced by the second of the above complete-information equilibria is ruled out.

¹⁵Our results also differ from the results of Galeotti and Vega-Redondo (2005) obtained under incomplete information. This is because, as discussed earlier in the paper, their payoff function does not satisfy Assumption A.

This indicates that information conditions can have an important role in network games and, in general, one may expect that more detailed information on the network should allow for a wider range of equilibrium outcomes – some consistent with the monotonicity established in Propositions 2 and 3 and others that are not. We shall return to this important issue in Section 6, where we also explore the implications of intermediate cases between incomplete and complete information.

4 Comparing Choices Across Networks: Adding Links

This section examines the effects of adding links in a network on individual behavior and social outcomes. Under incomplete information, we formalize this idea through the notion of First-Order Stochastic Dominance (FOSD). It is useful to divide the analysis between the two payoff scenarios considered: strategic complements and strategic substitutes. Whereas the first case admits sharp and general results, the second one is substantially more difficult to tackle and only allows us to obtain results for a restricted class of games. In what follows, we address each of these cases in turn, and then compare the analysis with the situation prevailing under complete information.

Consider two different situations where the underlying network displays degree distributions P and P' . If we want to think of P as embodying a higher connectivity than P' , then the key comparison from the viewpoint of a given player does not pertain to those distributions *per se* but to the distributions that reflect the degree of one of her typical neighbors; that is, the respective conditional (neighboring node) distributions \tilde{P} and \tilde{P}' given by (2). In what follows, we therefore focus on these conditional degree distributions.¹⁶

Let us say that an equilibrium σ dominates an equilibrium σ' if for every i and type t_i , $\sigma_i(t_i)$ FOSD $\sigma'_i(t_i)$.

¹⁶ It is important to note that first order stochastic dominance in the degree distribution does not imply a similar relation for the neighboring node degree distribution. To see this, consider a degree distribution P' in which degree 2 or 10 arise probability 1/2 each and a distribution P in which degrees 8 or 10 arise with probability 1/2 each. Clearly P FOSD P' . Next consider the conditional degree distributions, \tilde{P} and \tilde{P}' . Under \tilde{P}' , the probability that a neighbor has degree 10 is 5/6, while under \tilde{P} , the same probability is 5/9. Thus, \tilde{P} does not FOSD \tilde{P}' .

PROPOSITION 4 *Consider a game of incomplete information with strategic complements and suppose that \tilde{P} FOSD \tilde{P}' . Then for every equilibrium σ' under P' there exists a symmetric equilibrium σ under P which dominates it.*

To get some intuition for this result consider the case where players' choices are complements in the strict sense and let σ' be a symmetric equilibrium under P' . Proposition 2 assures us that this equilibrium is monotone. In particular, as we shift weight to higher degree neighbors by switching to the conditional degree distribution \tilde{P} , any player's highest best response to the original equilibrium profile would be at least as high as the supremum of her original strategy's support. We can now iterate this best response procedure. Since the action set is compact, this process converges and it is easy to see that the limit is a symmetric equilibrium which dominates the original one.

Consider now the effect on welfare of a FOSD shift, where this welfare is assessed by the expected payoff of a randomly chosen player (according to the prevailing degree distribution). Naturally, it must depend on whether the externalities are positive or negative. Suppose, for concreteness, that they are positive and let \tilde{P} FOSD \tilde{P}' . Then, from Proposition 4, we know that for every equilibrium σ' under P' there exists an equilibrium σ under P in which players' actions are all at least as high. Let us further assume that P FOSD P' . Then, since payoffs are non-decreasing in degree (cf. Proposition 3), the *ex-ante* expected payoff must rise when one moves from P' to P . But, of course, one could also have that P does *not* shift weight to high-degree nodes in such a clear-cut fashion despite the fact that \tilde{P} does (recall Footnote 16). In that case, the overall effect on welfare might be negative. We present an example in the appendix which illustrates this possibility.

Next, we turn to network games in which actions are strategic substitutes, where matters are substantially more complicated. The source of the complication is the following. In view of the monotonicity results in Proposition 2 an increase in degrees of a neighbor (on average) implies a fall in her action (on average), which, from strategic substitutes, suggests that the best response of a player should increase. However, this increase in action of every

degree may come into conflict with the expectation that neighbors must be choosing a lower action, on average. To make progress we restrict attention to *binary-action games* and to games with linear quadratic payoffs (where payoffs depend only on the sum of other players' actions).

The useful feature of binary-action games with strict strategic substitutes is that, in view of the monotonicity established by Proposition 3, there is a unique symmetric equilibrium strategy σ that is fully characterized by a threshold. That is, there exists some $t \in \{1, 2, \dots\}$ such that for all $t_i < t$ we have $\sigma(t_i) = 1$, for all $t_i > t$ we have $\sigma(t_i) = 0$, and for $t_i = t$ the induced $\sigma(t_i)$ may be some probability mixture between 0 and 1. The following result shows that FOSD changes in (conditional) degree distributions have clear cut effects on equilibrium behavior in such games.

PROPOSITION 5 *Consider a game of incomplete information with strict strategic substitutes and $X = \{0, 1\}$. If \tilde{P} FOSD \tilde{P}' then the equilibrium under P dominates that under P' but the probability that a randomly selected neighbor chooses 1 is lower.*

The intuition behind this result is as follows. Consider a symmetric strategy with threshold t' . As the distribution of neighbors' degrees shifts, in the FOSD sense, each player believes that it is more likely that her neighbors will have a higher degree. This means that the neighbor is less likely to choose 1. Since the game is one with strategic substitutes, each player's incentives to choose 1 increases and the first part of the result follows. Of course, if ultimately the probability of each player choosing 1 rises, then the incentives to choose 1 for each player are lower, which would generate an inconsistency. The second part of the result then follows.

Proposition 5 relies on the observation that the incentives of a player depend on the expected probability that each of her neighbors provides the public good. With this observation in hand, we can extend the conclusions of Proposition 5 to specific continuous action games – namely, to games in which the expected marginal payoff of a player can be written as a linear function of her own action and the total effort of each of her neighbors.

EXAMPLE 4 *Equilibria in Quadratic Games*

Consider the quadratic games from Example 2. The expected marginal payoff to a degree k player is proportional to $\gamma + 2(\alpha - \beta)x_i(k) + 2\lambda k\tilde{x}$, where \tilde{x} is the expected action of a random neighbor. Concavity in own actions and strategic substitutes imply that $\lambda < 0$ and $\alpha < \beta$. For an interior solution we require that $\gamma > 0$. With these observations we can now write, after some algebra, an interior equilibrium action for a degree k player as follows:

$$x_i(k) = \frac{\gamma}{2(\beta - \alpha)} \left[\frac{\beta - \alpha - \alpha(\tilde{k} - k)}{\beta - \alpha - \alpha\tilde{k}} \right] \quad (5)$$

where \tilde{k} is the expected degree of a neighbor, and β must be large enough to ensure this quantity is positive. It is now easily verified that if \tilde{P} FOSD \tilde{P}' , then for every degree k equilibrium action levels under P are higher than under P' while the conditional expected action of every neighbor is lower under \tilde{P} .

We now turn to effects of FOSD shifts in degree distributions on welfare. The general observation here is that these effects are ambiguous, for similar reasons as in the case of strategic complements – i.e., they require a comparison of both the absolute and the conditional degree distributions. To see this, let us focus for concreteness on the case of binary-action games studied in Proposition 5 (the same considerations apply to games with quadratic payoffs whose equilibrium strategy is given by (5)). Consider the case where the degree distribution switches from P' to P where their conditional counterparts satisfy \tilde{P} FOSD \tilde{P}' . From Proposition 5 we know that the probability that any given neighbor chooses 1 falls. Thus, if the externalities are positive, the expected payoff of a player with a given degree falls as well. The overall effect of this change on welfare now depends on how the new P redistributes the weight among the different degrees and this can go either way. In some instances, however, it is possible to derive definite results. This happens, for example, if the game exhibits positive externalities and, despite the fact that \tilde{P} FOSD \tilde{P}' , we have a converse comparison concerning the absolute degree distributions so that P' FOSD

P . The former FOSD relationship implies that, for any given player with a *fixed degree*, the expected payoff is lower under P than under P' (this happens because the probability that each of her neighbors chooses 1 decreases). Therefore, if P' FOSD P it follows from the latter observation and the fact that expected payoff is increasing in degree (due to positive externalities) that welfare is lower under P as compared to P' .

Finally, we note that the insights gathered from Propositions 4 and 5 extend to complete information in the case of strategic complements. This is stated in the following result, which considers how these equilibria are affected by the addition of links.

Let g be a network with $g_{ij} = 0$ and denote by $g' = g + ij$ the network obtained from g by adding the link between i and j .

PROPOSITION 6 *Consider a complete information game with strategic complements. For any equilibrium σ under g there exists an equilibrium σ' under g' that dominates it.¹⁷ Moreover, if X is connected, the game is of strict strategic complements, and σ is interior, then there exists an equilibrium σ' under g' in which all players in the component of i and j play strictly higher actions.*

Turning now to games of strategic substitutes under complete information, in order to draw the sharpest comparison with Proposition 5, let us focus on Best-Shot games (Example 3). In these games it is the unique best response to choose action 0 if any neighbor chooses 1 and it is the unique best response to choose 1 if all neighbors choose 0. As highlighted by Bramoullé and Kranton (2005) in a related class of games, this implies that the pure-strategy equilibria of these games are related to graph-theoretic objects termed independent sets.

An *independent set* for a network g is a set $I \subseteq N$ such that for any $i, j \in I$, $g_{ij} \neq 1$; so that no two players in I are linked.

It is clear that there is a one-to-one mapping between the (pure-strategy) Nash outcomes of a Best-Shot game played in a network g and its *maximal independent sets* – i.e., inde-

¹⁷In the present complete-information scenario, the general notion of dominance across strategies introduced above is understood as follows: for every $i \in N$, $\sigma'_i(g')$ FOSD $\sigma_i(g)$.

pendent sets that are not contained in any other independent set. Best-Shot games played on the empty network (no links) prescribe that all players choose action 1 in the unique equilibrium, while any Best-Shot game played on the full network (in which every pair of players has a direct link) has a set of n pure equilibria characterized by exactly one player choosing action 1. As it turns out, there is a gradual and monotone transition between these two extremes as links are added to the network.

PROPOSITION 7 *Consider a Best-Shot game played under complete information. Consider any pure strategy equilibrium σ of $g + ij$. Either σ is an equilibrium under g , or there exists an equilibrium under g in which a strict superset of players chooses 1. Moreover, if $g \neq g + ij$ then there are equilibria under g that are not equilibria of $g + ij$.*

The intuition of the proof is quite simple. Indeed, start with an equilibrium under $g + ij$. From the above discussion, this equilibrium corresponds to a maximal independent set in $g + ij$. Consider that set of nodes when the underlying network is given by g . Of course, it is still an independent set. If it is a maximal one, then it indeed identifies the original equilibrium strategies in $g + ij$ as equilibrium strategies when the prevailing network is g . Otherwise, it is a strict subset of a maximal independent set, which identifies an equilibrium under g in which a strict superset of players chooses the action 1.

This result can be interpreted as a natural complete-information counterpart of Proposition 4 for the class of Best-Shot games. Just as it was established for incomplete information, we find that the addition of new links tends to lower the prevalence of the high action at (pure-strategy Nash) equilibria under complete information.

5 Comparing Behavior Across Networks: Redistributing Links

In this section, we study how changes in the network structure in terms of redistributing links across players (in the sense of SOSD of the degree distribution) affects behavior. Given

the generality of the objective, we can only obtain full characterization results for binary-action games. For these games, we ascertain the equilibrium implications of *any* change of the degree distribution, both for the case of *strategic complements* as well as that of *strategic substitutes*. We also briefly discuss quadratic games.

As explained above, the key feature that simplifies the analysis of binary-action games is that their symmetric equilibria are threshold equilibria – that is, the choice of action solely depends on where the player’s type lies relative to a given threshold. Our analysis is summarized by the following two results, for each of the payoff scenarios under consideration: strategic complements and strategic substitutes.

In the following propositions P and P' are two different degree distributions. Denote by \tilde{F} and \tilde{F}' the induced cumulative distribution functions of the conditional degree distributions, \tilde{P} and \tilde{P}' , respectively. Let t and t' stand for the threshold types defining the (unique) threshold equilibria under P and P' , respectively, with m and m' denoting the probabilities with which, respectively, types t and t' choose action 0.

PROPOSITION 8 *Consider an incomplete information binary-action game with strict strategic complements. If $\tilde{F}'(t) \leq \tilde{F}(t) - \tilde{P}(t)(1-m)$ then there is an equilibrium with corresponding threshold type $t' \leq t$ whereas, conversely, if $\tilde{F}'(t) \geq \tilde{F}(t) - \tilde{P}(t)(1-m)$ there is an equilibrium with corresponding threshold type $t' \geq t$. Moreover, if the equilibrium threshold rises (falls), the probability that any given neighbor chooses 0 rises (falls).*

The key issue here is the change in the probability mass relative to the threshold. If the probability of types (degrees) equal or below the threshold goes down then the probability of action 1 increases and from strategic complements, the best response of threshold type t and all higher types must be 1. In other words, the threshold should fall weakly. Analogous considerations allow us to state the following result which pertains to games with strategic substitutes.

PROPOSITION 9 *Consider an incomplete information binary-action game with strict strategic substitutes. If $\tilde{F}'(t) \geq \tilde{F}(t-1) + \tilde{P}(t)(1-m)$ then $t' \leq t$ whereas, conversely, $\tilde{F}'(t) \leq$*

$\tilde{F}(t-1) + \tilde{P}(t)(1-m)$ implies $t' \geq t$. Moreover, if the equilibrium threshold rises (falls), the probability that any given neighbor chooses 1 falls (rises).

The novel contribution of these results is that they allow us to examine the effect of *any change of the degree distribution*. A natural and important example of such changes involves increasing the polarization of the degree distribution by shifting weights to the ends of the support of the degree distribution, as is done under a mean preserving spread of the degree distribution. In particular, the above results can be directly applied in the case of strong MPS shifts in degree distributions. We say that P is a *strong MPS* of P' if they have the same mean and there exists L and H such that $P(k) \geq P'(k)$ if $k < L$ or $k > H$, and $P(k) \leq P'(k)$ otherwise. Propositions 8 and 9 imply that, in the context of binary-action games, the equilibrium effects of any such change can be inferred from the relative values of the threshold t , and L and H .

Next, we briefly discuss our alternative leading context, namely, continuous action games with quadratic linear payoffs. In such games, as explained, the expected marginal returns are proportional to the degree of the player and the *expected* effort of each neighbor. Furthermore, the equilibrium actions (see (5)) are linear in degree. In this context, any redistribution of existing links (that keeps their number fixed) does not affect the average action if players keep playing according to the original equilibrium strategy that is linear in degree. This in turn entails that the original equilibrium strategy continues to define an equilibrium after the change. The equilibrium, in other words, is solely affected by the expected degree and no change in the degree distribution that affects only higher-order moments will have an effect on expected neighbor behavior. Of course, this includes mean-preserving spreads.

Finally, we turn to welfare concerns. In general, no clear-cut results can be found for the effect on average payoffs of general changes in the degree distribution, even for the games within our restricted classes (binary-action games or those with quadratic linear payoffs). The reason is that, after such changes, not only choice probabilities are affected (and thus the probabilities that players face concerning their neighbors' behavior) but this generally

impinges on players differentially depending on their types. Thus, in the end, the average effect must depend crucially on the precise weight given to each possible type by the original and revised degree distributions. The results, therefore, are bound to be ambiguous unless the type of changes in the degree distribution are restricted to some particular kind (e.g. to FOSD changes, as in Section 4).

6 Intermediate Information Structures

Our analysis has focused on two polar information cases: the “incomplete information” situation where players only know their own degree, and nothing about their neighbors and best respond to the anticipated actions of their neighbors based on the degree distribution, and the “complete information” situation where players best respond to the actual strategies of their neighbors. Other information conditions are also interesting for many applications. As a natural first step along these lines, we examine situations where a player knows not only how many neighbors she has, but also how many neighbors each of her neighbors has.

Our analysis of network games has been organized around two themes: a comparison of choices within a network (Section 3) and a comparison across networks (Sections 4 and 5). A cornerstone of this analysis was the fact that equilibrium strategies are monotone in player type/degree. As we already saw, while all (symmetric) equilibria are monotone under incomplete information, there is a multiplicity of equilibria with complete information, including some non-monotone equilibria. We now show that this multiplicity of equilibria, with some failing to be monotone, can arise as soon as players know the degree of their neighbors. Thus, adding a small amount of information to the incomplete information setting is enough to introduce a multiplicity and bring in some non-monotone equilibria.

Here we let d denote the information radius of a player. $d = 1$ indicates that a player knows how many neighbors she has. $d = 2$ indicates that a player knows how many neighbors she has, and how many neighbors each of them has. We use $d = \infty$ to indicate complete information.

So, consider a case where players know their own degree and the degrees of each of their neighbors ($d = 2$). The type space \mathcal{T}_i of a player i consists of elements of the form $(k; \ell_1, \ell_2, \dots, \ell_k)$ where $k \in \{0, 1, 2, \dots, n-1\}$ is the degree of the player and ℓ_j is the degree of neighbor j ($j = 1, 2, \dots, k$), where (in an anonymous setup where the identity of neighbors is ignored) we may assume without loss of generality that neighbors are indexed according to decreasing degree (i.e., $\ell_j \geq \ell_{j+1}$). Given the multidimensionality of types in this case, the question arises as to how one should define monotonicity in this case. In particular, the issue is what should be the order relationship \succeq on the type space underlying the requirement of monotonicity. For the case of strategic complements, it is natural to declare that two different types, $t = (k; \ell_1, \ell_2, \dots, \ell_k)$ and $t' = (k'; \ell'_1, \ell'_2, \dots, \ell'_k)$, satisfy $t \succeq t'$ iff $k \geq k'$ and $\ell_u \geq \ell'_u$ for all $u = 1, 2, \dots, k'$. On the other hand, for the case of strategic substitutes, we write $t \succeq t'$ iff $k \geq k'$ and $\ell_u \leq \ell'_u$ for all $u = k - k' + 1, \dots, k$. Given any such (partial) order on \mathcal{T}_i , we say that a strategy σ is monotone increasing if for all $t_i, t'_i \in \mathcal{T}_i$, $t_i \geq t'_i \Rightarrow \sigma(t_i)$ FOSD $\sigma(t'_i)$. The notion of decreasing monotonicity is defined analogously.

EXAMPLE 5 *Non-monotone Equilibria with Knowledge of Neighbors' Degrees ($d = 2$)*

Consider a setting where nodes have either degree 1 or degree 2, as given by the corresponding probabilities $P(1)$ and $P(2)$. Suppose that the game is binary-action with $X = \{0, 1\}$ and displays strategic complements. Specifically, suppose that the payoff of a player only depends on her own action x_i and the sum \bar{x} of her neighbors' actions as given by a function $v(x_i, \bar{x})$ as follows: $v(0, 0) = 0$, $v(0, 1) = 1/2$, $v(0, 2) = 3/4$, $v(1, 0) = -1$, $v(1, 1) = 1$, $v(1, 2) = 3$.

It is readily seen that, for any P with support on degrees 1 and 2, the following strategy σ defines a symmetric equilibrium: $\sigma(1; 1) = 1$; $\sigma(1; 2) = 0$; $\sigma(2; \ell_1, \ell_2) = 0$ for any $\ell_1, \ell_2 \in \{1, 2\}$. Here, two players that are only linked to each other both play 1, while all other players choose 0.

Similar non-monotonic equilibrium examples can be constructed for games where actions are strategic substitutes.

The previous example illustrates that whenever players enjoy some local network information, neighboring players can rely upon such enhanced information (which is partly shared) to extend the possibilities on which they may coordinate at equilibrium. This naturally suggests the question of whether, nevertheless, the existence of *some* monotone equilibrium strategy can still be guaranteed in general, both under strategic complements and substitutes. This is confirmed by the following result.

PROPOSITION 10 *In the setting where players know their own degree and the degrees of their neighbors, there exists a symmetric equilibrium which is monotone increasing (decreasing) if the game displays strict strategic complements (substitutes).*

The proof of the proposition, which appears in the appendix, follows standard approaches for establishing the existence of monotone equilibria. That is, we show that if all players use monotone strategies, then there exists a monotone best reply. Then working within monotone strategy spaces there exists an equilibrium, and it remains an equilibrium in the broader strategy space.

A direct implication of Proposition 10 is that there is always an equilibrium that, on average across the types $(k; \ell_1, \ell_2, \dots, \ell_k)$ consistent with each degree k , prescribes an (average) action that is monotone in degree. Equipped with the above monotonicity result, it is also possible to recover some of the insights gathered before under the assumption of incomplete information. For example, if we consider changes in network connectivity and the game displays strategic complements, it is not difficult to show that whenever two degree distributions satisfy that P FOSD P' , there are respective *monotone* equilibria σ and σ' such that $\sigma(t_i)$ FOSD $\sigma'(t_i)$ for every t_i .

Example 5 and Proposition 10 suggest that expanding the information radius may lead to a wider range of outcomes being supported in equilibrium. This motivates the question: are there games in which the equilibria actually shrink as the information radius expands? The answer is yes. Consider the following example:

EXAMPLE 6 *Complete Information ($d = \infty$) versus a Large Information Radius ($1 < d < \infty$)*

Suppose that players have either one or two neighbors and let:

$$\begin{aligned}\tilde{P}(1) &= \frac{P(1)}{2P(2) + P(1)} \equiv \phi \\ \tilde{P}(2) &= \frac{2P(2)}{2P(2) + P(1)} = 1 - \phi.\end{aligned}$$

Let $X = \{0, 1\}$, and suppose payoffs $v(x_i, \bar{x})$ only depend on a player's own action x_i and the sum \bar{x} of her neighbors actions, as given by $v(0, \cdot) = 0$, $v(1, 0) = v(1, 1) = -1$, $v(1, 2) = 1$.

First note that, independently of the degree distribution (i.e., of the value of ϕ), the game always has an equilibrium where every player, whatever her type, chooses action 0. Thus the issue here is under what conditions are there also other equilibria (where some types choose 1)? It is possible to show that for any finite information radius $d < \infty$ additional outcome profiles can be sustained, but that in the limit, at $d = \infty$, the equilibrium set actually shrinks and (roughly speaking) only the 0 action outcome can be sustained. The details of this construction are given in the appendix. Examples 5 and 6 illustrate that a greater radius of information can expand as well as shrink the set of outcomes which can be sustained in equilibrium, depending on the game that is being played.

We now turn to the important issue of how additional network information affects payoffs. The previous example provides us with an instance in which payoffs may decline as information expands (at least in the limit). Can the effects go the other way as well? To see how this may come about, note that in the best-shot game it is best for players to coordinate so that each neighborhood has a provider, but that no two neighbors provide the public good. With incomplete information such coordination is impossible. Thus, it is easy to construct examples where for a low value of d the only symmetric equilibrium involves too much provision, and this yields an inefficient outcome. On the other hand, the complete information equilibrium results in more efficient coordination (and with appropriate mixing,

one can get strict Pareto dominance). Indeed, this possibility can arise even with a slight increase in radius going from $d = 1$ to $d = 2$; we present an example in the appendix to illustrate this possibility. These observations suggest that additional network information can raise payoffs or lower payoffs depending on the game being played.

7 Concluding Remarks

We have examined a model of social interactions in which a player's payoff depends on her own action and the actions of her neighbors in an underlying network of connections. We have investigated how location within a fixed network as well as changes in overall network structure affect individual behavior. In particular, the paper makes two innovations: we allow for a rather general class of payoffs (which subsumes as special cases practically all the models studied so far in the literature) and we allow for incomplete information about network structures (in contrast to most existing work which assumes complete network information). Our results yield a number of insights about how network structure, location within a network, nature of the game (strategic substitutes versus complements and positive versus negative externalities), and the level of information (incomplete versus complete) shape individual behavior and payoffs.

Our results suggest some directions for further work. First, we have assumed that degrees of players in the network are independent. As we have pointed out in the discussion following Proposition 2, some of our results do carry over to particular cases of degree correlation. A general analysis of equilibria under degree correlation is clearly important as actual networks exhibit degree correlations. Second, we have assumed that payoffs satisfy Assumption A, which rules out payoff functions where the average actions of neighbors matters. One should not expect our results to extend to the "average" case, as then degree becomes largely irrelevant, but there are situations that fall between Assumption A and the average case. Extending the analysis to cover such cases appears to be an interesting avenue for further work. Third, we have examined problems where players care only about their

direct neighbors' actions. There are also contexts where the externalities in behavior extend more broadly (e.g., due to congestion), and players might care about broader play in the game.

8 Appendix: Proofs

Proof of Proposition 2: We present the proof for the case of strategic complements. The strategic substitutes case is analogous and omitted. Let $\{\sigma_k^*\}$ be a symmetric equilibrium of the network game. If $\{\sigma_k^*\}$ is a trivial profile with all degrees choosing action 0 with probability 1, the claim follows directly. Therefore, from now on, we shall assume that the equilibrium is non-trivial and that there is some k' and some $x' > 0$ such that $x' \in \mathbf{supp}(\sigma_{k'}^*)$.

Consider any $k \in \{0, 1, \dots, n\}$ and let $x_k = \sup[\mathbf{supp}(\sigma_k^*)]$. If $x_k = 0$, it trivially follows that $x_{k'} \geq x_k$ for all $x_{k'} \in \mathbf{supp}(\sigma_{k'}^*)$ with $k' > k$. So let us assume that $x_k > 0$. Then, for any $x < x_k$, the assumption of (strict) strategic complements implies that

$$v_{k+1}(x_k, x_{l_1}, \dots, x_{l_k}, x_s) - v_{k+1}(x, x_{l_1}, \dots, x_{l_k}, x_s) \geq v_k(x_k, x_{l_1}, \dots, x_{l_k}) - v_k(x, x_{l_1}, \dots, x_{l_k})$$

for any x_s , with the inequality being strict if $x_s > 0$. Averaging over all types, the fact that at least $x_k > 0$ implies that

$$U(x_k, \sigma_{-i}^*, k+1) - U(x, \sigma_{-i}^*, k+1) > U(x_k, \sigma_{-j}^*, k) - U(x, \sigma_{-j}^*, k).$$

On the other hand, note that from the choice of x_k ,

$$U(x_k, \sigma_{-j}^*, k) - U(x, \sigma_{-j}^*, k) \geq 0$$

for all x . Combining the aforementioned considerations we conclude:

$$U(x_k, \sigma_{-i}^*, k+1) - U(x, \sigma_{-i}^*, k+1) > 0$$

for all $x < x_k$. This in turn requires that if $x_{k+1} \in \mathbf{supp}(\sigma_{k+1}^*)$ then $x_{k+1} \geq x_k$, which of course implies that σ_{k+1}^* FOSD σ_k^* . Iterating the argument as needed, the desired conclusion follows, i.e., $\sigma_{k'}^*$ FOSD σ_k^* whenever $k' > k$. ■

Proof of Proposition 3: We present the proof for positive externalities; The proof for negative externalities is analogous and omitted. The claim is obviously true for a trivial equilibrium in which all players choose the action 0 with probability 1. Consider a non-trivial equilibrium σ^* . Suppose $x_k \in \text{supp}(\sigma_k^*)$ and $x_{k+1} \in \text{supp}(\sigma_{k+1}^*)$. By assumption,

$$v_{k+1}(x_k, x_{l_1}, \dots, x_{l_k}, 0) = v_k(x_k, x_{l_1}, \dots, x_{l_k})$$

for all x_{l_1}, \dots, x_{l_k} .

Since the payoff structure satisfies positive externalities, it follows that for any $x > 0$,

$$v_{k+1}(x_k, x_{l_1}, \dots, x_{l_k}, x) \geq v_k(x_k, x_{l_1}, \dots, x_{l_k}).$$

Looking at expected utilities, it follows that

$$U(x_k, \sigma_{-i}^*, k+1) \geq U(x_k, \sigma_{-j}^*, k).$$

Since σ_{k+1}^* is a best response in the network game being played,

$$U(x_{k+1}, \sigma_{-i}^*, k+1) \geq U(x_k, \sigma_{-j}^*, k)$$

and the result follows. ■

Proof of Proposition 4: Let σ be an equilibrium of the network game with underlying network characterized by P' . We first show that there exists an equilibrium in the game with degree distribution P' which dominates $\{\sigma_i(t)\}$ and is monotone. Indeed, start with the (symmetric) profile of actions prescribing each player to use her 1 action with probability 1. Now consider the best response profile for all players, placing a probability 1 on the highest possible action for each player who is indifferent. Clearly, we are left with a profile that dominates $\{\sigma_i(t)\}$. Furthermore, from strategic complementarities and Assumption A, the profile is monotone. Continuing iteratively in this manner, we converge to a symmetric pure equilibrium profile characterized by $\{x_k\}$ (each player i uses the strategy $\tilde{\sigma}_i(t)$, where $\tilde{\sigma}_i(k) = x_k$ for all i) which dominates $\{\sigma_i(t)\}$ and is monotone.

Since $\{x_k\}$ is a monotonic sequence, strategic complementarities then guarantee that for any $x \geq x_k$:

$$\sum_{l_1, \dots, l_{k-1}} \prod_{j=1}^{k-1} \tilde{P}(l_j) [v_k(x, x_{l_1}, \dots, x_{l_k})] \geq \sum_{l_1, \dots, l_{k-1}} \prod_{j=1}^{k-1} \tilde{P}'(l_j) [v_k(x, x_{l_1}, \dots, x_{l_k})].$$

In particular, if players are playing the symmetric profile $\{x_k\}$ in the network game with underlying degree distribution P , there is a best response of each degree k player which is at least as high as x_k . Consider the profile of best responses (and, as before, upon indifference, choose the highest best response to be played with probability 1). The new profile dominates $\{x_k\}$ and is monotone. Proceeding iteratively in that way, we converge to a symmetric equilibrium profile in the network game with degree distribution P that dominates the original equilibrium $\{\sigma_i(t)\}$. ■

Example of effects of adding links in games with strategic complements: Let $X = \{0, 1\}$ and suppose that the payoffs $v(x, y)$ of a typical player only depend on her own action x and the sum of her neighbors' actions y . More specifically, suppose that $v(1, y) = 1$ if $y > 1$, $v(1, y) = -\varepsilon$ for some small $\varepsilon > 0$ if $y \leq 1$, and $v(0, \cdot) = 0$. This game exhibits strategic complements and positive externalities. Now consider two different degree distributions P and P' given by $P(1) = 3/5$ and $P(5) = 2/5$ whereas $P'(1) = P'(3) = 1/2$. Clearly, P does *not* FOSD P' . However, $\tilde{P}(1) = 3/13$ and $\tilde{P}(5) = 10/13$ while $\tilde{P}'(1) = 1/4$ and $\tilde{P}'(3) = 3/4$. Thus, \tilde{P} FOSD \tilde{P}' and Proposition 4 applies.

In particular, consider the strategy σ' given by $\sigma'(k) = 0$ for $k \leq 1$ and $\sigma'(k) = 1$ for $k \geq 2$. This strategy is an equilibrium under P' for ε small enough. On the other hand, it is clear that the only strategy σ that dominates σ' and is also an equilibrium under P is $\sigma = \sigma'$. For P' the average welfare can be approximated (for small ε) as the probability that a randomly chosen node has degree 3 and at least two of its neighbors have this degree as well. This is $W' = \frac{1}{2} [1 - [(\frac{1}{4})^3 + 3(\frac{1}{4})^2 \frac{3}{4}]] \simeq 0.42$. Analogously, the average welfare W for P can be bounded above by the probability $P(5) = 2/5$ that a randomly chosen node has degree higher than one. Thus, we have that $W' > W$, which shows that a FOSD shift of the

conditional degree distribution may indeed lead to a welfare loss even if the game displays strategic complements and positive externalities. ■

Proof of Proposition 5: Suppose that the equilibrium has threshold t' under P' , where m' is the probability of a player of degree t' choosing action 0. Since we assume that \tilde{P} FOSD \tilde{P}' , from the monotone decreasing action property of the equilibrium strategy it follows that the equilibrium threshold under P cannot be lower than t' . To see this, note that for any player of degree k , the probability that $l \in \{0, 1, \dots, k\}$ of her neighbors choose action 1 is given by:

$$\binom{k}{l} \left(1 - \tilde{P}'(t')x - \sum_{k=t'+1}^{n-1} \tilde{P}'(k) \right)^l \left(\tilde{P}'(t')x + \sum_{k=t'+1}^{n-1} \tilde{P}'(k) \right)^{k-l}.$$

Since \tilde{P} FOSD \tilde{P}' ,

$$\tilde{P}(t')m' + \sum_{k=t'+1}^{n-1} \tilde{P}(k) \geq \tilde{P}'(t')m' + \sum_{k=t'+1}^{n-1} \tilde{P}'(k),$$

and so the (binomial) distribution of the number of neighbors choosing the 1 action under P' FOSD that under P . Thus, from strict strategic substitutes, the threshold t under P must be weakly higher, i.e. $t \geq t'$.

This implies that the probability of choosing action 1 increases for all types whenever $t \neq t'$. If instead $t = t'$, that probability remains equal for all types except, possibly, for t . So assume, for the sake of contradiction, that the probability for this type under P is $1 - m$ and $m > m'$. Then,

$$\tilde{P}(t)m + \sum_{k=t+1}^{n-1} \tilde{P}(k) > \tilde{P}'(t)m' + \sum_{k=t+1}^{n-1} \tilde{P}'(k),$$

and it would be a strict best response for t to choose 1, in contradiction.

Finally, we argue that the probability that any randomly selected neighbor chooses 1 must fall. For suppose it were to rise instead. Then, based on the former considerations, the probability of choosing action 1 should fall, which contradicts the first part of the proposition. ■

Proof of Proposition 6: Consider the first statement. Fix an equilibrium σ under network g . If σ is an equilibrium under g' then we are done. Otherwise, consider best responses to σ ; by strategic complements, there exists best responses which are weakly higher than the original profile. Iterations of best responses lead to weakly higher profiles at each iteration stage and, from compactness of X , converge. The limit strategy profile (say) σ' is an equilibrium under g' . This completes the proof of the first statement. On the other hand, for the second statement, note that if X is connected, there are strict complementarities, and the starting equilibrium is interior, then each iteration leads to a strictly higher profile of actions. Thus, the aforementioned iterations reach a strictly higher equilibrium profile under g' . ■

Proof of Proposition 7: Let us show the second statement first. Start with a set containing both i and j . This forms a (possibly non-maximal) independent set of g . Add a node that keeps it an independent set. Iterate until a maximal independent set is reached. This is a set S' which is a maximal independent set of g , but not of $g + ij$. Now, let us show that any maximal independent set of $g + ij$ which is not a maximal independent set of g is a subset of a maximal independent set of g . Consider any maximal independent set S of $g + ij$. It can have at most one of i and j in it. If it has neither in it, then it is also clearly a maximal independent set of g . Suppose that i is in S but that j is not in S . If some neighbor of j , other than i , is in S , then S is a maximal independent set of g . So consider the case where the only neighbor of j that is in S is i . Consider $S' \equiv S \cup \{j\}$. Then S' is a maximal independent set of g .

Proof of Proposition 8: Consider the first part and note that the condition $\tilde{F}'(t) \leq \tilde{F}(t) - \tilde{P}(t)(1 - m)$ is equivalent to

$$\sum_{\tau=t+1}^{n-1} \tilde{P}'(\tau) \geq \sum_{\tau=t+1}^{n-1} \tilde{P}(\tau) + \tilde{P}(t)(1 - m).$$

Then simply conduct an iterative process (along the lines of the proof of Proposition 4) that starts with an initial threshold strategy where a type τ chooses 1 iff $\tau \geq t$ and subsequently

applies repeatedly the *highest* best response mapping. The equilibrium attained in the limit has a threshold $t' \leq t$.

For the second case, note that $\tilde{F}'(t) \leq \tilde{F}(t) - \tilde{P}(t)(1 - m)$ is equivalent to

$$\sum_{\tau=t+1}^{n-1} \tilde{P}'(t) \leq \sum_{\tau=t+1}^{n-1} \tilde{P}(t) + \tilde{P}(t)(1 - m).$$

Then, start with an initial threshold strategy where a type τ chooses 1 iff $\tau \geq t+1$ and apply iteratively the *lowest* best response. The equilibrium attained in the limit has a threshold $t' \geq t$.

Finally, we want to show that $(t' - t)(\tilde{F}'(t')m' - \tilde{F}(t)m) \leq 0$. But this simply follows from the fact that if, say, $t' > t$ then the assumption of strategic complements rules out the possibility that $\tilde{F}'(t')m' < \tilde{F}(t)m$. ■

Proof of Proposition 9: With the notational conventions used in the statement of the proposition, suppose that $\tilde{F}'(t) \geq \tilde{F}(t-1) + \tilde{P}(t)(1 - m)$ but, contrary to what is claimed, $t' > t$. Then, when the prevailing degree distribution is P' , the probability $\tilde{F}'(t')$ perceived by a player that, at equilibrium, any one of her neighbors chooses action 1 is bounded below by $\tilde{F}'(t)$ and, therefore, by $\tilde{F}(t-1) + \tilde{P}(t)(1 - m)$. From the assumption of strategic substitutes, this implies that $t' \leq t$, a contradiction that establishes the first part of the result.

Now consider the converse case where $\tilde{F}'(t) \leq \tilde{F}(t-1) + \tilde{P}(t)(1 - m)$ and assume that $t > t'$. Then, the equilibrium probability $\tilde{F}'(t')$ that, under degree distribution P' , any one randomly selected neighbor chooses action 1 is bounded above by $\tilde{F}'(t)$ and, therefore, by $\tilde{F}(t-1) + \tilde{P}(t)(1 - m)$. Again from the assumption of strategic substitutes, this yields $t' \geq t$, a contradiction.

Finally, we want to show that $(t' - t)(\tilde{F}'(t')(1 - m') - \tilde{F}(t)(1 - m)) \leq 0$. But this simply follows from the fact that if, say, $t' > t$ then the assumption of strategic substitutes rules out the possibility that $\tilde{F}'(t')(1 - m') > \tilde{F}(t)(1 - m)$. ■

Proof of Proposition 10: Let us consider first the case of strategic complements and denote by \sum^m the set of monotone strategies. The proof is based on the following two

claims:

Claim 1: For any player i , if all other players $j \neq i$ use a strategy $\sigma_j \in \sum^m$ there is always a strategy $\sigma_i \in \sum^m$ that is a best response to it.

Claim 2: An equilibrium exists in the strategic-form game where all players' strategies are taken from \sum^m .

To establish Claim 1, we can rely on a direct adaptation of the approach used to prove Proposition 2. Consider a player i and let $t_i, t'_i \in \mathcal{T}_i$ such that $t'_i \succeq t_i$, where \succeq is the partial order applicable to the case of strategic complements (see Section 6). For any σ_{-i} where $\sigma_j \in \sum^m$ for every $j \neq i$, let $BR(\sigma_{-i}, t_i)$ be the set of best-response strategies by player i to σ_{-i} when her type is t_i . Let us assume that, for all $j \in N$, $\sigma(t_j) \neq 0$ for some $t_j \in \mathcal{T}_j$. (Otherwise, the desired conclusion follows even more directly, since the best-response correspondence is unaffected by being connected to a player whose strategy chooses action 0 uniformly.) By definition, for every $x_{t_i} \in BR(\sigma_{-i}, t_i)$, we must have that

$$\forall x \in X, \quad U(x_{t_i}, \sigma_{-i}, t_i) - U(x, \sigma_{-i}, t_i) \geq 0$$

Then, since $t'_i \succeq t_i$, the assumption of (strict) strategic complements readily implies that

$$\forall x \leq x_{t_i}, \quad U(x_{t_i}, \sigma_{-i}, t'_i) - U(x, \sigma_{-i}, t'_i) > 0. \quad (6)$$

Indeed, this follows from a two-fold observation:

(i) From Assumption A, if $t_i = (k, \ell_1, \ell_2, \dots, \ell_k)$ and $t'_i = (k', \ell'_1, \ell'_2, \dots, \ell'_k)$ and $t'_i \succeq t_i$ we can think of t_i involving k' neighbors with all neighbors indexed from $k + 1$ to k' (if any) choosing the action 0;

(ii) Since $\ell'_u \geq \ell_u$ the probability distribution over actions corresponding to each of her neighbors under t_i , $u = 1, 2, \dots, k$, is dominated in the FOSD sense by the corresponding neighbor under t'_i .

Let us now make use of (6) in the case where x_{t_i} is the highest best response by type t_i to σ_{-i} . Then, it follows that any $x_{t'_i} \in BR(\sigma_{-i}, t'_i)$ must satisfy:

$$x_{t'_i} > \sup\{x_{t_i} : x_{t_i} \in BR(\sigma_{-i}, t_i)\},$$

which establishes Claim 1.

To prove Claim 2, we can simply invoke the concavity postulated for each payoff function $v_k(\cdot, x)$ for any given $x \in X^k$ and the fact that the set of monotone strategies is compact and convex. To see the latter point, note that the monotonicity of a strategy σ is characterized by the condition:

$$\forall t_i, t'_i \in \mathcal{T}_i, \quad t'_i \succeq t_i \Rightarrow \sigma(t'_i) \text{FOSD} \sigma(t_i). \quad (7)$$

Clearly, if two different strategies σ and σ' satisfy (7), then any convex combination $\hat{\sigma} = \lambda\sigma + (1 - \lambda)\sigma'$ also satisfies it. Now, starting from any profile of monotonic strategies, iterations of best responses will remain within Σ^m and converge to an equilibrium in Σ^m as well.

Finally, to prove the result for the case where the game displays strategic substitutes, note that the above line of arguments can be applied unchanged, with the suitable adaptation of the partial order to be used to define the notion of monotonicity. In this second case, as explained in Section 6, we say that $t \succeq t'$ if and only if $k \geq k'$ and $\ell_u \leq \ell'_u$ for all $u = k - k' + 1, \dots, k$. ■

Example 6 continued: Complete Information ($d = \infty$) versus a Large Information Radius ($1 < d < \infty$): Consider a situation where all players have either one or two neighbors and the respective conditional probabilities for the degree of a neighboring node given by

$$\begin{aligned} \tilde{P}(1) &= \frac{P(1)}{2P(2) + P(1)} \equiv \zeta \\ \tilde{P}(2) &= \frac{2P(2)}{2P(2) + P(1)} = 1 - \phi. \end{aligned}$$

The game is of binary-actions and payoffs are given by a function $v(x_i, \bar{x})$ that only depends on a player's own action x_i and the sum \bar{x} of her neighbors actions, as given by $v(0, \cdot) = 0$, $v(1, 0) = v(1, 1) = -1$, $v(1, 2) = 1$.

Clearly, independently of the degree distribution (i.e., of the value of ϕ), the game always has an equilibrium where every player, whatever her type, chooses action 0. Thus the issue here is under what conditions there are also other equilibria (where some types choose 1). In this respect, let us consider the following cases:

(i) Incomplete Information ($d = 1$)

Here, if $\phi \leq 1 - 1/\sqrt{2}$, the strategy σ with $\sigma(1) = 0$ and $\sigma(2) = 1$ defines a (symmetric) equilibrium.¹⁸

(ii) Any Finite Information Radius Including Neighbor's Degrees ($1 < d < \infty$)

We can think of the connected component containing i as a horizontal line. In particular, the type space \mathcal{T}_i of a player i can be identified with a suitable subset of $\{1, 2\} \times \{0, 1, 2\}^{2(d-1)}$ such that a typical element of it is of the form

$$t_i = (r_0; r_{1+}, r_{1-}, \dots, r_{(d-1)+}, r_{(d-1)-}) \in T_i$$

where r_0 is interpreted as the degree of node i and r_{u+} and r_{u-} are the degrees of those players that are at distance u to the right and left of i , respectively, with the convention that $d_{u+} = 0$ (or $d_{u-} = 0$) means that there are no players to the right (or left) of i at distance u .¹⁹

In this context, we now argue that for all d , and provided $\phi \leq 1 - 1/\sqrt{2}$, the following strategy σ defines a symmetric equilibrium:

$$\sigma(t_i) = \begin{cases} 1 & \text{if } t_i = (2; 2, \dots, 2) \\ 0 & \text{otherwise.} \end{cases}$$

To see this, simply note that for a player i with two neighbors, the only relevant consideration is whether or not one of the following events occur: (a) player i observes an

¹⁸Simply note that a player with two neighbors has an expected payoff of playing action 1 given by $(1 - \phi)^2 \times 1 + (1 - (1 - \phi)^2) \times (-1)$, which is nonnegative iff $\phi \leq 1 - 1/\sqrt{2}$.

¹⁹Naturally, the following condition must hold: $[\exists u = 0, 1, \dots, r - 1 \text{ s.t. } d_{u+} = 1] \Rightarrow [\forall u' > u, d_{u'+} = 0]$ (and similarly for the left neighbors).

“end-player” with only one neighbor, or (b) any of her two neighbors, say j and k , observe an end-player (who would be at the outer end of their respective information range but whom player i does not observe). If (a) applies, a straightforward inductive argument implies that i must choose $a_i = 0$. If (b) applies to either of i 's neighbor, say j , then j will choose $a_j = 0$ and thus i should choose $a_i = 0$ as well. Now notice that the probability that neither j nor k (neighbors of i) observe an end-player not observed by i is $(1 - \phi)^2$. This indicates that the relevant threshold for player i to choose optimally $a_i = 1$ in case (b) is just as for $d = 1$. Combining all the aforementioned considerations, the above strategy is seen to be an equilibrium strategy if $\phi \leq 1 - 1/\sqrt{2}$.

The above considerations apply to *all* finite d . Thus, it follows that, for any such d , there is a lower bound n_0 on the population size such that if $n \geq n_0$ there is an equilibrium where, with probability bounded above zero, a positive fraction of players will choose 1. In fact, such a fraction (as well as the corresponding probability) will be close to 1 if n is large and ϕ is sufficiently low.

(iii) Complete Information ($d = \infty$)

The situation is qualitatively different if $d = \infty$. In this case, independently of the value of n and ϕ , all equilibrium outcomes involve everyone choosing action 0, except players who happen to be in components that only involve players of degree 2. Those components have equilibria where all players play 1 or all players play 0. However, any component with any player of degree 1 has a unique equilibrium where all players play 0.

This example illustrates that having more information may actually narrow the set of equilibria.²⁰ There is one equilibrium (all playing 0) which exists all along the sequence and is the only ex-post equilibrium. However, for less than complete information there also exist other equilibria that are more efficient. Here, having less information allows players to coordinate on an equilibrium which leads to higher expected utility for all players (weakly

²⁰We remark that although the multiplicity disappears at $d = \infty$, there still is a sort of continuity here in that the set of parameter values (ϕ 's) for which the non-zero equilibrium exists shrinks as d grows.

for the degree 1 players²¹). ■

Example with enhanced coordination for $d = 2$: The degree distribution has only players with degree 1 and 3 and conditional probabilities for neighboring nodes given by $\tilde{P}(1) \equiv \phi$, $\tilde{P}(3) = 1 - \phi$. Again, let the action space be $X = \{0, 1\}$ and payoffs only depend on own action x_i and the sum \bar{x} of neighbors' actions as given by the function $v(x_i, \bar{x})$ defined as follows: $v(0, \cdot) = 0$, $v(1, 0) = v(1, 1) = -c$, $v(1, 2) = v(1, 3) = 1$, where $c > 0$ is a suitably chosen parameter (see below).

Consider first the case of $d = 1$. Let

$$q_0 \equiv 1 - [\phi^3 + \binom{3}{2}\phi^2(1 - \phi)]$$

be the probability with which a player with three neighbors expects to have at least two neighbors of degree 3. Then, if

$$q_0 \times 1 - (1 - q_0) \times c < 0$$

or

$$q_0 < \frac{c}{1 + c},$$

it is clear that all players must choose action 0 at any equilibrium.

Next, consider a slight increase in information to $d = 2$. Consider a particular player i , who has at least two neighbors with degree 3, say j and k . Since $d = 2$, player i knows this fact. But also because $d = 2$, j and k know that one of their neighbors (namely i) has degree 3. The key question now is what is the probability perceived by i that *both* of these two neighbors have an *additional* neighbor (different from i) with degree three. This probability is

$$q_1 \equiv (1 - \phi^2)^2.$$

²¹The example is easily modified so that even the degree 1 players have higher utility if their neighbor chooses 1 instead of 0.

Comparing q_0 and q_1 we find

$$\begin{aligned} q_1 - q_0 &= [1 - 2\phi^2 + \phi^4] - [1 - (\phi^3 + 3\phi^2 - 3\phi^3)] \\ &= \phi^2 - 2\phi^3 + \phi^4 = \phi^2(1 - \phi)^2 > 0. \end{aligned}$$

Therefore, there is a value of c such that

$$q_0 < \frac{c}{1+c} \leq q_1. \quad (8)$$

In a case where (8) holds, as explained above under $d = 1$ any symmetric-equilibrium strategy σ must have all players choose 0; however, if $d = 2$, then the following strategy σ' also defines a symmetric equilibrium:

$$\begin{aligned} \sigma'(1; \cdot) &= 0 \\ \sigma'(3; 3, 3, \cdot) &= 1. \end{aligned}$$

Finally, we point that, at the above equilibrium, there is a positive fraction of players who will choose action 1 with positive probability. In fact, it can be readily shown that, if $p(3) > 1/4$, the set of those players includes a giant component (i.e. a component of the network that, even for arbitrarily large n , includes a fraction of players that is bounded away from zero).²² ■

²²This follows from the Molloy-Reed condition

$$\sum_{\kappa} P(\kappa)\kappa^2 - 2 \sum_{\kappa} P(\kappa)\kappa > 0$$

for the existence of a giant component in random networks. For our example, this condition is equivalent to $p(3) > 1/4$.

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