Incentives, Risk Sharing and Wealth: A Model of Intrinsic Cycles

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Abstract

The paper shows that the interactions between incentives, risk sharing, and wealth may drive economic cycles in the steady state, where the economy oscillates between two equilibrium modes. When the economy is poor, it is in the incentivized mode, where the young agents take risks, work hard, and are more productive. Consequently, the economy gets richer, making it harder to incentivize the young generation. Eventually the economy falls into the disincentivized mode, where young agents obtain full insurance, shirk, and are less productive. As a result, the economy becomes poorer and eventually falls back into the incentivized mode. Such oscillations arise only when the productivity of the economy falls in a medium range.

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Key words: Steady State Cycles; Incentives; Risk Sharing; Wealth; Contractual Mode Switch

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1 Introduction

Could intrinsic frictions of an economy, such as ubiquitous moral hazard problems, drive economic fluctuations? A strand of literature, starting with Suarez and Suzzman (1997), shows they could, via mechanisms where the moral hazard problems exert influences by affecting the economic agents' capability to raise external finance. This paper shares with the literature in emphasizing the importance of the intrinsic frictions to economic fluctuations, but uncovers a complementary mechanism, at the core of which is the trade-off between incentives and insurance, a fundamental trade-off underlined by the economic theory in relation to moral hazard. By focusing on this trade-off, the paper neatly links together the cyclic movements of investment (both quantity and return), wealth, incentives and risk sharing.

The paper considers an overlapping generations economy where young agents provide labor and older agents provide capital to produce the consumption good. After earning the wage from this production, young agents engage in projects to produce capital. The projects are risky and subject to a typical moral hazard problem: the project of an agent succeeds with a higher (lower) probability if he works hard (shirks), and the choice of effort is his private information. The risks of the projects are idiosyncratic. This naturally gives rise to arrangements in which agents pool together their projects and obtain mutual insurance. Driven by the typical trade-off between insurance and incentive, two types of insurance contracts could prevail in equilibrium: (a) incentivizing contracts with which the participating agents work hard but bear part of the idiosyncratic risks of their projects; and (b) disincentivizing contracts with which they get full insurance and hence shirk.

The main result of the paper is that in the steady state the economy fluctuates between two distinct modes of equilibrium: an incentivized mode where the incentivizing contracts prevail, and a disincentivized mode where the disincentivizing contracts prevail. During periods when the capital stock is low and the economy is thus poor, the incentivized mode reigns: the young agents work hard and, as a result, the economy keeps becoming richer in terms of higher capital stocks. However, when the capital stock surpasses a threshold, the incentivized mode is not sustainable. Then the disincentivized mode prevails: the young agents now shirk and, as a result the economy keeps becoming poorer. But when the capital stock falls below a threshold, the disincentivized mode is not sustainable and the incentivized mode reigns again, reversing the economy back onto a upward path.

The switches in the equilibrium mode are driven by the interaction between optimal contracting determined by at the micro level and the wealth of young agents determined at the macro level. When the economy is rich, the high level of the capital stock gives rise to a high wage rate in the labour market, thus creating larger wealth for the young agents. They now, in order to smooth consumption inter-temporally, seek to transfer more wealth to the future across all contingencies, which comes into direct conflict with the provision of incentives. As a result, the optimal insurance contracts are the disincentivizing ones, which allow for full insurance and perfect inter-temporal consumption smoothing. On the other hand, when the capital stock is low, young agents earn a low wage rate and hence they have little to pass on to the future even under full insurance. Consequently, they prefer the incentivizing contracts, with which they work hard and produce more capital for the next period.

The paper predicts that the scale of investment is larger during booms than it is during busts, but the productivity (or the rate of return) of investment is lower during booms than during busts. The latter result is because of incentives: in the paper agents work less hard during booms than they do during bust; it is thus in line with a finding by Favara (forthcoming) that recession features productivity-improving activities "due to the strengthened incentives".

The paper contributes to the literature where cycles may arise *in the absence of exogenous shocks* and because of information frictions;[1] see Aghion and Banerjee (2005), Aghion, Banerjee and Piketty (1999), Favera (forthcoming), Matsuyama (2007), Myerson (2010), and Suarez and Suzzman (1997), among others, and for a survey of this literature, see Brunnermeier, Eisenbach, and Sannikov (2012) and Tirole (2005).[2] The closest paper to ours is Suarez and Suzzman (1997): in both papers, how stressful the moral hazard problems are, reflected in the level of effort chosen, varies with the economic cycles; and in both papers, globally stable cycles arise in the steady state. But differently from his paper and other ones in the literature, which all underline the

[1]There is another strand of literature that examines how information frictions amplify fluctuations that *originate* from exogeneous shocks; see Bernanke and Gertler (1989), Bernanke, Gertler and Gilchrist, (1999), Brunnermeier and Sannikov (2011), Cordoba and Ripoll (2004), Kiyotaki and Moore (1997), and Krishnamurthy (2003), and for a survey see Brunnermeier, Eisenbach, and Sannikov (2012), and Chapters 13 and 14 of Tirole (2005).

[2]The early literature, seeking to endogenize cycles, resorts to the curvature of the utility or production functions, imperfect competition, or dynamic complementarities; see Boldrin (1992), Kyotaki (1988), Grandmont (1985) and Reichlin (1986), among others; and for a survey see Guesnerie and Woodford (1992).

constraints on the capacity of raising external finance, the mechanism for cycles in our paper is driven by the interaction between wealth, incentives and risk sharing. In particular, our paper has three novel features. First, the incentivizing and disincentivizing contractual regime alternate between booms and busts. Second, saving plays a key role in the mechanism driving cycles in our paper, with higher savings reducing the incentives of agents to work hard and toppling the economy into the disincentivized mode, while in the literature, the information frictions affect the capability to raise external finance required for investment, which might have implications for saving, but saving by itself does not play any role in the mechanisms for fluctuations. Third, we show that economies within a middle range of productivity are prone to the cyclic steady state.

This last result also sheds new light on the peculiarities of middle-income countries, as discussed by, for example, Abramovitz (1986), Baumol (1986), Dowrick and Gemmel (1991), and Ogaki, Ostry and Reinhart (1996). Moreover, we find that the saving rate is higher in the disincentivized mode when the wage is higher, which is consistent with the claim by Ogaki, Ostry and Reinhart (1996), that "[T]he hypothesis that the saving rate, and its sensitivity to the interest rate, are a rising function of income finds strong empirical support."

The rest of the paper is organized as follows. We set up the model in Section 2. In Section 3 we focus on the case of logarithmic utility, for which clean, explicit results can be obtained. The case of CRRA functions is considered in Section 4, where we show long-period cycles may exist. Section 5 concludes. In Appendix A we show that our modelling of insurance captures a broad extent of realistic insurance contracts, such as those for hedging. All proofs are relegated to Appendix B.

2 The Model

The economy lasts for an infinite number of periods, $t = 0, 1, ...$, and consists of overlapping generations of agents, each of whom live for two periods. There is one perishable consumption good, used as numeraire, and one capital good. In each period, there is a continuum of 1 unit of young and 1 unit of old agents, so that the total population is fixed at 2 units. In period 0, there is an initial old generation, with capital stock K_0. The utility function of young agents is

$$U(c_1) + \beta U(c_2),$$

where c_i is their consumption in the ith period for $i = 1, 2$, $\beta \in (0, 1)$ is the discount factor, and $U(\cdot)$ is increasing and strictly concave, with $U' > 0$ and $U'' < 0$.

The Production of Consumption Good

In each period t, young agents contribute labour and old agents contribute capital to produce the consumption good. The aggregate production function is $Y_t = AK_t^\alpha L_t^{1-\alpha}$, in which K_t and L_t denote the aggregate supply of capital and labor, respectively. We assume A is constant over time to abstract away the external aggregate shocks. Each young agent supplies 1 unit of labor: $L_t = 1$ for all t.

There are perfectly competitive markets for hiring labor and for renting capital. Therefore, the young agents earn wage $W_t = A(1-\alpha)K_t^\alpha$ for their unit labor and the old agents earn rental rate $R_t = A\alpha K_t^{\alpha-1}$ per unit of their capital holding.

The Production of Capital and Moral Hazard

In each period, after earning the wage, each young agent engages in a project that produces capital from the consumption good. The project could fail, and then returns nothing. If the project succeeds, it produces I units of capital *in the next period* from I units of the consumption good of this period. Thus I measures the scale of the investment. The probability of success depends on the effort of the agent: it is q if he shirks and p if he works hard, with $p > q$. If an agent works hard on a project of scale I, he incurs disutility of θI. Whether an agent shirks or works hard is his private information and is not observed by others. Hence, the choice of effort is subject to moral hazard.

The outcomes of investment projects are independent across the young agents and over time. Capital depreciates completely after one period. Therefore all the capital used in period t comes from the investment in period $t-1$ and is owned by the old agents of period t. The capital stock in period t, K_t, equals the aggregate of successful investments in period $t-1$. For example, if all the projects have scale I_{t-1}, then $K_t = pI_{t-1}$ when the agents all work hard and $K_t = qI_{t-1}$ when they all shirk.

Mutual Insurance and Contracting

Since the outcomes of the projects are independent, by the Law of Large Numbers, if a large number of projects are pooled together, the average outcome is almost certain. This opens up the possibility of insurance and risk sharing between young agents, who are all risk averse. To capture in a simple way the essence of pooling for risk sharing, we assume the insurance is arranged as follows.

Mutual insurance companies (*MICs*) are established and they offer insurance contracts to young agents against the risks of their projects.[3] As the project of any agent has two states (success or failure), an insurance contract to the agent is thus characterized by two variables: the amount the agent gets from the company when his project fails, denoted by L, and the amount he pays the company when his project succeeds, which is a function of L and thus denoted by $H(L)$. Thus, the contracts offered by the MIC are of a *menu* $\{(L, H(L)|L \in \Phi)$, with L varying in a feasible set Φ and a particular L defining a particular contract. The market competition drives all MICs to earn zero profit. Therefore, if the subscriber succeeds with probability s, then

$$(1-s)L = sH(L).$$

This equation has two implications.

(1): Function $H(L)$ is linear. $H(L) = \tau L$, where

$$\tau = \frac{1-s}{s}. \tag{1}$$

We call this τ as the insurance price, which characterizes the menu of insurance contracts.

(2): The probability of success $s = p$ if the subscriber works hard, and $s = q$ if he shirks. Therefore, an MIC could provide at most two menus, characterized respectively by $\tau = (1-p)/p$ and $\tau = (1-q)/q$, depending on whether it expects the subscriber to work hard or to shirk.

Besides price τ, in another dimension does the two menus differ, namely Φ, the set of L from which the MIC allows subscribers to choose. The success probability of a project is never below q. Therefore, when an agent chooses a contract from the menu with $\tau = (1-q)/q$, the MIC can never make loss and thus the agent is allowed to choose any $L \geq 0$, namely, $\Phi = R^+$. But when

[3]The MICs are players in the paper, in the same way as the banks are players in Diamond and Dybvig (1983), although in essence, the MICs here and the banks in that paper are institutions through which the agents contract with each other.

an agent subscribes to the menu with $\tau = (1-p)/p$, the MIC, in order not to make a loss, must ensure that he will work hard on the project. His effort choice, however, is not observed by the MIC. Therefore, the menu with $\tau = (1-p)/p$ must provide the subscribers with the incentives to work hard, which, as will be demonstrated later, implies that the MIC imposes an upper bound on the permissible L, namely $\Phi = [0, \overline{L}]$.

Putting the two dimensions together, the MICs offer either or both of the following menus:

$$\{(L, \frac{1-q}{q} \cdot L | 0 \leq L\}; \quad \{(L, \frac{1-p}{p} \cdot L | 0 \leq L \leq \overline{L}\}. \tag{2}$$

In Appendix A we demonstrate that this modelling of mutual insurance actually captures a variety of real-life contracting for insurance and hedging.

An Agent's Life Cycle and Decisions

Given the wage (or wealth) a young agent has obtained from the production of the consumption good, W, and his rational expectation of the return rate of capital in the next period, R, he determines the scale of his capital-producing project, I, and subscribes to an insurance contract, $(L, \tau L)$, and then decides whether to work hard or shirk on his project.[4] The consumption in the young period is then:

$$C = W - I. \tag{3}$$

In the old period, if the agent's project succeeds, then he possesses I units of capital and earns rent IR, but he has to pay τL to the MIC by the contract he chose. His consumption in this contingency, denoted by C^g, is therefore

$$C^g = IR - L\tau. \tag{4}$$

If his project fails, then he gets no capital but the MIC gives him L units of the consumption good by the contract he chose. His consumption in this contingency, denoted by C^b, is therefore

$$C^b = L. \tag{5}$$

We assume that an agent can only contract with one MIC and that the MIC observe the scale of his project, I. This is important, because $\overline{L}$, the upper bound associated with price $(1-p)/p$, depends on I, as we will see.

[4]The old agents have no decisions to make; they just rent their capital and consume, then their capital depreciate completely and they retire.

Equilibrium Definition and Contractual Mode Switch

There are two levels of interactions. At the micro level, the MICs offer insurance contracts to the young agents. At the macro level, there are markets for hiring labor and renting capital, which determines wage, W, and capital rental rate, R. The two levels are interlocked. On the one hand, the contracting at the micro level depends on the price variables (i.e. W and R) determined at the macro level. On the other hand, the price variables are determined by the aggregate contracting and investment decision by individual agents at the micro level. As such, equilibrium entails (a) at the macro level the markets clear and the law of motion for the capital stock is derived from the optimal investment and effort decisions by the young agents; and (b) at the micro level the MICs offer the optimal contracts to the young agents and earn zero profit.

As for (a), the clearing of the labor market and the capital market implies:

$$W_t = A(1-\alpha)K_t^{\alpha}; \tag{6}$$

$$R_{t+1} = A\alpha K_{t+1}^{\alpha-1}. \tag{7}$$

Furthermore, if γ fraction of the young agents shirk and $1-\gamma$ of them work hard and the former invests I^0 and the latter I^1, then the capital stock at the next period, by the Law of Large Numbers, is:

$$K_{t+1} = \gamma q I^0 + (1-\gamma) p I^1. \tag{8}$$

As for (b), the zero profit condition has been captured by (1) and implies, as we saw, that each MIC at most offers two menus of insurance contracts, given by (2).

Among these two menus, the optimal contracting means that MICs offer the one(s) with which young agents attain the highest utility, taken as given their wage, W, and the interest rate at the next period, R. The highest utility with each of the two menus is characterized below.

If an agent subscribes to menu $\{(L, \frac{1-q}{q} \cdot L | 0 \leq L\}$, the MIC does not care if the agent works hard or shirk. If he shirks, his life cycle utility is $U(C) + \beta q U(C^g) + \beta(1-q)U(C^b)$. By (3), (4), and (5), with $\tau = (1-q)/q$, and given W and R, the highest attainable utility is:

$$V^0(W,R) \equiv \max_{I,L} U(W-I) + \beta q U(IR - \frac{1-q}{q}L) + \beta(1-q)U(L). \tag{9}$$

And *let the solution of the problem for I denoted by* $\widetilde{I}^0(W,R)$.

If he works hard, his life cycle utility is $U(C) + \beta p U(C^g) + \beta(1-p)U(C^b) - \theta I$, and the highest utility attainable is:

$$V^S(W,R) \equiv \max_{I,L} U(W-I) + \beta p U(IR - \frac{1-q}{q}L) + \beta(1-p)U(L) - \theta I. \qquad (10)$$

Therefore, if the agent chooses a contract from the menu with $\tau = (1-q)/q$, the highest attainable utility is $\max\{V^0(W,R), V^S(W,R)\}$.

If an agent subscribes to the menu with $\tau = (1-p)/p$, the MIC has to ensure he will work hard. The MIC does not observe his choice of effort and thus has to provide him with the necessary incentives. If he works hard, his continuation payoff is $\beta[pU(C^g) + (1-p)U(C^b)] - \theta I$, while if he shirks, his continuation payoff is $\beta[qU(C^g) + (1-q)U(C^b)]$. The agent has incentives to work hard, if and only if the former is not smaller than the latter, or equivalently,

$$U(C^g) - U(C^b) \geq \frac{\theta}{\beta(p-q)} I.$$

Since $C^g = IR - L\frac{1-p}{p}$ and $C^b = L$ from (4) and (5), and $\tau = (1-p)/p$, we have the following incentive compatible (IC) constraint:

$$U(IR - L\frac{1-p}{p}) - U(L) \geq \frac{\theta}{\beta(p-q)} I. \qquad (11)$$

This IC constraint places an upper bound on L, denoted by $\overline{L}$ in (2). The upper bound depends on R and I, both of which the MIC observes. The highest attainable profit with the menu is thus:

$$V^1(W,R) \equiv \max_{I,L} U(W-I) + \beta p U(IR - L\frac{1-p}{p}) + \beta(1-p)U(L) - \theta I, \ s.t.(11). \qquad (12)$$

And *let the solution of the problem for I denoted by* $\widetilde{I}^1(W,R)$.[5]

We assume

$$\frac{(1-q)p}{(1-p)q} \geq e^{\frac{1}{p-q}},$$

which ensures $V^1(W,R) \geq V^s(W,R)$ for the case of logarithmic utility,[6] and therefore excludes the case described by equation (10) where the agent subscribes to the menu with $\tau = (1-q)/q$,

[5] The problem is similar to that given in (10), except that the premium here is $(1-p)/p$ instead of $(1-q)/q$ and the agent here is subject to the IC constraint, (11), instead of any L available.

[6] This claim is put as Claim A1 in Appendix B and proved therein, where a comparison between $V^1(W,R)$ and $V^S(W,R)$ for a general case is also discussed.

but *works hard*. Therefore, if the agent subscribes to that menu, he will shirk. As such the menu, namely, $\{(L, \frac{1-q}{q} \cdot L | 0 \leq L\}$, is called as the *disincentivizing menu* and menu $\{(L, \frac{1-p}{p} \cdot L | L$ subject to (11)$\}$ is called as the *incentivizing menu*.

By encompassing both features (a) and (b) explicated above, we now formally define our concept of equilibrium as follows (recalling that W_t denotes the wage in period t, R_{t+1} the interest rate and K_{t+1} the capital stock in the next period, γ the proportion of the young agents who choose the disincentivizing menu, I^0 their investment scale, and I^1 the investment scale of those agents who choose the incentivizing menu).

Definition 1 *In a period with initial capital stock* K_t*, profile* $\{W_t, R_{t+1}, K_{t+1}; \gamma, I^0, I^1\}$ *forms an equilibrium if and only if:*

(i): The labor market and the capital market clear, namely, (6) and (7) hold true;

(ii): The capital stock in the next period results from aggregating the outcomes of the individual investments, namely, (8) holds true;

(iii): $\gamma = 0$ *if* $V^1(W_t, R_{t+1}) > V^0(W_t, R_{t+1})$, $\gamma = 1$ *if* $V^0(W_t, R_{t+1}) > V^1(W_t, R_{t+1})$, *and* $V^0(W_t, R_{t+1}) = V^1(W_t, R_{t+1})$ *if* $0 < \gamma < 1$.

(iv): $I^0 = \widetilde{I}^0(W_t, R_{t+1})$ *and* $I^1 = \widetilde{I}^1(W_t, R_{t+1})$.

Conditions (i) and (ii) are what equilibrium entails at the macro level. Condition (iii) captures optimal contracting and embeds the choice of the contractual menu in equilibrium. It says that given W_t and R_{t+1}, if the incentivizing menu gives the agents a higher payoff than the disincentivizing menu (i.e. $V^1(W_t, R_{t+1}) > V^0(W_t, R_{t+1})$), then no agents choose the latter (i.e. $\gamma = 0$), while they all choose it (i.e. $\gamma = 1$) if it offers a higher payoff than the incentivizing menu (i.e. $V^0(W_t, R_{t+1}) > V^1(W_t, R_{t+1})$); moreover, they must be indifferent between the two menus if both are chosen in equilibrium (i.e. $0 < \gamma < 1$). Lastly, condition (iv) says that the choice of investment scale is part of the optimal decision by the young agents.

For a particular period, if the equilibrium features $\gamma = 0$, we call the period is in the *incentivized mode (IM)*, because in the period the incentivizing menu prevails and all the young agents are incentivized to work hard. If the equilibrium features $\gamma = 1$, we call the period is in the *disincentivized mode (DIM)*, because in the period the disincentivizing menu prevails and all the young agents shirk. Similar, the equilibrium with $0 < \gamma < 1$ is called the *mixed mode*.

Henceforth we use superscription "0" to represent the DIM, "1" the IM, "01" the switch from the former to the latter, and "10" the switch in the reverse direction.

We examine the case of logarithmic utility in the next section, focusing especially on the switch in contractual mode and the dynamics of capital stock.

3 The Case of Logarithmic Utility

We pick logarithmic utility function for two reasons. One, it makes the exposition direct and simple. The other, and more importantly, use of logarithmic function, with which the income effect exactly is exactly offset by the substitution effect, clearly differentiates our paper from a big strand of literature that relies on a strong income effect to generate endogenous cycles.[7] As will be shown below, in our paper it is the switch in contractual mode, not the income effect, that drives cycles in the steady state; if the economy were always in the IM or the DIM, then the capital stock would *monotonically* converge to the unique steady-state level, namely, no cycles would arise even in the process of convergence.

3.1 The Switches in Contractual Mode

To demonstrate switches in contractual mode, we first figure out the conditions under which one or the other mode rules, as is described by condition (iii) of the definition above. For that purpose, we determine the properties of $V^j(W_t, R_{t+1})$ and $\widetilde{I}^j(W_t, R_{t+1})$ for $j = 0, 1$, in order below.

First, for the case of $j = 0$, namely, that associated with the disincentivizing menu, the decision problem of young agents is given by (9). The solution for it is characterized below.

Lemma 1 *If an agent subscribes to the disincentivizing menu, then he gets full insurance:*

$$C^g = C^b = L = qIR. \tag{13}$$

And his investment is determined by

$$U'(W - I) = \beta qRU'(qIR). \tag{14}$$

[7] See Guesnerie and Woodford (1992) for a survey and detailed references.

The full insurance result is standard: risk averse agents, when facing the fair insurance price, want to be fully insured. Note, however, that the full insurance is feasible with the disincentivizing menu, because the agents are not required to work hard; were they to be incentivized, the consumption from success would need to be higher than that from failure.

The full insurance result suggests that the agents subscribing to the disincentivizing menu are actually facing a problem of saving with a return rate of qR: they give up I units of the consumption good today in exchange of $qR \cdot I$ units of the consumption good tomorrow no matter what happens to their projects. Moreover, when the agents choose the disincentivizing menu, the amount they can save, namely the size of I and L, is unrestrained and therefore they can perfectly smooth consumption across the two periods of their life.

These two features, namely the full insurance and perfect inter-temporal smoothing, command that choosing the disincentivizing menu leads to the following saving problem:

$$V^0(W,R) = \max_I U(W-I) + \beta U(I \cdot qR) \tag{15}$$

Therefore, the optimal allocation of I is ruled by (14), which says the marginal utility of consumption are equalized across the two periods.

When $U(\cdot) = \log(\cdot)$, the optimal investment (or saving) is independent of the interest rate:

$$\widetilde{I}^0(W,R) = \frac{\beta}{1+\beta} W, \tag{16}$$

from which it follows:

$$V^0(W,R) = \log\{W^{1+\beta} \frac{\beta^\beta}{(1+\beta)^{1+\beta}} q^\beta R^\beta\}. \tag{17}$$

Second, we turn to the case of the incentivizing menu, of which the decision problem of young agents is given by (12). Let

$$\delta \equiv \frac{\theta}{\beta(p-q)}.$$

Lemma 2 *Suppose $U(\cdot) = \log(\cdot)$. (i) If an agent subscribes to the incentivizing menu, then his investment, $\widetilde{I}^1(W,R)$, is independent of R and determined by the following equation:*

$$\frac{1-p}{p} \cdot \frac{qI + \frac{\phi}{\delta}}{(1-q)I - \frac{\phi}{\delta}} = e^{\delta I}, \tag{18}$$

where $\phi \equiv \frac{W - \frac{1+\beta}{\beta} I}{W-I}$ is between 0 and 1; and his value is

$$V^1(W,R) = \log\{(W-\widetilde{I}^1)[\frac{p}{1-p}((1-q)\widetilde{I}^1 - \frac{\phi}{\delta})]^\beta e^{\beta q \delta \widetilde{I}^1} R^\beta\} \tag{19}$$

(ii) $d\widetilde{I}^1/dW > 0$ *and* $\widetilde{I}^1 < \frac{\beta}{1+\beta}W$ *(i.e.* $\widetilde{I}^0$*).*

The independence of $\widetilde{I}^1(W,R)$ with R comes from the feature of logarithmic utility, that the income effect exactly offsets the substitution effect. Due to this independence, we write $\widetilde{I}^1(W,R)$ as $\widetilde{I}^1(W)$.

Finally, we come to determine when the contractual mode switches.

Equations (17) and (19) together implies that $V^0(W,R) = V^1(W,R)$ if and only if

$$W^{1+\beta}q^{\beta}\frac{\beta^{\beta}}{(1+\beta)^{1+\beta}} = (W - \widetilde{I}^1(W))[\frac{p}{1-p}((1-q)\widetilde{I}^1(W) - \frac{\phi}{\delta})]^{\beta}e^{\beta q\delta\widetilde{I}^1(W)}, \tag{20}$$

and $V^0(W,R) > V^1(W,R)$ if and only the left hand side is bigger than the right hand side. The key feature is that the comparison between $V^0(W,R)$ and $V^1(W,R)$ is independent of R and depends only on W, which is the greatest convenience conferred by logarithmic utility.

Denote the root of (20) by $\widehat{W}$, whose existence will be shown soon. As the wealth (or wage) of young agents is positively linked to the capital stock through $W_t = A(1-\alpha)K_t^{\alpha}$, we have $W_t < \widehat{W}$ if and only if the initial capital stock $K_t < \widehat{K}$, where

$$\widehat{K} = (\frac{\widehat{W}}{A(1-\alpha)})^{\frac{1}{\alpha}}. \tag{21}$$

Proposition 1 *The root of equation (20),* $\widehat{W}$*, exists and is independent of* A*, and* $V^0(W_t, R_{t+1}) < V^1(W_t, R_{t+1})$ *if and only if* $W_t < \widehat{W}$*, namely, if and only if* $K_t < \widehat{K}$*.*

That is, in period t all young agents choose the incentivizing menu and work hard, so the IM rules, if the initial capital stock, K_t, is below $\widehat{K}$; if $K_t > \widehat{K}$, they all choose the disincentivizing menu and shirk, so the DIM rules. Thus, in equilibrium the contractual mode switches from the incentivizing menu to the disincentivizing menu with the wealth of the economy surpassing a threshold. For an intuition, we compare between the two menus more closely.

As was noted, the disincentivizing menu, $\{(L, \frac{1-q}{q} \cdot L | 0 \leq L\}$, delivers two benefits: (1) it allows the young agents to be fully insured against the risks of their projects; and (2) it also allows them to perfectly smooth consumption inter-temporally, because it imposes no constraint upon the amount of insurance permitted to subscribe, or put differently, it allows any amount of wealth to be transferred into the next period.

By contrast, both benefits are forsaken with the incentivizing menu in exchange for providing incentives to work hard. First, the agents want to work hard only if they get more when their projects succeed than when the projects fail (i.e. $C^g > C^b$), that is, they will not have full insurance against the risks of their projects. Second, the necessity to honor the IC constraint, (11), also seriously obstructs the passage of wealth on to the old period, as shown below.

Lemma 3 $\lim_{W\to\infty} \widetilde{I}^1(W) \equiv I_\infty < \infty$. *And for any* W, $C^g < R_\infty(qI_\infty + \frac{1}{\delta})$ *and* $C^b < \frac{pR_\infty}{1-p}[(1-q)I_\infty - \frac{1}{\delta}]$, *where* R_∞ *is a positive and finite number.*

By the lemma, *however large the wealth* (W) *at the young period is, the amount of wealth the agents can pass on to the old period in an incentive-compatible way is upper-bounded.* This is because the scale of investment is upper-bounded ($I < I_\infty$), which is in turn because of the need to honour the IC constraint, (11). As $U(IR) > U(IR - L\frac{1-p}{p})$ and $U(L) > -\infty$ under the agents' optimal decisions, the constraint implies that $U(IR) > \delta I + d$ for some constant d. The interest rate R is in the order of $K^{\alpha-1}$, thus of $I^{\alpha-1}$, in equilibrium. Hence IR is in the order of I^α. Therefore, the IC constraint commands that the investment scale, I, must satisfy, for some constants d and d',

$$U(d'I^\alpha) > \delta I + d,$$

and is hence upper-bounded, considering U is concave.

The trade-off between the disincentivizing menu and the incentivizing menu is therefore as follows. The former gives full insurance and perfect inter-temporal consumption smoothing, while the latter entails a high yield rate of producing the capital. When W is very small, the agents have not much to pass on to the future even under full insurance and perfect inter-temporal smoothing; and therefore, the incentivizing menu dominates. On the other hand, when W is very large, the agents prefer to pass a lot on to the future in all contingencies. This is feasible only with the disincentivizing menu, which, therefore, dominates.

The switch in contractual mode shown in Proposition 1 may drive the economy into fluctuations in *the steady state*, even though there are no macroeconomic shocks. This is to be shown in the next section.

3.2 The Dynamics of Capital Stock

By Proposition 1, the evolution of capital stock is ruled by the dynamics of the DIM if $K_t > \widehat{K}$ and by that of the IM if $K_t > \widehat{K}$. These dynamics are found out in order below.

The next period capital stock, $K_{t+1,}$ is the aggregate outcomes of the successful projects. In the DIM, all young agents invests $\widetilde{I}^0$, which, by (16), equals $\frac{\beta}{1+\beta}W_t$, and their projects succeed, independently, with probability q (as they all shirk). By the law of large numbers, therefore, $K_{t+1} = q\widetilde{I}^0 = q\frac{\beta}{1+\beta}W_t$. As $W_t = A(1-\alpha)K_t^\alpha$, the dynamics of the DIM are then described by

$$K_{t+1} = \frac{q\beta(1-\alpha)A}{1+\beta}(K_t)^\alpha \equiv M^0(K_t; q). \tag{22}$$

The (non-zero) steady state of the DIM, denoted by K^0, is to be found by substituting $K_{t+1} = K_t = K^0$ into (22). Thus,

$$K^0 = (\frac{\beta(1-\alpha)qA}{1+\beta})^{\frac{1}{1-\alpha}}. \tag{23}$$

In the IM, all young agents invests $\widetilde{I}^1(W_t)$, which is determined by (18), and their projects succeed, independently, with probability p (as they all work hard in the IM). By the law of large numbers, therefore, $K_{t+1} = p\widetilde{I}^1(W_t)$. As $W_t = A(1-\alpha)K_t^\alpha$, the dynamics of the DIM are then described by

$$K_{t+1} = p\widetilde{I}^1(A(1-\alpha)K_t^\alpha; \delta, p) \equiv M^1(K_t; \delta, p). \tag{24}$$

The (non-zero) steady state of the dynamics, denoted by K^1, satisfies the following equation:

$$K^1 = pI^1(A(1-\alpha)(K^1)^\alpha; \delta, p). \tag{25}$$

That gives K^1 as a function of p and δ, denoted by $K^1(\delta, p)$.

Proposition 2 *(i) If $q \leq 0.5$, both $M^0(K, q)$ and $M^1(K; \delta, p)$ are increasing and concave with K and have a unique steady state (namely K^0 and K^1) which is globally stable.*

(ii) Both K^0 and K^1 increases with A.

Result (i) says that if there were no switch in contractual mode and the economy were always in the IM or the DIM, then the capital stock would *monotonically* converge to the unique steady-state level, that is, there would be no oscillation even in the process of convergence. Therefore, in our paper it is the switch in contractual mode that drives cycles in the steady state, if there are any.

To find the dynamics of capital stock of the economy, by Proposition 1, we shall consider the dynamics in the IM, $M^1(K_t;\delta,p)$, if $K_t < \widehat{K}$ and those in the DIM, $M^0(K_t;q)$, if $K_t > \widehat{K}$. The steady state of the former dynamics is $K^1(\delta,p)$ and that of the latter is $K^0(q)$. The properties of the economy's dynamics are determined by the relative positions between K^0, K^1, and $\widehat{K}$, of which the following lemma describes one case.

Lemma 4 *There exists $\overline{\delta}$, $\underline{A}$ and $\overline{A}$ such that if $0 < \delta < \overline{\delta}$ and $\underline{A} < A < \overline{A}$, then $K^0 < \widehat{K} < K^1$ and $M^1(K_t;\delta,p) > M^0(K_t;q)$ for $K_t < K^0(p)$ where $K^0(p)$ is defined by (23) with q replaced with p.*

Intuitively, we obtain the relative position of $M^1(K_t;\delta,p)$ to $M^0(K_t;q)$ and that of K^1 to K^0 by manipulating δ, and have $\widehat{K}$ sit between K^0 and K^1 by manipulating A. When δ approaches 0, the allocation of the IM converges to that of the DIM except that the probability of success is p instead of q. Especially $M^1(K_t;\delta,p)$ approaches $M^0(K_t;p)$ and sits above $M^0(K_t;q)$, and $K^1(\delta,p)$ approaches $K^0(p)$ and sits to the right of $K^0(q)$, illustrated as below.

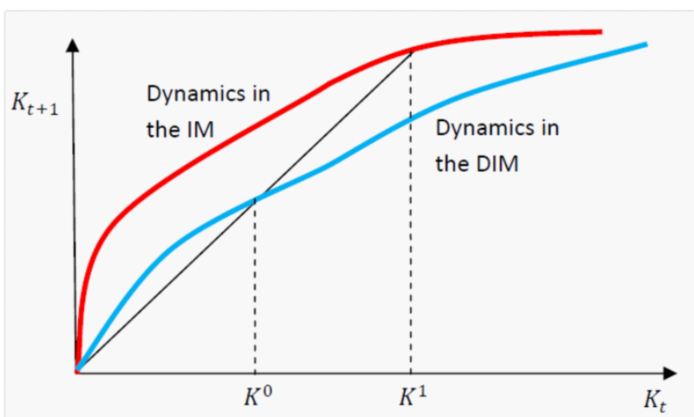

Figure 1: The Dynamics in the IM and that in the DIM

And we obtain the position of $\widehat{K}$ relative to K^0 and K^1 by manipulating A. By Proposition 2 (ii), K^0 and K^1 both continuously increases with A, but $\widehat{K} = (\widehat{W}/A(1-\alpha))^{\frac{1}{\alpha}}$ decreases with A in the order of $A^{-\frac{1}{\alpha}}$ since by Proposition 1, $\widehat{W}$ is independent of A. Therefore, if A is in a medium range, we have $K^0 < \widehat{K} < K^1$.

The dynamics of capital stock are ruled by $K_{t+1} = M^1(K^t; p, \delta)$ for $K_t < \widehat{K}$, and by $K_{t+1} = M^0(K^t; q)$ for $K_t > \widehat{K}$ and are therefore illustrated by a figure that is obtained by inserting the threshold $\widehat{K}$ into Figure 1 above, as below. The figure below also illustrates that in the steady state of the dynamics the economy oscillates between the two stocks of capital, $\underline{K}$ and $\overline{K}$:

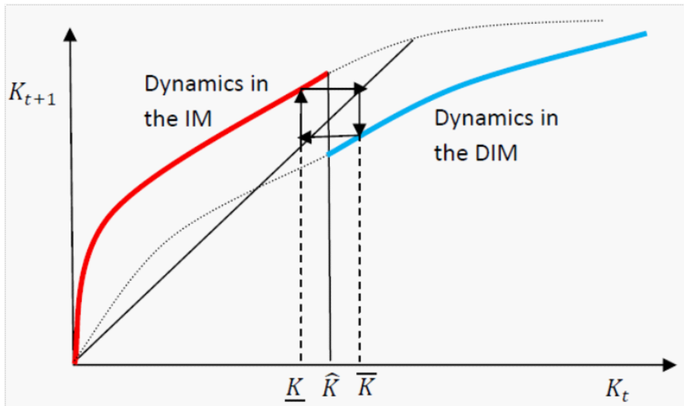

Figure 2: The Dynamics where $K^0 < \widehat{K} < K^1$

Mathematically, $\underline{K}$ and $\overline{K}$ are determined by:

$$\begin{aligned} \overline{K} &= M^1(\underline{K}; p, \delta) \\ \underline{K} &= M^0(\overline{K}) \end{aligned}$$

This steady state of cycle $(\underline{K}, \overline{K})$ is globally stable. That is, whatever the initial capital stock is, the dynamics of capital stock converge to this cycle, as the following proposition states.

Proposition 3 *Assume $0 < \delta < \overline{\delta}$ and $\underline{A} < A < \overline{A}$, so that by Lemma 4 $K^0 < \widehat{K} < K^1$. For any initial capital stock K_0, there exists T such that the sequence $K_T, K_{T+2}, K_{T+4}, ..$ converges to $\underline{K}$, and the sequence $K_{T+1}, K_{T+3}, K_{T+5}, ..$ converges to $\overline{K}$; and $K_{T+2n+1} = M^1(K_{T+2n}) > K_{T+2n}$ and $K_{T+2n+2} = M^0(K_{T+2n+1}) < K_{T+2n+1}$ for integer $n \geq 0$.*

That is, the economy is in a boom at $t = T + 2n + 1$ (i.e. K_t close to $\overline{K}$), when all the young agents shirk since the economy is rich ($K_t > \widehat{K}$). This all-shirking topples the economy into a bust the next period (i.e. K_{T+2n+2} close to $\underline{K}$), when, since the economy is poor ($K_{T+2n+2} < \widehat{K}$)), the young agents then all work hard, lifting the economy back into a boom the next period. This fluctuation is not smoothed out over time and stays there forever: in the steady state the capital stock permanently oscillates between $\underline{K}$ and $\overline{K}$.

We compare between what occurs in steady-state booms and busts in the following proposition, where we use an upper bar to denote the case associated with booms and a lower bar the case associated with busts.

Proposition 4 *(i)* $\overline{I} > \underline{I}$*, but* $q\overline{I} < p\underline{I}$*. (ii)* $\overline{I}/\overline{W} > \underline{I}/\underline{W}$*.*

Result (i) says that the scale of the investment is higher during booms than it is during busts, but the yield rate of it is lower ($q < p$), and furthermore, the weakness in the yield rate over-offsets the strength in the scale, making a boom followed by a bust and a bust by a boom. Result (ii) says that the higher scale of investment during booms comes not only from a higher wealth (i.e. $\overline{W} > \underline{W}$), but also from a higher saving rate.

The steady-state cycle examined above arises only when $\underline{A} < A < \overline{A}$, that is the productivity level is in a medium range. Note that $\widehat{K}$ decreases with A (by 21), whereas both K^0 and K^1 increase with A (by Proposition 2.ii). Therefore, if $A > \overline{A}$, then $\widehat{K} < K^0 < K^1$. For this case, the dynamics is illustrated as follows.

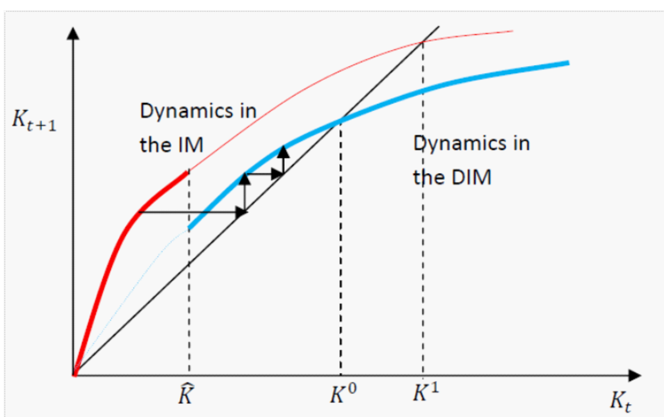

Figure 3: The Dynamics When A Is Large

It is easy to see from Figure 3 that the unique steady state is K^0, the steady state in the DIM, where agents enjoy leisure and full insurance, and no cycles arise.

On the other hand, if $A < \underline{A}$, then $K^0 < K^1 < \widehat{K}$ and the dynamics is illustrated as follows.

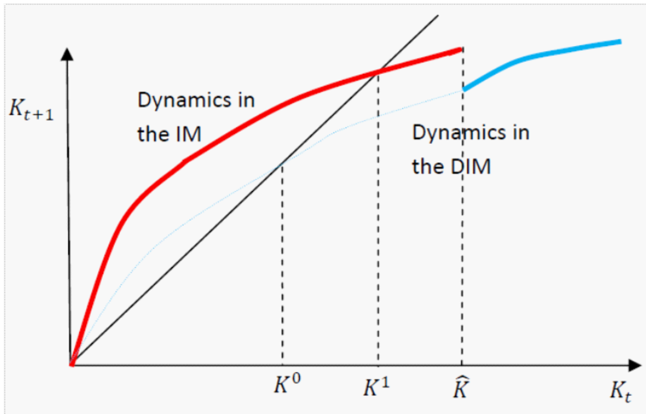

Figure 4: The Dynamics When A Is Small

It is easy to see from Figure 3 that the unique steady state is K^1, the steady state in the IM, no cycles arising either.

Having focused on the case of logarithmic utility, which gives rise to the clean results, we move on to the case of general Constant Relative Risk Aversion (CRRA) functions.

4 The Case of CRRA Utility Function

In this section, we accommodate two more features. One, unlike in the case of logarithmic utility, where the steady-state cycle spans only two periods (one in a boom, the other in a bust), now it could consist of many periods of booms followed by many periods of busts. The other is that in some periods there could be the mixed mode where part of agents work hard and the rest shirks, although ex ante they are identical.

In this section, the period utility of agents is described by a CRRA function, that is,

$$U(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}.$$

We assume

$$\sigma\alpha < 1 \tag{26}$$

and that $\sigma > \frac{1}{2}$, which ensures the following important property:

P: $U \circ D$ is concave, where $D(\cdot)$ is the inverse function of $\frac{1}{U'(\cdot)}$.

For a general CRRA function, the comparison between $V^0(W_t, R_{t+1})$ and $V^1(W_t, R_{t+1})$ depends on both W_t and R_{t+1}, and thus a key convenience of logarithmic utility is lost. The wealth,

W_t, is pinned down by the initial capital stock. The next period interest rate, R_{t+1}, is pinned down by the next period capital stock, which depends, in turn, on the effort choice by young agents, and thus on the equilibrium mode of this period hinging on the comparison between $V^0(W_t, R_{t+1})$ and $V^1(W_t, R_{t+1})$. This interdependence commands that to find the equilibrium mode for a given period, we shall first suppose a mode (say the DIM), then figure out the interest rate in the mode (say R^0_{t+1}) , and then check back the mode supposed at the beginning indeed rules (that is $V^0(W_t, R^0_{t+1}) > V^1(W_t, R^0_{t+1})$. This is the approach taken below.

4.1 The DIM: When It Rules and the Dynamics in It

Suppose a period t is in the DIM. Then all young agents invest $\widetilde{I}^0(W_t, R_{t+1})$, which, mathematically, is the solution for I of (9), the optimization problem they face if choosing the disincentivizing menu. Their projects succeed independently with probability q. The next period capital stock is thus $K_{t+1} = q\widetilde{I}^0(W_t, R_{t+1})$.Since the interest rate is pinned down by capital stock through $R = A\alpha K^{\alpha-1}$, we have

$$R_{t+1} = A\alpha(q\widetilde{I}^0(W_t, R_{t+1}))^{\alpha-1}.$$

This equation determines the interest rate as a function of W_t, the wealth of the young agents, in the DIM, which is denoted by $R^0(W_t)$. And define

$$I^0(W_t) \equiv \widetilde{I}^0(W_t, R^0(W_t)),$$

which, therefore, denotes the scale of investment as a function of W_t in the DIM.

As $W_t = A(1-\alpha)K_t^\alpha$, the dynamics of capital stock is then:

$$K_{t+1} = qI^0(A(1-\alpha)K_t^\alpha)$$

with its steady state, K^0, determined by

$$K^0 = qI^0(A(1-\alpha)(K^0)^\alpha). \tag{27}$$

For $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, $R^0(W)$ is implicitly determined by

$$(\frac{R^0}{A\alpha})^{-\frac{1}{1-\alpha}} = \frac{q\beta^{\frac{1}{\sigma}}(qR^0)^{\frac{1-\sigma}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}(qR^0)^{\frac{1-\sigma}{\sigma}}}W, \tag{28}$$

and the following lemma holds.

Lemma 5 $\frac{\partial K_{t+1}}{\partial K_t} > 0$; $\lim_{K_t \to 0} \frac{\partial K_{t+1}}{\partial K_t}(K_t, q) = \infty$; $\frac{\partial K_{t+1}}{\partial q} > 0$; *and the dynamics have a unique steady state,* K^0 *and* $K^0 = T(\alpha, \beta, \sigma)(Aq)^{\frac{1}{1-\alpha}} < (A(1-\alpha)q)^{\frac{1}{1-\alpha}}$.

By the lemma, the dynamics are increasing and shift upward when q rises, and the steady state is unique and stable and increases with A and q in the order of $(Aq)^{\frac{1}{1-\alpha}}$. These properties help settle down the relative position between $M^0(K_t)$ and $M^1(K_t)$, as was in the case of logarithmic utility.

Lastly, we check the consistency of the supposition at the beginning, namely that the period is ruled by the DIM. That is, indeed given W_t, the agents prefer the disincentivizing menu to the incentivizing one, $V^0(W_t, R_{t+1}) \geq V^1(W_t, R_{t+1})$. In the DIM, the next period's interest rate $R_{t+1} = R^0(W_t)$. Therefore, the DIM indeed rules if and only if $V^0(W_t, R^0(W_t)) \geq V^1(W_t, R^0(W_t))$. *We define* W^{01} *as the root of*

$$V^0(W, R^0(W)) = V^1(W, R^0(W)). \tag{29}$$

As was in the case of logarithmic utility, the DIM delivers the benefit of full insurance and that of perfect inter-temporal consumption smoothing, which the agents prefer when their wealth, W, is large. Therefore, the DIM rule in a period t if $W_t > W^{01}$.

We proceed to the examination of the IM.

4.2 The IM: When It Rules and the Dynamics in It

Suppose a period t is in the IM. Then all young agents invest $\widetilde{I}^0(W_t, R_{t+1})$, which, mathematically, is the solution for I of (12), the optimization problem they face if choosing the incentivizing menu. Their projects succeed independently with probability p, which is higher than q as they are working hard. The next period capital stock is thus $K_{t+1} = p\widetilde{I}^1(W_t, R_{t+1})$.Since the interest rate is pinned down by capital stock through $R = A\alpha K^{\alpha-1}$, in the IM

$$R_{t+1} = A\alpha(p\widetilde{I}^1(W_t, R_{t+1}))^{\alpha-1}. \tag{30}$$

This equation determines *the equilibrium interest rate,* R_{t+1}*, as a function of* W_t*, which is denoted by* $R^1(W_t)$. And define

$$I^1(W) \equiv \widetilde{I}^1(W, R^1(W)).$$

This determines the scale of investment as a function of wealth in the IM. Note that both the equilibrium interest rate and the equilibrium investment scale are also functions of δ and p. When it is necessary to make this point explicit, we use notations $R^1(W;\delta,p)$ and $I^1(W;\delta,p)$.

As $W_t = A(1-\alpha)K_t^\alpha$, the dynamics of capital stock in the IM is then:

$$K_{t+1} = pI^1(A(1-\alpha)K_t^\alpha),$$

with its steady state, K^1, determined by

$$K^1 = pI^1(A(1-\alpha)(K^1)^\alpha).$$

As was in the case of logarithmic utility, in the IM the agents suffer the problem of restrained inter-temporal consumption smoothing: however large W is, the amount of wealth that can be passed on to the future in the IM is bounded from above:

Lemma 6 *With property P, in the IM* $\lim_{W\to\infty} I^1 < \infty$; $\lim_{W\to\infty} \widehat{C^g} < \infty$; *and* $\lim_{W\to\infty} \widehat{C^b} < \infty$.

The intuition is the same as was given in the discussion following Lemma 3. The need to provide incentives imposes the constrains not only upon the extent of insurance the agent can obtain, but, more importantly, also upon the amount of wealth they can pass on to the future.

Lastly, we check the consistency of the supposition at the beginning, namely that the period is ruled by the IM. That is, indeed given W_t, the agents prefer the incentivizing menu to the dis-incentivizing one, $V^0(W_t, R_{t+1}) \leq V^1(W_t, R_{t+1})$. In the IM, the next period's interest rate $R_{t+1} = R^1(W_t)$. Therefore, the DIM indeed rules if and only if $V^0(W_t, R^1(W_t)) \leq V^1(W_t, R^1(W_t))$. Define W^{10} as the root of

$$V^0(W, R^1(W)) = V^1(W, R^1(W)). \tag{31}$$

Given the two problems of partial insurance and restrained inter-temporal consumption smoothing, at a period the IM is indeed the equilibrium mode only if $W_t < W^{10}$.

4.3 The Dynamics and Multi-Period Cycles

For the case of logarithmic utility, $W^{01} = W^{10} = \widehat{W}$, because there the comparison between $V^0(W,R)$ and $V^1(W,R)$ is independent of R. But in general, $W^{01} \neq W^{10}$, because the equilibrium interest rate under IM and DIM are different. For W^{01}, the equilibrium concerned is the

DIM where the interest rate in it is $R^0(W)$. By contrast, for W^{10}, the equilibrium concerned is the IM where the interest rate is $R^1(W)$. And $R^0(W) \neq R^1(W)$ even for the case of logarithmic.

The corresponding thresholds in capital stock, K^{01} and K^{10}, are pinned down by the thresholds in wealth, W^{01} and W^{10}, through

$$K = (\frac{W}{A(1-\alpha)})^{\frac{1}{\alpha}}. \tag{32}$$

Then, *the DIM rules at period t if $K_t > K^{01}$, and the IM rules at period t if $K_t < K^{10}$.*

As was in the case for logarithmic utility, the properties of the dynamics of capital stock in the economy are determined by the relative positions between the steady state in the DIM (K^0) and that in the IM (K^1) and K^{01} and K^{10}. Again, for the relative position between K^0 and K^1, we can manipulate δ. When $\delta \to 0$, the dynamics of the IM and its steady state converge to the counterparts of the DIM, with the probability of success being p instead of q, that is,

$$\begin{aligned} \lim_{\delta \to 0} M^1(K_t; \delta, p) &= M^0(K_t, p) \\ \lim_{\delta \to 0} K^1(\delta, p) &= K^0(p). \end{aligned}$$

By Lemma 5, $M^0(K_t, p) > M^0(K_t, q)$ and $K^0(p) > K^0(q)$. Therefore, when δ is close to 0, $M^1(K_t)$ sits above $M^0(K_t)$ and K^1 sits to the right of K^0, as was in the case for logarithmic utility and illustrated by figure 1.

To accommodate the two thresholds, K^{01} and K^{10}, we manipulate A, as we did in the logarithmic case. By Lemma 5, K^0 is in the order of $(Aq)^{\frac{1}{1-\alpha}}$. K^1 also increases with A. What we need then is:

Lemma 7 *Under Assumption (26), the two thresholds, K^{01} and K^{10}, decrease with A.*

By the lemma and the fact that K^0 and K^1 both increase with A, when A is in a medium range, K^{01} and K^{10} stand between the two steady states, K^0 and K^1. That is, $K^0 < K^{01}, K^{10} < K^1$.

The last thing to resolve is the position of K^{01} relative to K^{10}. Unlike the case of logarithmic utility, $K^{01} = K^{10}$ no longer holds. We have thus two cases to consider, depending on the relative position between the two thresholds.

Case 1: $K^{10} > K^{01}$ and Multiple-Period Cycles

For this case, in a period t such that $K^{10} > K_t > K^{01}$, both the IM and the DIM are an equilibrium at the period. In the paper, we use the following way of selecting equilibrium, which could be called *Inertia Standard*: when both the IM and the DIM are an equilibrium in a given period, the mode of equilibrium in this period follows that in the last period, for example, if the equilibrium in the last period is the DIM, then the equilibrium in this period is the DIM. This inertia can be justified by an inter-generation interaction which is not explicitly modelled in the paper. The idea is that if the young agents of the last period live in an atmosphere of shirking and enjoying life, then they role-model the young agents of this period into the same life style so long as it is sustainable (i.e. the DIM is an equilibrium); and similarly for the IM.

If this inertial standard of selecting equilibrium is applied, then the steady state of dynamics features intrinsic cycles. Furthermore, unlike in the case of logarithmic utility, the cycles could now run up and down for many periods, rather than one period, as is illustrated in the figure below.

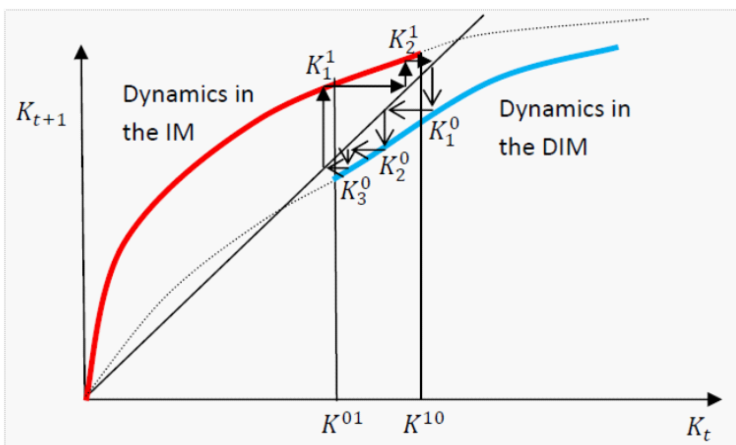

Figure 5: A Steady State Cycle over Five Periods: $K_1^1, K_2^1, K_1^0, K_2^0$, and K_3^0

Formally, the steady state is a profile of $\{K_j^1, K_n^0\}_{j=1,2,..J;n=1,2,...,N}$ defined by the following two groups of conditions:

A. *The inequality conditions*: $K_1^1 < K_2^1 < ... < K_J^1 < K^{10} < K_1^0$ and $K_1^0 > K_2^0 > ... > K_N^0 > K^{01} > K_1^1$;

B. *The equality conditions*:

$$\begin{aligned} K_1^1 &= M^0(K_N^0) \\ K_j^1 &= M^1(K_{j-1}^1) \text{ for } j = 2, ..., J \\ K_1^0 &= M^1(K_J^1) \\ K_n^0 &= M^0(K_{n-1}^0) \text{ for } n = 2, ..., N \end{aligned}$$

That is, the cycle consists of a J-period rising path in the IM and a N-period declining path in the DIM, hence overall running over $J + N$ periods.

To state a condition for such a cycle to exist, let ${}^T M^1$ denote T times compound of function M^1, that is, ${}^T M^1(\cdot) = M^1(M^1...M^1(\cdot)...))$ for T compounds. Similarly is ${}^T M^0$ defined. Note that for any $K > K^{01}$, there exist a unique positive integer $N(K)$ such that ${}^N M^0(K) < K^{01} <^{N-1} M^0(K)$: since $K^{01} > K^0$, this $K > K^0$; thus ${}^T M^0(K)$ decreases with T and converges to K^0 with $T \to \infty$; and therefore, at a unique time N, the sequence $\{{}^T M^0(K)\}_{T=0,1,..}$ just pass K^{01}. Similarly, for any $K < K^{10}$, there exist a unique positive integer $J(K)$ such that ${}^J M^1(K) > K^{10} >^{J-1} M^1(K)$. Note that N is non-decreasing with K: the further the staring point is away from K^{01}, the more steps needed to have the sequence pass K^{01}. Similarly, $J(K)$ is non-increasing with K. Therefore, $N(K^{10}) \leq N(M^1((K^{10}))$ and $J(K^{01}) \leq J(M^0(K^{01}))$.

With these notations, the condition is

$$N(K^{10}) = N(M^1((K^{10})) \equiv N^* \tag{33}$$

$$J(K^{01}) = J(M^0(K^{01})) \equiv J^* \tag{34}$$

Condition (33) says that if the economy just comes out of the IM and enters the DIM (thus its capital stock is between K^{10} and $M^1((K^{10}))$, then always after exactly N^* periods its capital stock falls below K^{01} and thus it enters the IM again. Similar, Condition (34) says that if the economy just comes out of the DIM, then always after exactly J^* periods it enters the DIM again.

Proposition 5 *If both M^0 and M^1 are concave with K and conditions (33) and (34) hold, then the profile $\{K_j^1, K_n^0\}_{j=1,2,..J^*;n=1,2,...,N^*}$ defined by conditions A and B above exists, that is, there is a cycle of $J^* + N^*$ periods. Moreover, it is of multi-period if $K^{10} > M^1(K^{01})$ or if $K^{01} < M^0(K^{10})$.*

The reason for the latter part of the proposition is that if $K^{10} > M^1(K^{01})$, then by definition of $J(K^{01})$, we have $J^* \geq 2$, namely the cycle runs for more than one period in the IM. Similarly, if $K^{01} < M^0(K^{10})$, then the cycle runs for more than one period in the DIM.

Case 2: $K^{10} < K^{01}$ and the Mixed Mode

For this case, when K_t is between K^{10} and K^{01}, neither the DIM nor the IM is the equilibrium. The period is thus in the mixed mode where a proportion γ of young agents shirk and $1-\gamma$ of them work hard, with $0 < \gamma < 1$. The dynamics are illustrated by the figure below.

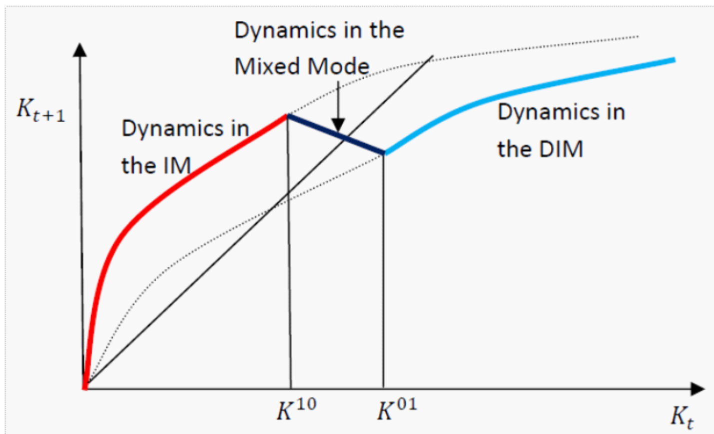

Figure 6: The Dynamics when $K^{10} < K^{01}$

Note that the K_{t+1} could decrease with K_t in the mixed equilibrium, because the proportion of the agents who work hard could decrease with it. There are no cycles in the steady state, but in the process of convergence the economy oscillates.

5 Conclusion

The market economies feature volatility, cyclical movements, and instability. The paper demonstrates that these fluctuations may be triggered by the switches in contractual arrangements through which the agents obtain insurance and handle the incentive problems in connection with moral hazards. When the economic conditions change, agents have incentives to write a new mode of contracts for insurance and incentives, which, through its effects on investment and its interactions with labour and capital markets, brings about new market conditions. As such, cycles may endogenously arise from the intrinsic dynamics of the economy.

Furthermore, the paper finds that a necessary condition for such cycles to arise is that the productivity level of the economy is in a medium range, which forces the economy to change the contractual mode before it reaches the steady state with the given mode. By contrast, if the productivity is high enough, the economy, even though for the initial periods its agents might have to work hard, will eventually converge to the steady state in which the agents all enjoy leisure and full insurance.

Appendix A: Interpretation of Mutual Insurance

Here we demonstrate that our modeling of mutual insurance captures both insurance and hedging in real life, which are the two main ways of obtaining insurance.

First, even though the mutual insurance contracts in the paper demand no ex ante insurance premium, they are equivalent to real-life insurance policies. Suppose, as in a real-life insurance contract, an agent in our paper pays to the MIC an insurance premium, Z, ex ante, namely out of his wage income in the first period, and he thus only has $I - Z$ of his own fund for investment; in exchange, the MIC repays him μZ units of the consumption good when his project fails and nothing when it succeeds. The MIC will invest their income of premiums in the projects of young agents because they are the only asset used for transferring wealth over time.[8] Thus, in equilibrium Z will go back to the agent's project. The investment by the MIC demands the fair return rate, $1/s$, where s is the probability of success. From this agent, the MIC gets $1/s \cdot Z$ units of capital, namely $1/s \cdot Z \cdot R$ units of the consumption good, when his project succeeds, and pays out to him μZ units of the consumption good when it fails. The zero profit condition thus commands that $s \times (1/s \cdot Z \cdot R) = (1 - s) \times \mu Z$. Thus $\mu = R/(1 - s)$. Then, the consumption of the agent is

$$C^g = (I - 1/s \cdot Z)R; \quad C^b = \mu Z = R/(1 - s) \cdot Z.$$

If we let $L \equiv R/(1-s) \cdot Z$, then the agent exactly gets the same consumption profile, given by (4) and (5), as he would get from mutual insurance contract in our paper $(L, \tau L)$, with $\tau = (1-s)/s$.

[8] Adding a risk free asset will not change anything because its return rate will be equalized to the marginal return rate of the investment in the projects.

Second, our modelling of mutual insurance also captures the gist of hedging. In the paper hedging could be carried out with following "futures" contracts. Out of the I units of investment, the agent hedges Z units, only allowing the remained $I - Z$ subject to the idiosyncratic risk of his project, and the future contract is that no matter what happens to his project, he gets from the MIC Z' units of capital. Since the project succeeds with probability s, the zero profit condition commands $Z' = s \cdot Z$. Thus, with such a future contract, the consumption of the agent is

$$C^g = (I - Z + Z')R = (I - (1-s)Z)R; \quad C^b = Z'R = sZR.$$

If we let $L \equiv sZR$, then this replicates the consumption profile with mutual insurance contract $(L, \tau L)$ of our paper, given by (4) and (5).

Appendix B: Proofs

Claim A1: When $U(\cdot) = \log(\cdot)$, $V^1(W,R) > V^S(W,R)$ if $\frac{(1-q)p}{(1-p)q} \geq e^{\frac{1}{p-q}}$.

Proof: It is sufficient to show that if $\frac{(1-q)p}{(1-p)q} > e^{\frac{1}{p-q}}$, the solution of problem (10) automatically satisfies the IC constraint, (11): If so, then both this problem and problem (12) can be unified as:

$$V^U(W,R;\tau) \equiv \max_{I,L} \log(W-I) + \beta p \log(IR - L\tau) + \beta(1-p)\log L - \theta I, \ s.t.(11);$$

$V^U(W,R;\tau = \frac{1-p}{p}) = V^1(W,R)$ and $V^U(W,R;\tau = \frac{1-q}{q}) = V^S(W,R)$; and as $V^U(W,R;\tau)$ decreases with τ by Envelop Theorem, $V^1(W,R) > V^S(W,R)$.

The first order conditions of problem (10) are:

$$\frac{\beta p R}{IR - L\frac{1-q}{q}} = \frac{1}{W-I} + \theta \tag{35}$$

$$\frac{1}{IR - L\frac{1-q}{q}} = \frac{1}{L} \cdot \frac{(1-p)q}{(1-q)p}. \tag{36}$$

The second equation implies that $L = (IR - L\frac{1-q}{q}) \cdot \frac{(1-p)q}{(1-q)p} \Leftrightarrow L = IR \cdot \frac{(1-p)q}{(1-q)p} - L \cdot \frac{1-p}{p} \Leftrightarrow L \cdot \frac{1}{p} = IR \cdot \frac{(1-p)q}{(1-q)p} \Leftrightarrow L = (1-p)IR \cdot \frac{q}{1-q}$. Therefore, $IR - L\frac{1-q}{q} = pIR$. Substitute it into the left hand side of (35) and note that its right hand side is greater than θ. We have: $\frac{\beta}{I} > \theta \Leftrightarrow$

$$1 > \frac{\theta I}{\beta}. \tag{37}$$

On the other hand, in case of logarithmic utility, the IC constraint, (11), is

$$\log \frac{IR - L\frac{1-q}{q}}{L} \geq \frac{\theta I}{\beta(p-q)}.$$

By (36), $\frac{IR - L\frac{1-q}{q}}{L} = \frac{(1-q)p}{(1-p)q}$, and by (37), $\frac{1}{p-q} > \frac{\theta I}{\beta(p-q)}$. Therefore, the above inequality, namely, the IC constraint, is implied by $\log \frac{(1-q)p}{(1-p)q} > \frac{1}{p-q} \Leftrightarrow \frac{(1-q)p}{(1-p)q} > e^{\frac{1}{p-q}}$. Q.E.D.

In general, compare between problem (10) and problem (12), the former concerning the agent working hard under the disincentivizing menu, $\{(L,(1-q)//q|0 \leq L\}$, the latter concerning the agent under the incentivizing menu, $\{(L,(1-q)//q|L$ subject to (11)$\}$. The difference is in the following two aspects, one to the advantage of the disincentivizing menu, the other to its disadvantage, but both related to the fact that with the menu the agent is paying the premium $(1-q)//q$. The disadvantage is that this premium is too high and not fair for the agent if he works hard: It is based on the probability of success being q, but his actual probability of success, given he works hard, is $p > q$. On the other hand, the advantage is that exactly because he is paying this premium, the MIC that he contracts with does not bother to impose an upper bound upon L, that is, the optimization problem (10) is not subject to the IC constraint, (11), whereas problem (12) is.

Given that the agent wants to work hard, the incentivizing menu dominates the disincentivizing menu, namely, $V^1(W,R) > V^S(W,R)$, if θ is small enough or $p-q$ is large enough. On the one hand, for any $q < p$, if $\theta = 0$, then, by the comparison expounded above, the IC constraint is never binding, and thus the disincentivizing menu loses its advantage of not being subject to the constraint, but suffers the disadvantage of paying the too high premium. On the other hand, given θ, if $p-q$ is large enough, then the cost of paying the too high premium is large enough, and in particular if $q \to 0$, the premium $\frac{1-q}{q} \to \infty$, then the cost goes to infinity. Thus in both cases, the disincentivizing menu is dominated.

Proof of Lemma 1:

Proof. We prove the lemma by solving problem (9). The problem is replicated below:

$$V^0(W,R) \equiv \max_{I,L} U(W-I) + \beta q U(IR - \frac{1-q}{q}L) + \beta(1-q)U(L).$$

The first order conditions (FOCs) of the problem are:

$$q \cdot \frac{1-q}{q} \cdot U'(IR - \frac{1-q}{q}L) = (1-q) \cdot U'(L) \quad (38)$$

$$U'(W-I) = \beta q R U'(IR - \frac{1-q}{q}L). \quad (39)$$

Equation (38) captures the consumption smoothing across the two future contingencies. Equation (39), on the other hand, captures the inter-temporal consumption smoothing.[9]

By (38), $IR - \frac{1-q}{q}L = L$, that is, $C^g = C^b$, as the left hand side (LHS) is C^g, the right hand side (RHS) C^b. From the equation, $L = qIR$.

With $L = qIR$, (39) becomes (14). ■

Proof of Lemma 2:

To prove the lemma, we first find the first order conditions of problem (12) for a general utility function as below, which will be used to prove some lemmas later. Then we apply them to the case of log utility.

Lemma A1: For the solution of problem (12), the IC constraint is binding, that is,

$$U(IR - L\frac{1-p}{p}) - U(L) = \delta I. \quad (40)$$

Furthermore,

$$\frac{1}{U'(C^g)} = R[\frac{q\beta(1-\phi)}{U'(W-I)} + \frac{\phi}{\delta}] \quad (41)$$

$$\frac{1}{U'(C^b)} = \frac{pR}{1-p}[\frac{(1-q)\beta(1-\phi)}{U'(W-I)} - \frac{\phi}{\delta}] \quad (42)$$

where $C^g = IR - L\frac{1-p}{p}$ and $C^b = L$; and

$$\phi \equiv 1 - \frac{r}{pR} \quad (43)$$

$$r \equiv (\frac{p}{\beta U'(IR - L\frac{1-p}{p})} + \frac{1-p}{\beta U'(L)})U'(W-I). \quad (44)$$

Moreover,

$$0 < \phi < 1. \quad (45)$$

[9] Hence, the choice of I, the scale of investment projects, essentially mirrors the decisions on saving.

Proof: Let λ be the multiplier for the IC constraint. Then the first order conditions of the problem for (I, L, λ) include:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial I} &= -U'(W-I) + (\beta p + \lambda) R U'(C^g) - (\theta + \lambda\delta) = 0 \\ \frac{\partial \mathcal{L}}{\partial L} &= -(\beta p + \lambda)\frac{1-p}{p} U'(C^g) + (\beta(1-p) - \lambda) U'(L) = 0.\end{aligned}$$

First, from $\frac{\partial \mathcal{L}}{\partial L} = 0$, we find

$$\lambda = \frac{\beta p(1-p)(U'(L) - U'(C^g))}{(1-p)U'(C^g) + pU'(L)}. \tag{46}$$

Since $C^b = L$ and $C^b < C^g$ as implied by the IC constraint, we have $\lambda > 0$. Therefore, the IC constraint is binding, which gives rise to (40). Intuitively, the risk averse agents, when facing the fair insurance price, want to smooth consumption across the two contingencies as much as possible, until the IC constraint is binding.

Second, from (46), $\beta p + \lambda = \frac{\beta p U'(L)}{(1-p)U'(C^g) + pU'(L)}$. And $\theta + \lambda\delta|_{\theta = \beta\delta(p-q)} = \frac{\beta\delta}{(1-p)U'(C^g) + pU'(L)}(p(1-q)U'(L) - q(1-p)U'(C^g))$. Substitute all these into $\frac{\partial \mathcal{L}}{\partial I} = 0$, rearrange, and we find:

$$U'(W-I) + \frac{\beta\delta(p(1-q)U'(L) - q(1-p)U'(C^g))}{(1-p)U'(C^g) + pU'(L)} = \frac{\beta p R U'(L) U'(C^g)}{(1-p)U'(C^g) + pU'(L)}$$

Multiple both sides by $\frac{(1-p)U'(C^g) + pU'(L)}{\beta U'(L) U'(C^g)}$, rearrange, and remember $C^g = IR - L\frac{1-p}{p}$, then we get*:*

$$pR + \frac{q(1-p)\delta}{U'(L)} = \frac{p(1-q)\delta}{U'(IR - L\frac{1-p}{p})} + \left(\frac{p}{\beta U'(IR - L\frac{1-p}{p})} + \frac{1-p}{\beta U'(L)}\right)U'(W-I). \tag{47}$$

Equation (47) is quite complex. The way to handle it is to split it into two equations by introducing r as is defined by (44), which then reproduces the following Rogerson's equation (see Rogerson 1985):

$$\frac{1}{U'(C)} = \frac{1}{\beta r} E \frac{1}{U'(C_+)},$$

where C_+ denotes the random variable of the old period consumption. Expressed with ϕ, Rogerson's equation and (47) respectively becomes

$$\begin{aligned}\frac{\beta(1-\phi)pR}{U'(C)} &= \frac{p}{U'(C^g)} + \frac{1-p}{U'(C^b)} \\ \phi p R &= \frac{p(1-q)\delta}{U'(C^g)} - \frac{q(1-p)\delta}{U'(C^b)}\end{aligned} \tag{48}$$

Both equations are linear with $\frac{1}{U'(C^g)}$ and $\frac{1}{U'(C^b)}$. Solve them out and we have (41) and (42).

Finally, by definition, $r > 0$, which implies $\phi < 1$. Moreover by (48), $\phi pR = \frac{p(1-q)\delta}{U'(C^g)} - \frac{q(1-p)\delta}{U'(C^b)} > \frac{\delta}{U'(C^g)}(p(1-q) - q(1-p)) = \frac{\delta}{U'(C^g)}(p-q) > 0$. Therefore, $0 < \phi < 1$. Q.E.D.

Now we come to prove Lemma 2 below.

Proof. (i) *When* $U(\cdot) = \log(\cdot)$, then, $U'(C) = \frac{1}{C}$. By (44), $r = \frac{I}{\beta(W-I)}pR$ and by (43), $\phi = \frac{W - \frac{1+\beta}{\beta}I}{W-I}$. With this ϕ, $\frac{\beta(1-\phi)}{U'(W-I)} = \beta(1-\phi)(W-I) = I$. Substituting this into (41) and (42), we have

$$C^g = R(qI + \frac{\phi}{\delta}) \tag{49}$$

$$C^b = \frac{pR}{1-p}[(1-q)I - \frac{\phi}{\delta}]. \tag{50}$$

Substitute these two equations into the binding IC constraint, (40), which, with logarithmic utility, is equivalent to $\frac{C^g}{C^b} = e^{\delta I}$. Then, the agents' optimal choice of I at given R, namely $\widetilde{I}^1(W,R)$, is determined by:

$$\frac{1-p}{p} \cdot \frac{qI + \frac{\phi}{\delta}}{(1-q)I - \frac{\phi}{\delta}} = e^{\delta I},$$

with $\phi = \frac{W - \frac{1+\beta}{\beta}I}{W-I}$. Since ϕ is independent of R, so is $\widetilde{I}^1(W,R)$.

By (50) $C^b = \frac{pR}{1-p}[(1-q)I - \frac{\phi}{\delta}]$, by the binding the IC constraint $\frac{C^g}{C^b} = e^{\delta I^1}$, and $\beta p\delta - \theta = \beta q\delta$. Substitute all these into the formula for $V^1(W,R)$, and we find:

$$V^1(W,R) = \log\{(W - I^1)[\frac{p}{1-p}((1-q)I^1 - \frac{\phi}{\delta})]^\beta e^{\beta q\delta I^1} R^\beta\}.$$

(ii): To show $\widetilde{I}^1(W)$ is increasing, let $F(I,\phi) \equiv \frac{qI+\frac{\phi}{\delta}}{(1-q)I-\frac{\phi}{\delta}}$. Then $\widetilde{I}^1(W)$ is implicitly defined by $F(I,\phi(I,W)) - \frac{p}{1-p}e^{\delta I} = 0$, with $\phi(I,W) = \frac{W-\frac{1+\beta}{\beta}I}{W-I}$. By implicit function theorem, $\widetilde{I}^{1\prime}(W) = \frac{-F_\phi \phi_W}{F_I + F_\phi \phi_I - \frac{p}{1-p}\delta e^{\delta I}}$. Note that $F_\phi > 0$, $F_I < 0$, $\phi_I < 0$ (for $I < W$), and $\phi_W > 0$. Therefore, the nominator $-F_\phi \phi_W < 0$ and the denominator $F_I + F_\phi \phi_I - \frac{p}{1-p}\delta e^{\delta I} < F_I + F_\phi \phi_I < F_\phi \phi_I < 0$. It follows that the fraction for $\widetilde{I}^{1\prime}(W)$ is positive.

To show $\widetilde{I}^1(W) < \frac{\beta}{1+\beta}W$, note that we saw $\phi > 0$ in (45), and for log utility, $\phi = \frac{W-\frac{1+\beta}{\beta}I}{W-I}$. And certainly $W > I$ (i.e the young period consumption $C = W - I > 0$). Altogether we have $W - \frac{1+\beta}{\beta}\widetilde{I}^1 > 0 \Leftrightarrow \widetilde{I}^1 < \frac{\beta}{1+\beta}W$. ■

Proof of Proposition 1:

Proof. To prove the existence of the root of (20), it suffices to show that (a) when $W \to \infty$, the left hand side (LHS) of (20) dominates its right hand side (RHS); and (b) when $W \to 0$, the RHS dominates the LHS.

(a) is simple. When $W \to \infty$, by Lemma 3 (proved below), I^1 goes to some finite I_∞. Therefore, the RHS increases with W in the speed of W, while the LHS increases in the speed of $W^{1+\beta}$ which dominates the RHS.

We show (b) in four steps. Step 1, by Lemma 3, when $W \to 0$, $I^1 < \frac{\beta}{1+\beta}W$ also goes to 0.

Step 2, we prove $\lim_{W\to 0} \frac{\phi}{I^1} = \delta(p-q)$. As $I^1 \to 0$, the RHS of (18), which determines $I^1(W)$, converges to 1. By this equation, then, $\chi \equiv \frac{\phi}{I^1}$ converges to the root of

$$\frac{1-p}{p} \cdot \frac{q + \frac{\chi}{\delta}}{(1-q) - \frac{\chi}{\delta}} = 1,$$

which is $\delta(p-q)$.

Step 3, as $I^1 \to 0$ and $\frac{\phi}{I^1} \to \delta(p-q)$, we have $\phi \to 0$. Since $\phi = \frac{W - \frac{1+\beta}{\beta}I}{W-I}$, it follows that $I^1 = \frac{\beta}{1+\beta}W + o(W)$.

Step 4, since $e^{\beta q \delta I^1} \approx 1$, $(1-q)I^1 - \frac{\phi}{\delta}|_{\frac{\phi}{I^1}\approx\delta(p-q)} \approx (1-p)I^1$ and $I^1 \approx \frac{\beta}{1+\beta}W$, the RHS of (20), $(W - I^1)[\frac{p}{1-p}((1-q)I^1 - \frac{\phi}{\delta})]^\beta e^{\beta q \delta I^1} \approx W^{1+\beta}p^\beta \frac{\beta^\beta}{(1+\beta)^{1+\beta}} > W^{1+\beta}q^\beta \frac{\beta^\beta}{(1+\beta)^{1+\beta}}$, the LHS term.

Therefore, the root of (20), $\widehat{W}$, exists. Moreover, $\widehat{W}$ is independent of A, because $I^1(W)$, determined by (18) is independent of A and thus so is the whole equation of (20).

The argument above also shows that if $W > \widehat{W}$, the LHS of (20) dominates the RHS and thus $V^0(W_t, R_{t+1}) > V^1(W_t, R_{t+1})$; and if $W < \widehat{W}$, the RHS of (20) dominates the LHS and thus $V^0(W_t, R_{t+1}) < V^1(W_t, R_{t+1})$. ■

Proof of Lemma 3:

Proof. (a): We showed in Lemma 2 that $\widetilde{I}^1(W)$ is increasing. Thus $\lim_{W\to\infty} \widetilde{I}^1(W)$ exists and is either finite or infinity. Suppose $\widetilde{I}^1 \to \infty$. Then, since ϕ is always between 0 and 1, the left hand side of (18) converges to $\frac{1-p}{p} \cdot \frac{q}{1-q} < 1$, whereas the right hand side goes to $+\infty$, a contradiction. Therefore, $\lim_{W\to\infty} \widetilde{I}^1 < \infty$. It follows that $\lim_{W\to\infty} \phi = \lim_{W\to\infty} \frac{W - \frac{1+\beta}{\beta}I}{W-I} = 1$. Then equation (18) converges to

$$\frac{1-p}{p} \cdot \frac{qI + \frac{1}{\delta}}{(1-q)I - \frac{1}{\delta}} = e^{\delta I}, \tag{51}$$

which thus determines $I_\infty \equiv \lim_{W\to\infty} I^1$.

(b): Both C^g and C^b increase with W. Thus $C^g < \lim_{W\to\infty} C^g$ and $C^b < \lim_{W\to\infty} C^b$. By (49) $C^g = R(qI + \frac{\phi}{\delta})$ and then $\lim_{W\to\infty} C^g = R_\infty(qI_\infty + \frac{1}{\delta})$, where R_∞ is the interest rate in the next period when all the agents invest I_∞ and work hard. Thus it is a positive, finite number. Similarly, by (50) $C^b = \frac{pR}{1-p}[(1-q)I - \frac{\phi}{\delta}]$ and then $\lim_{W\to\infty} C^b = \frac{pR_\infty}{1-p}[(1-q)I_\infty - \frac{1}{\delta}]$. ■

Proof of Proposition 2:

Proof. (i) All the claims concerning $M^0(K) = \frac{q\beta(1-\alpha)A}{1+\beta}K^\alpha$ is self evident: it is increasing and concave and has a unique fixed point. Only the claims concerning $M^1(K)$ needs proof. Note $M^1(K) = pI^1 \circ W(K)$, where $W(K) = A(1-\alpha)K^\alpha$ is increasing and concave and $I^1(W)$ is determined by (18). Since a compound of increasing functions is increasing and a compound of concave functions is concave, in order to prove $M^1(K)$ is increasing and concave, it suffices to show $I^1(W)$ is increasing and concave. That it is increasing has been proved in Lemma 2. We show it is concave here. That is equivalent to that its inverse function $W(I)$, which is found out explicitly below, is convex. As $\phi = (W - \frac{1+\beta}{\beta}I)/(W-I)$, we find

$$W = \frac{(1+\beta)/\beta - \phi}{1-\phi} I \equiv h(\phi)I,$$

where ϕ as a function of I is found from (18):

$$\phi(I) = \delta\frac{(1-q)\lambda(I) - q}{1+\lambda(I)} I \equiv \delta g(\lambda) I, \tag{52}$$

with

$$\lambda(I) \equiv \frac{p}{1-p} e^{\delta I}.$$

Claim: $f(x)x$ is convex over $x > 0$ if $f' > 0$ and $f'' > 0$.

Proof: $[f(x)x]'' = f''x + 2f' > 0$ if $f' > 0$ and $f'' > 0$.

By the claim, to show $W(I)$ is convex, it suffices to show $h(\phi(I))$ is increasing and convex, which, because $h(\phi) \equiv \frac{(1+\beta)/\beta - \phi}{1-\phi}$ is increasing and convex with ϕ over $\phi \in (0,1)$, follows from $\phi(I)$ being increasing and convex, which is shown in order below.

$\frac{\phi'(I)}{\delta} = g + I \cdot g(\lambda(I))' > 0$ since $g(\lambda(I))' = g'(\lambda)\lambda'(I) = \frac{1}{(1+\lambda)^2} \cdot \delta\lambda > 0$.

$\frac{\phi''(I)}{\delta} = I \cdot [g''(\lambda)(\lambda'(I))^2 + g'(\lambda)\lambda''(I)] + 2g'(\lambda)\lambda'(I)|_{\lambda'=\delta\lambda;\lambda''=\delta^2\lambda} = I \cdot [g''(\lambda)(\delta\lambda)^2 + g'(\lambda)\delta^2\lambda] + 2g'(\lambda)\delta\lambda > 0|_{2g'=-(1+\lambda)g''>0} \Leftrightarrow I \cdot [\frac{-2}{1+\lambda}(\delta\lambda)^2 + \delta^2\lambda] + 2\delta\lambda > 0 \Leftrightarrow I \cdot [\frac{-2}{1+\lambda}\delta\lambda + \delta] + 2 > 0 \Leftrightarrow$

$$\delta I < \frac{2(\lambda+1)}{\lambda - 1}. \tag{53}$$

To prove this inequality, note that by (52) and $\phi < 1$, we have $\delta I < \frac{1+\lambda}{(1-q)\lambda-q}$. Therefore, (53) follows from $\frac{1+\lambda}{(1-q)\lambda-q} \leq \frac{2(\lambda+1)}{\lambda-1} \Leftrightarrow \lambda - 1 \leq 2(1-q)\lambda - 2q \Leftrightarrow 0 \leq (1-2q)(1+\lambda)$, which holds true if $q \leq 0.5$.

We now prove $M^1(K)$ has a unique non-zero fixed point, or equivalently, $f(K) \equiv M^1(K) - K$ has a unique non-zero root. To show its existence, note that $\lim_{K\to\infty} f(K) < 0$ because $\lim_{W\to\infty} I^1 < \infty$. On the other hand, $\lim_{K\to 0} f(K) > 0$: in the proof of Proposition 1 we show $I^1 = \frac{\beta}{1+\beta} W + o(W)$ at $W \approx 0$; then at $K \approx 0$, $f(K) = pI^1 \circ W(K) - K \approx p\frac{\beta}{1+\beta} \cdot A(1-\alpha)K^\alpha - K > 0 \Leftrightarrow p\frac{\beta}{1+\beta} \cdot A(1-\alpha)K^{\alpha-1} > 1$, which holds true since $K^{\alpha-1} \to \infty$ if $K \to 0$. The uniqueness of non-zero root comes from the fact that any concave function has at most two roots and $f(K)$ is concave (since M^1 is concave) and another root of f is 0.

The global stability of the steady state in the DIM or the IM follows from the uniqueness of the steady state and concavity of the dynamics.

(ii): It is straightforward that $K^0 = (\frac{\beta(1-\alpha)qA}{1+\beta})^{\frac{1}{1-\alpha}}$ increases with A. By the argument for the unique existence of K^1 above, $f(K) > 0$ for $K < K^1$ and $f(K) < 0$ for $K > K^1$, which implies $f'(K^1) < 0$.Then, applying the implicit function theorem to $M^1(K, A) - K = 0$, we have $dK^1/dA = -\frac{\partial M^1}{\partial A}/f'(K^1) > 0$, since $\frac{\partial M^1}{\partial A} = p \cdot dI^1/dW \cdot (1-\alpha)K^\alpha > 0$. ■

Proof of Lemma 4:

Proof. Let us establish the relative position of between K^0 and K^1 first. When δ (namely θ) equals 0, the IC constraint will not be binding and the dynamics of the IM will collapse into the dynamics of the DIM, except that the probability of success is p instead of q. Therefore, $\lim_{\delta\to 0} M^1(K_t; \delta, p) = M^0(K_t, p)$ and $\lim_{\delta\to 0} K^1(\delta, p) = K^0(p)$. When $U(\cdot) = \log(\cdot)$, $M^0(K_t, q)$ is given by (22) and $K^0(q)$ by (23), both increasing with q. And we know $M^1(K_t; \delta, p)$ and $K^1(\delta, p)$ are both continuous with δ. Thus,

$$\lim_{\delta\to 0} M^1(K_t; \delta, p) = \frac{\beta(1-\alpha)pA}{1+\beta}(K_t)^\alpha > M^0(K_t, q); \tag{54}$$

$$\lim_{\delta\to 0} K^1(\delta, p) = (\frac{\beta(1-\alpha)pA}{1+\beta})^{\frac{1}{1-\alpha}} > K^0(q). \tag{55}$$

Furthermore, the higher is K_t, the slower is the convergence in (54), because a higher K_t gives advantage to the DIM and thus makes $M^1(K_t; \delta, p)$ less likely sit above $M^0(K_t, q)$. In mathematical terms, if $|M^1(K; \delta, p) - M^0(K, p)| < \varepsilon$ at a given K when $\delta < \epsilon$, then for $K_t < K$, $|M^1(K_t; \delta, p) - M^0(K_t, p)| < \varepsilon$ when $\delta < \epsilon$. Therefore, given a bound K, say $K^0(p)$, there exists

$\overline{\delta}$, such that if $\delta < \overline{\delta}$, $M^1(K_t; \delta, p) > M^0(K_t, q)$ for any $K_t < K$ and $K^1(\delta, p) > K^0(q)$.

Now we show how to have $\widehat{K}$ sit between K^0 and K^1. All the three are functions of A. By (23) $K^0 = (\frac{\beta(1-\alpha)qA}{1+\beta})^{\frac{1}{1-\alpha}}$; by (21) $\widehat{K} = (\frac{\widehat{W}}{A(1-\alpha)})^{\frac{1}{\alpha}}$, with $\widehat{W}$ independent of A; and K^1 as a function of A is implicitly defined by (25).

Let $\underline{A}$ be the root of $\widehat{K} = K^1$, namely,

$$(\frac{\widehat{W}}{\underline{A}(1-\alpha)})^{\frac{1}{\alpha}} = K^1(\underline{A}).$$

And $\overline{A}$ be the root of $\widehat{K} = K^0$, namely,

$$(\frac{\widehat{W}}{\overline{A}(1-\alpha)})^{\frac{1}{\alpha}} = (\frac{\beta(1-\alpha)q\overline{A}}{1+\beta})^{\frac{1}{1-\alpha}}.$$

Then we have two observations:

First, both roots exist uniquely, because by Proposition 1, $\widehat{K}(A)$ decreases with A in the order of $A^{-\frac{1}{\alpha}}$, whereas by $K^0(A)$ increases with A in the order of $A^{\frac{1}{1-\alpha}}$ and $K^1(A)$ also increases with A and to infinity with $A \to \infty$.

Second, $\underline{A} < \overline{A}$, because $K^1 > K^0$.

Therefore, if $\underline{A} < A < \overline{A}$, then due to the decreasing of $\widehat{K}$ with A, $\widehat{K}(\underline{A}) > \widehat{K}(A) > \widehat{K}(\overline{A})$, equivalent to $K^1 > \widehat{K} > K^0$ by the definition of $\underline{A}$ and $\overline{A}$. ■

Proof of Proposition 3:

Proof. The proposition depends on following lemmas, which we prove first before proceeding to the proof of the proposition.

Lemma P1: If $K_{t+1} > \widehat{K} > K_t$, then $K_{t+2} < \widehat{K}$.

Lemma P2: If $K_{t+1} < \widehat{K} < K_t$, then $K_{t+2} > \widehat{K}$.

Proof: For Lemma P1, note that as $\widehat{K} > K_t$, the dynamics applicable to K_t are the dynamics of the IM, M^1, and hence $K_{t+1} = M^1(K_t)$ and as $K_{t+1} > \widehat{K}$, the dynamics applicable to K_{t+1} are M^0 and hence $K_{t+2} = M^0(K_{t+1})$. Therefore, $K_{t+2} = M^0 \circ M^1(K_t)$, where $M^0 \circ M^1(\cdot) \equiv M^0(M^1(\cdot))$ is the compound of the two functions. $M^0 \circ M^1(\cdot)$ is increasing, because both $M^0(\cdot)$ and $M^1(\cdot)$ are increasing. As $K_t < \widehat{K}$, therefore, $K_{t+2} = M^0 \circ M^1(K_t) < M^0 \circ M^1(\widehat{K})$. Then, to prove $K_{t+2} < \widehat{K}$, it suffices to prove that $M^0 \circ M^1(\widehat{K}) < \widehat{K}$. For this inequality, just note that $\underline{K}$ is the unique and stable steady state of dynamics $x_{t+1} = M^0 \circ M^1(x_t)$ and $\widehat{K} > \underline{K}$ and sits on the declining path, which altogether imply $M^0 \circ M^1(\widehat{K}) < \widehat{K}$.

For Lemma P2, if $K_{t+1} < \widehat{K} < K_t$, by a parallel argument, $K_{t+2} = M^1 \circ M^0(K_t)$, and, as $K_t > \widehat{K}$, we have $K_{t+2} > M^1 \circ M^0(\widehat{K})$. To prove the lemma, it suffices to prove $M^1 \circ M^0(\widehat{K}) > \widehat{K}$. This inequality holds true, similarly, because $\overline{K}$ is the unique steady state of $M^1 \circ M^0(\cdot)$ and $\widehat{K} < \overline{K}$ and thus sits on the rising path. Q.E.D.

We now come to prove the proposition. We consider only the case where $K_0 < \widehat{K}$; the case where $K_0 \geq \widehat{K}$ can be proved in a parallel way. Given $K_0 < \widehat{K} < K^1$, the capital stock first follows the dynamics of the IM and increasingly converges to the steady state K^1. Since $K^1 > \widehat{K}$, there must exists a time T such that $K_{T+1} > \widehat{K} > K_T$, namely, the contractual regime switches at T. Then, by Lemma P1, $K_{T+2} < \widehat{K}$. We have already shown that $\widehat{K} < K_{T+1}$. Therefore, $K_{T+2} < \widehat{K} < K_{T+1}$. Then, by lemma P2, $K_{T+3} > \widehat{K}$, which together with $\widehat{K} > K_{T+2}$ in turn, by Lemma P1, implies $K_{T+4} < \widehat{K}$. By this line of reasoning, we have $\{K_T, K_{T+2}, K_{T+4}, ...\}$ on the left hand side of $\widehat{K}$ and $\{K_{T+1}, K_{T+3}, K_{T+5}, ...\}$ on its right hand side. Moreover, the former sequence, $\{K_{T+2n}\}_{n=0,1,2,...}$ follows the dynamics of $M^0 \circ M^1(\cdot)$ which has a unique and stable steady state, $\underline{K}$. Therefore, $K_{T+2n} \rightarrow \underline{K}$ with $n \rightarrow \infty$. Similar, the latter sequence, $\{K_{T+2n+1}\}_{n=0,1,2,...}$ follows the dynamics of $M^1 \circ M^0$, which has a unique and stable steady state $\overline{K}$. Therefore, $K_{T+2n+1} \rightarrow \underline{K}$ with $n \rightarrow \infty$. ■

Proof of Proposition 4:

Proof. The boom is in the DIM and thus by (16), $\overline{I} = \frac{\beta}{1+\beta}\overline{W}$, while the bust in the IM and thus by Lemma 2(ii) $\underline{I} < \frac{\beta}{1+\beta}\underline{W}$. Thus, $\overline{I}/\overline{W} = \frac{\beta}{1+\beta} > \underline{I}/\underline{W}$, which is (ii). For (i), $\overline{I} > \underline{I}$ follows from $\overline{W} > \underline{W}$, which follows from $\overline{K} > \underline{K}$. Moreover, since the boom is followed by the bust, we have $q\overline{I} = \underline{K}$, and the bust is followed by the boom, thus $p\underline{I} = \overline{K}$. Therefore $q\overline{I} = \underline{K} < \overline{K} = p\underline{I}$.
■

Proof Lemma 5:

Proof. For the first assertion, just note that $\frac{dK_{t+1}}{dK_t} = \frac{dK_{t+1}}{dI^0} \cdot \frac{dI^0}{dW} \cdot \frac{dW}{dK_t}$ and all the derivatives on the RHS are positive.

For the second assertion: Note that $\widetilde{I}^0(W_t, R_{t+1})$ for a general utility function is determined by (14), which, with $W_t = A(1-\alpha)K_t^\alpha$ and $R_{t+1} = A\alpha(K_{t+1})^{\alpha-1}$, is equivalent to

$$U'(A(1-\alpha)K_t^\alpha - \frac{K_{t+1}}{q}) = \beta q A\alpha(K_{t+1})^{\alpha-1}U'(A\alpha(K_{t+1})^\alpha).$$

Let $F(\cdot)$ be the inverse function of $U'(\cdot)$. Then, $A(1-\alpha)K_t^\alpha = \frac{K_{t+1}}{q} + F(\beta q A\alpha(K_{t+1})^{\alpha-1}U'(A\alpha(K_{t+1})^\alpha))$.

With $U'(c) = c^{-\sigma}$ and thus $F(y) = y^{-\frac{1}{\sigma}}$, it follows that

$$A(1-\alpha)K_t^\alpha = \frac{K_{t+1}}{q} + (\beta q)^{-\frac{1}{\sigma}} (A\alpha)^{\frac{\sigma-1}{\sigma}} (K_{t+1})^{\alpha+\frac{1-\alpha}{\sigma}} \tag{56}$$

If $K_t \to 0$, then $K_{t+1} \to 0$ and the the RHS of (56) is in the order of $\xi = \min(1, \alpha + \frac{1-\alpha}{\sigma})$. Obviously $\xi > \alpha$. Therefore, at $K_t \approx 0$, $K_{t+1} \approx (K_t)^{\frac{\alpha}{\xi}}$ and thus $\frac{dK_{t+1}}{dK_t} = (K_t)^{\frac{\alpha}{\xi}-1} \to \infty$.

The third assertion, $\partial K_{t+1}/\partial q > 0$, is derived by applying the implicit function theorem to (56).

For the last assertion, substitute $K_{t+1} = K_t = K^0$ into (56) and we find the steady state K^0 is determined by

$$A(1-\alpha)(K^0)^\alpha = \frac{K^0}{q} + (\beta q)^{-\frac{1}{\sigma}} (A\alpha)^{\frac{\sigma-1}{\sigma}} (K^0)^{\alpha+\frac{1-\alpha}{\sigma}}.$$

Divided on both sides by $(K^0)^\alpha$ and with $x = \frac{(K^0)^{1-\alpha}}{\alpha q A}$ and proper rearrangement, the equation is equivalent to

$$\frac{1-\alpha}{\alpha} - x = (\beta)^{-\frac{1}{\sigma}} x^{\frac{1}{\sigma}}.$$

There is a unique solution of x, because the LHS decreases with x while the RHS increases with it. The solution depends on α, β, and σ, but is independent of q and A, denoted by $x(\alpha, \beta, \sigma)$. Then, $(K^0)^{1-\alpha} = x(\alpha, \beta, \sigma)\alpha q A \Leftrightarrow K^0 = (x(\alpha, \beta, \sigma)\alpha)^{\frac{1}{1-\alpha}} (qA)^{\frac{1}{1-\alpha}} \equiv T(\alpha, \beta, \sigma)(qA)^{\frac{1}{1-\alpha}}$. Certainly, $x(\alpha, \beta, \sigma) > 0$. For the upper bound, note that by the equation, $\frac{1-\alpha}{\alpha} - x > 0 \Leftrightarrow x < \frac{1-\alpha}{\alpha} \Leftrightarrow K^0 < (A(1-\alpha)q)^{\frac{1}{1-\alpha}}$. ■

Proof of Lemma 6:

Proof. As $I^1(W) = \widetilde{I}^1(W, R^1(W))$, to prove the lemma, below in (57) of Lemma A9, we give an upper bound on $\widetilde{I}^1(W, R)$, the agents' optimal choice of I when choosing the incentivizing menu under given W and R.

The inequality (57) cannot hold if $\lim_{W\to\infty} I^1 = \infty$. In the IM, $R^1 = A\alpha(pI^1)^{\alpha-1}$. If $I^1 \to \infty$, then $R^1 \to 0$; and with $W \to \infty$, C^b is lower bounded, and so is $\frac{1}{U'(C^b)}$, which implies that $(U \circ D)'(\frac{1}{U'(C^b)})$ is upper-bounded since $(U \circ D)'$ is decreasing as $U \circ D$ is concave. These altogether imply that $(U \circ D)'(\frac{1}{U'(C^b)}) \cdot \frac{R^1}{\delta(1-p)} \to 0$ and cannot be bigger than δI^1 if it goes to infinity, a contradiction to (57). Hence, we prove the first assertion of the proposition.

As for the rest part, note that by (4) $C^g < I^1 R^1$ and $C^b < C^g < I^1 R^1$. Since $\lim_{W\to\infty} I^1$ is finite, $\lim_{W\to\infty} R^1$ is positive and finite, and therefore so are $\lim_{W\to\infty} C^g$ and $\lim_{W\to\infty} C^b$. ■

Lemma A9:

$$\delta \widetilde{I}^1(W,R) < (U\circ D)'(\frac{1}{U'(C^b)})\cdot\frac{R}{\delta(1-p)}. \quad (57)$$

Proof. Let $D(\cdot)$ be the inverse function of $\frac{1}{U'(\cdot)}$. Then, from (41) and (42), we have $C^g = D(\frac{q\beta(1-\phi)R}{U'(W-I)}+\frac{R\phi}{\delta})$ and $C^b = D(\frac{p}{1-p}(\frac{(1-q)\beta(1-\phi)R}{U'(W-I)}-\frac{R\phi}{\delta}))$. Substitute them into the binding IC constraint, $U(C^g)-U(C^b)=\delta I$, and we have

$$U\circ D(\frac{q\beta(1-\phi)R}{U'(W-I)}+\frac{R\phi}{\delta}) - U\circ D(\frac{p}{1-p}(\frac{(1-q)\beta(1-\phi)R}{U'(W-I)}-\frac{R\phi}{\delta})) = \delta I.$$

If f is concave and increasing and $\Delta>0$, then $f(x+\Delta)-f(x)<f'(x)\Delta$. Apply this inequality to the LHS of the equation above and let $Q\equiv\frac{\beta(1-\phi)R}{U'(W-I^1)}$, so $Q>0$. Then, $\delta I < (U\circ D)'(\frac{1}{U'(C^b)})\cdot[(q-\frac{p(1-q)}{1-p})Q+\frac{R\phi}{\delta}(1+\frac{p}{1-p})] = (U\circ D)'(\frac{1}{U'(C^b)})\cdot[\frac{q-p}{1-p}Q+\frac{R\phi}{\delta(1-p)}]|_{q<p} < (U\circ D)'(\frac{1}{U'(C^b)})\cdot\frac{R\phi}{\delta(1-p)}|_{\phi<1} < (U\circ D)'(\frac{1}{U'(C^b)})\cdot\frac{R}{\delta(1-p)}$. ■

Proof of Lemma 7:

Proof. It suffices to prove that both W^{01} and W^{10} increases with A no faster than A with power 1. Note that $V^1(W,R)=U(W-I^1)+\beta pU(C^g)+\beta(1-p)U(C^b)-\theta I^1 < U(W)+\beta U(pC^g+(1-p)C^b) = U(W)+\beta U(I^1\cdot pR)$, as $C^g = IR-L\frac{1-p}{p}$ and $C^b=L$. Hence, if $V^0(W,R)>U(W)+\beta U(I^1\cdot pR)$, then $V^0(W,R)>V^1(W,R)$.

Let $W(a,b)$ defined by:

$$\max_S U(W-S)+\beta U(Sa) = U(W)+\beta U(b)$$

Note that $V^0(W,R)=\max_S U(W-S)+\beta U(S\cdot qR)$. Therefore, $W(qR,I^1pR)$ is the threshold for

$$V^0(W,R)=U(W)+\beta U(I^1\cdot pR).$$

$\frac{\partial W(a,b)}{\partial a}<0$ and $\frac{\partial W(a,b)}{\partial b}>0$: the higher is the saving rate, the smaller is the threshold beyond which the value from the optimal saving dominates; and the greater the old period consumption without saving, the higher the threshold.

With $W(a,b)$ so defined, we have $W^{10} < W(qR^1, I^1pR^1)|_{R^1=Aa(pI^1)^{\alpha-1}} = W(qR^1, Aa(pI^1)^\alpha) < W(qR_\infty, Aa(pI_\infty)^\alpha)$ and $W^{01} < W(qR^0(W), I^1pR^0(W))$. To prove the lemma, it suffices to show that both upper bounds increases with A in a speed no faster than A with power 1. For this purpose, let us first find the formula for $W(a,b)$.

Claim A10-1: When $U(c)=\frac{c^{1-\sigma}-1}{1-\sigma}$, $W(a,b)=(\frac{\xi(a,\sigma)-1}{\beta})^{\frac{1}{\sigma-1}}b$, where $\xi(a,\sigma)=(1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}})^\sigma$.

Proof: For $U(c)=\frac{c^{1-\sigma}-1}{1-\sigma}$, the solution of $\max_S U(W-S)+\beta U(Sa)$ bears $W-S=\frac{1}{1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}}}W$ and $Sa=\frac{\beta^{\frac{1}{\sigma}}a^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}}}W$. Then $W(a,b)$ is determined by $(\frac{1}{1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}}}W)^{1-\sigma}+\beta\left(\frac{\beta^{\frac{1}{\sigma}}a^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}}}W\right)^{1-\sigma}=W^{1-\sigma}+\beta b^{1-\sigma}\Leftrightarrow(\frac{1}{1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}}})^{1-\sigma}+\beta\left(\frac{\beta^{\frac{1}{\sigma}}a^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}}}\right)^{1-\sigma}=1+\beta(\frac{b}{W})^{1-\sigma}\Leftrightarrow W(a,b)=(\frac{\xi(a)-1}{\beta})^{\frac{1}{\sigma-1}}b$, where $\xi(a,\sigma)=(\frac{1}{1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}}})^{1-\sigma}+\beta\left(\frac{\beta^{\frac{1}{\sigma}}a^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}}}\right)^{1-\sigma}=(1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}})^{\sigma-1}+\frac{\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}}}{(1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}})^{1-\sigma}}=(1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}})^{\sigma-1}(1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}})=(1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}})^{\sigma}$. Q.E.D.

Then, we move on to estimating the speed of $W(qR_\infty, Aa(pI_\infty)^\alpha)$ and $W(qR^0(W(a,b)), I^1pR^0(W(a,b)))$ increasing with A, the former first. We use "$x\propto y$" to denote that x is the same order of y, that is, $\frac{x}{y}$ converges to some positive finite number when $A\to\infty$.

Claim A10-2: If $\sigma\alpha<1$, then $W(qR_\infty, Aa(pI_\infty)^\alpha)$ increases with A no faster than A with power 1. That is, $\frac{W(qR_\infty,Aa(pI_\infty)^\alpha)}{A}\to 0$ if $A\to\infty$.

Proof: By claim A10-1, $W(a,b)=(\frac{\xi(a,\sigma)-1}{\beta})^{\frac{1}{\sigma-1}}b$, where $a=qR_\infty$ and $b=Aa(pI_\infty)^\alpha=pI_\infty R_\infty$. We first estimate the speed of I_∞ increasing with A. In the IM, by (57), $\delta I^1<U\circ D'(\frac{1}{U'(C^b)})\cdot\frac{R^1}{\delta(1-p)}<BR^1$, where B is the upper bound of $U\circ D'(\frac{1}{U'(C^b)})\frac{1}{\delta(1-p)}$, which is shown to exist and be independent of A and W in the proof of Lemma 6. Substitute $R^1=A\alpha(pI^1)^{\alpha-1}$ into the inequality, and we see $\delta I^1<B\alpha Ap^{\alpha-1}(I^1)^{\alpha-1}\Leftrightarrow(I^1)^{2-\alpha}<B'A\Leftrightarrow I^1<B''A^{\frac{1}{2-\alpha}}\Rightarrow I_\infty\le B''A^{\frac{1}{2-\alpha}}$, that is, I_∞ increases no faster than $A^{\frac{1}{2-\alpha}}$. Then, $a=qR_\infty=qA\alpha(I_\infty)^{\alpha-1}\ge B'''A\cdot A^{-\frac{1-\alpha}{2-\alpha}}=B'''A^{\frac{1}{2-\alpha}}\to\infty$ and $b=Aa(pI_\infty)^\alpha$ increases no faster than $A^{1+\frac{\alpha}{2-\alpha}}=A^{\frac{2}{2-\alpha}}$. Then, consider two cases.

Case 1 where $\sigma<1$: Then $a^{\frac{1-\sigma}{\sigma}}\to\infty$. Hence, $\xi(a,\sigma)=(1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}})^\sigma\approx(\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}})^\sigma=\beta a^{1-\sigma}\to\infty$. Then $W(a,b)=(\frac{\xi(a)-1}{\beta})^{\frac{1}{\sigma-1}}b|_{\xi\to\infty}\approx(\frac{\xi(a)}{\beta})^{\frac{1}{\sigma-1}}b\approx(\frac{\beta a^{1-\sigma}}{\beta})^{\frac{1}{\sigma-1}}b=a^{-1}b|_{a=q\widehat{R}_\infty;b=p\widehat{I}_\infty\widehat{R}_\infty}=\frac{p}{q}\widehat{I}_\infty\propto A^{\frac{1}{2-\alpha}}<<A$, as we need.

Case 2 where $\sigma>1$: Then $a^{\frac{1-\sigma}{\sigma}}\to 0$. Hence, $\xi(a,\sigma)=(1+\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}})^\sigma\approx 1+\sigma\beta^{\frac{1}{\sigma}}a^{\frac{1-\sigma}{\sigma}}$. Then $W(a,b)=(\frac{\xi(a)-1}{\beta})^{\frac{1}{\sigma-1}}b\approx(\sigma\beta^{\frac{1}{\sigma}-1}a^{\frac{1-\sigma}{\sigma}})^{\frac{1}{\sigma-1}}b=(\sigma\beta^{\frac{1}{\sigma}-1})^{\frac{1}{\sigma-1}}a^{-\frac{1}{\sigma}}b|_{a=q\widehat{R}_\infty;b=p\widehat{I}_\infty\widehat{R}_\infty}\propto\widehat{I}_\infty(\widehat{R}_\infty)^{1-\frac{1}{\sigma}}\propto\widehat{I}_\infty(A\widehat{I}_\infty^{\alpha-1})^{1-\frac{1}{\sigma}}=A^{1-\frac{1}{\sigma}}(\widehat{I}_\infty)^{1+\frac{(\alpha-1)(\sigma-1)}{\sigma}}<B''A^{1-\frac{1}{\sigma}+\frac{1}{2-\alpha}\cdot\frac{\sigma+(\alpha-1)(\sigma-1)}{\sigma}}=B''A^{\frac{2\sigma-1}{\sigma(2-\alpha)}}$. Therefore, if $\frac{2\sigma-1}{\sigma(2-\alpha)}<1$, or equivalently, if $\sigma\alpha<1$, then $W(qR_\infty, Aa(pI_\infty)^\alpha)$ increases with A at a speed no faster than A. Q.E.D.

We then consider $W(qR^0(W(a,b)), I^1pR^0(W(a,b)))$, where $a=qR^0(W(a,b))$ and $b=I^1pR^0(W(a,b))$ both go to infinity with A, as the a and b in the former case do.

Claim A10-3: If $\sigma\alpha<1$, then $W(qR^0(W(a,b)), I^1pR^0(W(a,b)))$ increases with A no faster than A with power 1.

Proof: By (28), $R^0(W)$ is determined by

$$(\frac{R}{A\alpha})^{-\frac{1}{1-\alpha}} = \frac{q\beta^{\frac{1}{\sigma}}(qR)^{\frac{1-\sigma}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}(qR)^{\frac{1-\sigma}{\sigma}}}W \tag{58}$$

With $A \to \infty$, $R^0 \to \infty$. We consider, as in Claim A10-1, two cases.

Case 1 where $\sigma < 1$: Then, $(qR^0)^{\frac{1-\sigma}{\sigma}} \to \infty$ and the right hand side of (58) $\approx qW$ and (58) becomes $(\frac{R^0}{A\alpha})^{-\frac{1}{1-\alpha}} \approx qW \Leftrightarrow R^0 \propto A \cdot W^{\alpha-1}$. On the other hand, as we saw in the examination of Case 1 of Claim A10-2, for the current case, $W \approx a^{-1}b|_{a=qR^0;b=I^1pR^0} = \frac{p}{q}I^1 \propto I^1 \propto R^0$, with the last $\propto$ implied by (57) in which $R = R^0$ when it is applied in the DIM. Put together, $W \propto A \cdot W^{\alpha-1} \Leftrightarrow W^{2-\alpha} \propto A \Leftrightarrow W \propto A^{\frac{1}{2-\alpha}} << A$, as we need.

Case 2 where $\sigma > 1$: Then, $(qR^0)^{\frac{1-\sigma}{\sigma}} \to 0$ and the right hand side of (58) $\approx q\beta^{\frac{1}{\sigma}}(qR^0)^{\frac{1-\sigma}{\sigma}}W \propto (R^0)^{\frac{1-\sigma}{\sigma}}W$. The equation thus implies that $(\frac{R^0}{A})^{-\frac{1}{1-\alpha}} \propto (R^0)^{\frac{1-\sigma}{\sigma}}W \Leftrightarrow (R^0)^{\frac{\sigma-1}{\sigma}-\frac{1}{1-\alpha}} \propto W \cdot A^{-\frac{1}{1-\alpha}}$. On the other hand, as we saw in the examination of Case 2 of Claim A10-2, $W \propto I^1(R^0)^{1-\frac{1}{\sigma}} \propto (R^0)^{2-\frac{1}{\sigma}} \Leftrightarrow R^0 \propto W^{\frac{\sigma}{2\sigma-1}}$ (note that we have assumed $\sigma > \frac{1}{2}$) for the current case. Put together, $W^{\frac{\sigma}{2\sigma-1}\times(\frac{\sigma-1}{\sigma}-\frac{1}{1-\alpha})} \propto W \cdot A^{-\frac{1}{1-\alpha}} \Leftrightarrow W^{-\frac{1+\alpha\sigma-\alpha}{(2\sigma-1)(1-\alpha)}-1} \propto A^{-\frac{1}{1-\alpha}} \Leftrightarrow W^{-\frac{\sigma(2-\alpha)}{(2\sigma-1)(1-\alpha)}} \propto A^{-\frac{1}{1-\alpha}} \Leftrightarrow W \propto A^{\frac{1}{1-\alpha}\times\frac{(2\sigma-1)(1-\alpha)}{\sigma(2-\alpha)}} \Leftrightarrow W \propto A^{\frac{2\sigma-1}{\sigma(2-\alpha)}}$. Therefore, as was in Case 2 of the last claim, if $\sigma\alpha < 1$, then $W(qR^0(W), I^1pR^0(W))$ increases with A at a speed no faster than A. Q.E.D.

Hence, $W^{10} < W(qR_\infty, Aa(pI_\infty)^\alpha)$ and $W^{01} < W(qR^0(W), I^1pR^0(W))$ both increase with A at a speed no faster than A. Actually, the proofs above show that the two $W(a,b)$ are in the same order of A, which hints that W^{10} and W^{01} are in the same order of A. ■

The proof of Proposition 5:

Proof. First note that if the steady state $\{K_j^1, K_n^0\}_{j=1,2,..J;n=1,2,...,N}$ exists, K_1^1, the starting period in the IM, is the fixed point of map ${}^N M^0 \circ^J M^1$: if the capital stock starts with K_1^1 and evolves first according to the dynamics of the IM for J periods and then according to the dynamics of the DIM for N periods, then it will come back to the same level (i.e. K_1^1). Similarly, K_2^1 is the fixed point of map $M^1 \circ^N M^0 \circ^{J-1} M^1$, etc. As such we prove the proposition in two steps. At step 1, we show for any J and N, map ${}^N M^0 \circ^J M^1$ has a unique fixed point over (K^0, K^1), and so does map $M^1 \circ^N M^0 \circ^{J-1} M^1$, etc. These fixed points, $\{K_j^1, K_n^0\}_{j=1,2,..J;n=1,2,...,N}$, satisfy the equality conditions. At step 2, we show that for $J = J^*$ and $N = N^*$, these fixed points satisfy the inequality conditions.

Step 1: We only show that map ${}^{N}M^0 \circ^{J} M^1$ has a unique fixed point over (K^0, K^1); the proof for other maps such as $M^1 \circ^{N} M^0 \circ^{J-1} M^1$ will be similar. The uniqueness comes from the concavity of ${}^{N}M^0 \circ^{J} M^1$, which follows from the concavity of M^0 and M^1 and the fact that a compound of concave functions is concave. For existence, let

$$f_1^1(x) \equiv^{N} M^0 \circ^{J} M^1(x) - x.$$

Then it suffices to show f_1^1 has a root over (K^0, K^1). First note that $K' \equiv^{J} M^1(K^0) > K^0$ because $K^0 < K^1$ and ${}^{J}M^1(x) > x$ for $x < K^1$. Therefore, $f_1^1(K^0) =^{N} M^0 \circ^{J} M^1(K^0) - K^0 =^{N} M^0(K') - K^0 > 0$, because $K' > K^0$ and $M^0(x) > K^0$ for any $x > K^0$. Second, as K^1 is the fixed point of M^1, we have ${}^{J}M^1(K^1) = K^1$. Therefore, $f_1^1(K^1) =^{N} M^0 \circ^{J} M^1(K^1) - K^1 =^{N} M^0(K^1) - K^1 < 0$ because $K^1 > K^0$ and ${}^{N}M^0(x) < x$ for any $x > K^0$. Therefore, f_1^1 has a root over (K^0, K^1).

Step 2: To ensure $\{K_j^1, K_n^0\}_{j=1,2,..J^*;n=1,2,...,N^*}$ satisfy the inequality conditions, we only need to check

$$K_1^1(\;=\; {}^{N^*}M^0(K_1^0)) < K^{01} <^{N^*-1} M^0(K_1^0) \tag{59}$$

$$K_1^0(\;=\; {}^{J^*}M^1(K_1^1)) > K^{10} >^{J^*-1} M^1(K_1^1) \tag{60}$$

namely, the cycle falls into the DIM exactly on having been running for J periods in the IM and it leaps into the IM exactly on having been running for N periods in the DIM; the rest of the inequalities are satisfied automatically: since all the fixed points are between $K^0$and K^1, the sequence is increasing if it is in the IM and is decreasing if it is in the DIM.

To check (59), first we show that $K_1^1 < K^{01}$, which, since K_1^1 is the unique root of $f_1^1(x)$, with $f_1^1(K^0) > 0$ (shown above), is equivalent to $f_1^1(K_1^0) < 0 \Leftrightarrow^{N^*} M^0 \circ^{J^*} M^1(K_1^0) < K_1^0$. Since $J^* = J(K^{01})$ by condition (34), by the definition of $J(K^{01})$, we have ${}^{J^*}M^1(K^{01}) > K^{10}$. Moreover, by the same definition, ${}^{J^*-1}M^1(K^{01}) < K^{10}$, which implies ${}^{J^*}M^1(K^{01}) < M^1(K^{10})$. Therefore, ${}^{J^*}M^1(K^{01}) \in (K^{10}, M^1(K^{10}))$. Then by condition (33), ${}^{N^*}M^0({}^{J^*}M^1(K^{01}) < K^{01}$, namely $f_1^1(K^{01}) < 0$.

Second, we show $K_1^0 > K^{10}$ (the former part of 60), which, since K_1^0 is the unique root of $f_1^0(x) \equiv^{J^*} M^1 \circ^{N^*} M^0(x) - x$, with $f_1^0(K^1) < 0$ (which could be shown in a similar way to showing $f_1^1(K^1) < 0$), is equivalent to $f_1^0(K^{10}) > 0 \Leftrightarrow^{J^*} M^1 \circ^{N^*} M^0(K^{10}) > K^{10}$. Since $N^* = N(K^{10})$ by condition (33), by the definition of $N(K^{10})$, ${}^{N^*}M^0(K^{10}) < K^{01}$ and moreover ${}^{N^*-1}M^0(K^{10}) >$

K^{01}, with the latter implying ${}^{N^*}M^0(K^{10}) > M^0(K^{01})$. Therefore, ${}^{N^*}M^0(K^{10}) \in (M^0(K^{01}), K^{01})$. Then by condition (34), ${}^{J^*}M^1({}^{N^*}M^0(K^{10})) > K^{10}$, namely $f_1^0(K^{10}) > 0$.

Third, we show $K^{01} <^{N^*-1} M^0(K_1^0)$ (the latter part of 59), which, as $K^{10} < K_1^0$ (the second point) and ${}^{N^*-1}M^0(\cdot)$ is increasing, follows from $K^{01} <^{N^*-1} M^0(K^{10})$, which in turn comes from the definition of $N(K^{10}) = N^*$.

Fourth, we show $K^{10} >^{J^*-1} M^1(K_1^1)$ (the latter part of 60), which, as $K^{01} > K_1^1$ (the first point) and ${}^{J^*-1}M^1(\cdot)$ is increasing, follows from $K^{10} >^{J^*-1} M^1(K^{01})$, which in turn comes from the definition of $J(K^{01}) = J^*$.

So we finish step 2 and show the existence of cycle $\{K_j^1, K_n^0\}_{j=1,2,..J^*;n=1,2,...,N^*}$.

Finally, if $K^{10} > M^1(K^{01})$, then by definition of $J(K^{01})$, we have $J(K^{01}) \geq 2$, that is, $J^* \geq 2$. Similarly, if $K^{01} < M^0(K^{10})$, then $N^* \geq 2$. ■

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