Bank runs, liquidity, and macro-prudential regulation*

Toni Ahnert†

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Abstract

This paper examines the role of liquidity in an economy with many banks subject to runs and systemic liquidation costs. First, the presence of liquidity drives a wedge between the amount of withdrawals and liquidation. This restores multiple equilibria even when a global game refinement is used. Second, systemic liquidation costs imply that one bank's liquidity holding reduces the liquidation costs of other banks. The positive implication is the partial substitutability of private liquidity holdings as banks free-ride on other banks' liquidity. The normative implication is that banks hold insufficient liquidity relative to a constrained planner, interpreted as a macro-prudential authority that internalizes the system-wide effects of liquidity. Comparative statics analyses with respect to the expected investment return and the liquidation cost are performed. [124 words]

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†London School of Economics and Political Science, Financial Markets Group and Department of Economics, Houghton Street, London WC2A 2AE. Part of this research was conducted while being a Visiting PhD Student at the Department of Economics at New York University. I am indebted to Douglas Gale for sponsoring this visit. Email: `t.ahnert@lse.ac.uk`.

1 Introduction

A crucial concept in economics and finance is liquidity. An asset is liquid if it can be converted into cash quickly and at a low cost.[1] Holding enough liquid assets is important for financial intermediaries, as they may face sudden withdrawals from their investors. This applies to both the classical case of a run on a commercial bank by retail investors (Diamond and Dybvig (1983)) and modern-day runs, such as institutional investors who withdraw from money market mutual funds (e.g. Wermers (WP)). Liquidity also plays a major role in recent proposals for the regulation of financial intermediaries.[2]

This paper examines the role of liquidity in an economy with many banks subject to runs and systemic liquidation costs. Banks invest in a long-term project and hold some liquidity to prepare for early withdrawals from investors. The risky project has a higher expected return but is costly to liquidate before maturity. Liquidity drives a wedge between the amount of withdrawals from investors and the amount of liquidation, thereby trading off the opportunity cost of the higher expected investment return with the benefit from reducing costly liquidation. The profitability of a bank's project depends on aggregate economic conditions such as business cycle movements.[3] Investors have the option to withdraw before the maturity of the investment project. They receive noisy private news about the aggregate economic condition before deciding whether to withdraw. A bad economic condition results in a large number of investors with bad signals and therefore many withdrawals. This leads to run on a bank that has insufficient liquidity to serve all withdrawing investors and has to liquidate the project at a cost. Systemic liquidation costs, whereby one bank's liquidation cost increases in the other bank's liquidation volume, are also explored to analyze the system-wide dimension of liquidity.

[1] An example is US sovereign debt, whose market is characterized by a large trading volume. Therefore, selling a given quantity can be realized fast and with a low price impact.

[2] This is in contrast to the financial regulation during the last three decades that focused mainly on capital. The current proposals include the introduction of rules governing the composition of banks' balance sheets envisaged under the Basel Committee on Banking Supervision's proposed Liquidity Coverage Ratio (LCR) or Net Stable Funding Ratio (The Basel Committee (2010a,b)). Both regulatory tools seek to impose limits on the degree of liquidity mismatch on a bank's balance sheet by, for example, imposing a lower bound on banks' liquidity ratios.

[3] Business cycle movements affect the default rates of the loan portfolio of banks. Alternatively, there is a shock to an asset class in which banks are invested, such as asset-backed securities.

The first contribution is to show that the presence of liquidity restores multiple equilibria – even if a global game refinement is used (Proposition 1). When banks hold some liquidity to prepare for withdrawals from investors, there exists an equilibrium without liquidation if the economic condition is good. Some investors receive bad news about the economic condition, infer that their bank's profitability is low, and withdraw. But since the true state of the economy is good, this is a small number of investors and the available liquidity suffices to serve them. Most investors receive good news and do not withdraw, as such costly liquidation is avoided. Likewise, an equilibrium with liquidation exists if the economic condition is weak. Then many investors receive bad news and withdraw. As a result, liquidity is exhausted and costly liquidation occurs. Taken together, I show that there exists an interim range of the economic condition that supports both equilibria.

The equilibrium with liquidation corresponds to the unique Bayesian equilibrium in other global game models of bank runs, such as Goldstein and Pauzner (2005) and Morris and Shin (2000). Why is the other equilibrium without liquidation absent in these papers? As investors receive noisy news, some investors will always receive a bad signal and withdraw – even if the economic condition is good. Without liquidity, there is always positive liquidation to serve withdrawing investors, ruling out the possibility of an equilibrium without liquidation. In fact, I show formally that the equilibrium without liquidation vanishes as the level of liquidity vanishes (Corollary 1). Therefore, liquidity is crucial for re-establishing multiple equilibria in bank run coordination games. The multiplicity result does not rely on endogenous information acquisition, which has also been shown to break uniqueness (e.g. Hellwig and Veldkamp (2009), Ahnert (WP)).

The second contribution is to demonstrate a role for a macro-prudential regulation of liquidity. To this end, I compare the privately optimal and socially constrained efficient levels of liquidity.[4] When liquidation costs are systemic, insufficient liquidity at one bank means more liquidation for a given amount of withdrawals and therefore a higher liquidation cost for other banks in the system. The positive implication is the partial substitutability of private liquidity holdings as banks free-ride on other banks' liquidity (Proposition 2). The normative implication is that the private banking system holds insufficient liquidity relative to a constrained planner (Proposition 3). As a planner in-

[4] In order to analyze the effects of ex-ante liquidity holdings, I need to select an equilibrium for economic conditions that support both equilibria. To focus on the macro-prudential implications of liquidity, I select the equilibrium with liquidation.

ternalizes the system-wide effects of liquidity, this planner is naturally interpreted as a macro-prudential authority.

Proposition 4 summarizes comparative static results that illustrate the intuition of private and social liquidity choices. The level of liquidity held by a bank trades off the marginal cost in terms of foregone expected investment return with the marginal benefits in terms of avoiding costly liquidation, which reduces coordination failure among investors. A higher expected investment return (better economic conditions on average) increases the opportunity cost of liquidity and therefore reduces a bank's optimal liquidity level. By contrast, a higher liquidation cost increases the marginal benefit from avoiding liquidation and therefore increases a bank's optimal liquidity level. By extension, the comparative statics for the constrained efficient liquidity choice yield the same signs as the constrained planner faces the same trade-off, just with a higher marginal benefit from liquidity.

Systemic liquidation costs, which generate a positive externality from liquidity and are at the core of my normative result, are micro-founded by a body of literature. Limited participation in asset markets can lead to cash-in-the-market pricing and therefore underpricing of assets (Allen and Gale (1994)). Similarly, liquidation values are depressed after an industry-specific shock since distress sales take place to unlevered industry outsiders who value industry-specific assets less (Shleifer and Vishny (1992)). Finally, financial arbitrageurs cannot pick up assets in fire sales since they are constrained by losses and outflows themselves (Gromb and Vayanos (2002)).

The paper closest in terms of methodology is Morris and Shin (2000), who build on the seminal work of Carlsson and van Damme (1993), using global games techniques to analyze a withdrawal game in the spirit of Diamond and Dybvig (1983).[5] The Bayesian equilibrium of Morris and Shin (2000), which features runs on illiquid but solvent banks, is unique. By contrast, I show in my first contribution how the presence of liquidity breaks equilibrium uniqueness by allowing for another equilibrium without runs. Furthermore, I extend the analysis to multiple banks to explore the effect of systemic liquidation costs on ex-ante incentives to hold liquidity.[6] Vives (WP) and Morris and Shin (WP) also

[5]Multiple equilibria in Diamond and Dybvig (1983) occur because of the self-fulfilling beliefs. If an investors fears withdrawals by other investors, then this will imply costly liquidation of the bank's assets that reduces a non-withdrawing investor's payoff. Therefore, each investor finds it optimal to withdraw, constituting a bank-run equilibrium. Likewise for the no-run equilibrium.

[6]Goldstein and Pauzner (2005) also use global games techniques to generate a unique equilibrium in a

analyze investor withdrawal games and the effect of liquidity. However, they abstract from conditions that can induce the no-liquidation equilibrium and are not concerned with the ex-ante portfolio choice.

Allowing for multiple banks, my second contribution is to examine the consequences of systemic liquidation costs for ex-ante liquidity choices, both privately and socially. Other consequences of systemic liquidation costs have already been analyzed. Wagner (2011) studies the diversification-diversity trade-off in the ex-ante portfolio choice. Since joint liquidation is costly ex-post, investors have an incentive to hold diverse portfolios.[7] In contrast, I examine the consequences for ex-ante liquidity holdings in the presence of systemic liquidation costs and analyze the consequences for financial intermediaries that may be subject to runs. Uhlig (2010) analyzes endogenous liquidation costs in a model with outside investors and a two-tiered banking sector. The arising system-wide externality generates strategic complementarities in the depositors' withdrawal decisions also present in my model. His focus is on a positive analysis of the previous financial crisis and discusses some ex-post policy interventions. By contrast, my focus is on optimal (liquidity) regulation from an ex-ante perspective. Studying ex-ante policy has the advantage of precluding the issue of moral hazard arising from an ex-post policy intervention, a theme also stressed by Farhi and Tirole (2012).

The literature on macro-prudential regulation is growing fast. Korinek (2011) analyzes risk-taking in an economy in which systemic externalities take the form of pecuniary fire sales and provides a micro-founded rationale for macro-prudential policy, such as a Pigouvian tax on risk-taking or capital requirements. Korinek (WP) contrasts ex-ante macro-prudential regulation with ex-post policy interventions. In line with the present paper, Farhi and Tirole (2012) highlight the importance of a macro-prudential approach to contain a crisis ex-ante.

setup closer to the original model of Diamond and Dybvig (1983), for example preserving the sequential service constraint. The same comments apply.

[7]Wagner (2009) also stresses the role of endogenous liquidation costs, showing that they give rise to cross-bank externalities. The implications for optimal bank portfolios are ambiguous, however, as banks may be 'too correlated' (as in the standard case) or 'too diversified' under laissez faire, implying that regulatory treatment should be heterogeneous.

2 The Model

The economy extends over three dates labelled as initial ($t = 0$), interim ($t = 1$), and final ($t = 2$), and it is inhabited by a continuum of investors and two banks $n \in \{A, B\}$. The notion of financial intermediation provided by banks is not limited to the traditional case of retail investors and commercial banks but incorporates, for instance, money market mutual funds and investment banks.[8] There is a single physical good used for consumption and investment.

Investors There is a unit mass of initially identical investors $i \in [0, 1]$ with idiosyncratic uncertainty about their consumption needs (Diamond and Dybvig (1983)). All investors are uncertain at the initial date and privately learn their consumption preference $\theta_i \in \{0, 1\}$ at the interim date. Each investor is either early ($\theta_i = 1$) and wishes to consume at the interim date or late ($\theta_i = 0$) and wishes to consume at the final date. Investors can store between the interim and the final date. The ex-ante probability of being an early investor is $\lambda \equiv \Pr\{\theta_i = 1\} \in (0, 1)$, which is identical across investors and also the share of early investors by the law of large numbers. A investor's utility function is:

$$U_i(c_1, c_2) = \theta_i c_1 + (1 - \theta_i) c_2 \tag{1}$$

where c_t is consumption at date t, and θ_i represents an idiosyncratic liquidity shock. Investors are endowed with two units at the initial date and randomly deposit at either bank; as such each bank receives one unit of deposits.

Investment opportunities Two investment opportunities in the form of constant-return-to-scale technologies are available at the initial date (Table 1). First, storage is universally available and yields a unit safe return. Since it matures after one period, storage is referred to as *liquidity*. Second, an investment project, such as lending to a productive sector, is available to banks.[9] A project matures at the final date and yields

[8]Also, the investors and banks of this model can be interpreted as local and global banks in the spirit of Uhlig (2010). Then a prematurely withdrawing investor represents a run of a local bank on a global bank, an arguably reasonable feature of the recent financial crises.

[9]The motive for the existence of banks is different from Diamond and Dybvig (1983). While these authors demonstrate a role for a bank as provider of liquidity insurance for risk-averse investors, banking in this model arises from a bank's enhanced access to investment projects because of an advantage in monitoring, for example.

Asset	$t = 0$	$t = 1$	$t = 2$
Storage ($0 \to 1$)	-1	1	
Project ($0 \to 2$)	-1	l_n	$(1 - l_n)(r - \chi(l_n, l_{-n}))$

Table 1: Investment technologies

a stochastic return with mean $\bar{r} > 1$. Premature liquidation of the project at the interim date is costly. Similar to Morris and Shin (2000), liquidation of an amount $l_n \in [0, 1]$ by bank n at par reduces the final-date return by $\chi(l_n, l_{-n})$, where $\chi(\cdot) \geq 0$ is the cost function of premature liquidation. The reduction to the final-date payoff is implied by a lender-of-last-resort policy, for instance. Liquidation costs are modelled to be proportional to the total amount of liquidation:

$$\chi(l_n, l_{-n}) \equiv \chi[l_n + d\, l_{-n}] \tag{2}$$

where $\chi \in (0, 1)$ measures the cost of liquidation, and $d \in \{0, 1\}$ is a dummy that is one when systemic liquidation costs are present. To avoid strict dominance of the project, $\bar{r} < 1 + 2\chi$ is assumed throughout.

Bank-specific liquidation costs are the source of strategic complementarity between late investors of a given bank. According to **?**, individual liquidation costs are discussed in James (1991) and Mullins and Pyle (1994). These costs comprise direct liquidation expenses and a reduction in the 'going concern' value of bank assets under distress. The empirical literature typically finds these liquidation costs to be large: of the order of 30% of bank assets on average.[10]

Systemic liquidation costs or fire-sales, if present, are the source of strategic complementarity between late investors across banks and create an externality in a bank's liquidity choice. Liquidation costs are systemic if there is limited liquidity in the market

[10]Mullins and Pyle (1994) and Brown and Epstein (1992) present estimates of direct liquidation expenses of around 10%, varying between 17% for assets relating to owned real estate to 0% for liquid securities for assets in receivership at the FDIC. Adding to direct expenses losses associated with forced liquidation, James (1991) gives an average total cost of 30% of a failed bank's assets. Similar orders of magnitude are reported in Bennett and Unal (WP), whose sample runs for much longer, covering 1986-2007.

(Allen and Gale (1994)), a fire sale to industry outsiders (Shleifer and Vishny (1992)), or financial constraints of arbitrageurs (Gromb and Vayanos (2002)). As liquidation depresses not only a given bank's asset liquidation value, but also another bank's liquidation value, there is a negative externality from liquidation.

Information structure The investment return $\boldsymbol{r}$, a measure of the economic condition such as a key macroeconomic variable, is distributed normally with precision $\alpha \in (0, \infty)$:

$$\boldsymbol{r} \sim \mathcal{N}\left(\bar{r}, \frac{1}{\alpha}\right)$$

The investment return is realised at the interim date but not publicly observed. However, each investor receives a private signal x_i about the return:

$$\begin{aligned} \boldsymbol{x}_i &\equiv \boldsymbol{r} + \boldsymbol{\epsilon}_i \\ \boldsymbol{\epsilon}_i &\sim \mathcal{N}\left(0, \frac{1}{\gamma}\right) \end{aligned}$$

where the idiosyncratic noise $\boldsymbol{\epsilon}_i$ has zero mean, precision $\gamma \in (0, \infty)$, and is independently and identically distributed across investors and independent of the investment return. All distributions are common knowledge.

Banks At the initial date banks simultaneously choose their liquidity holdings (y_A, y_B) and invest the remainder in the project. The liquidity choice is publicly observed. Investors that withdraw at the interim date receive unity, while investors that wait for the final date receive a pro-rata payment of a bank's assets, which includes the proceeds from investment in the project.[11] Abstracting from a misalignment of incentives between the bank manager and investors, a bank's objective is to maximize the expected utility of investors.[12]

[11]See, for example, Dasgupta (2004), Goldstein (2005), and Shapiro and Skeie (WP) for a similar assumption on the interim-date withdrawal payment. Therefore, banks are viable at the interim date as the promised payment does not exceed the liquidation value. The focus of the present paper is on the effect of liquidity on equilibrium multiplicity as well as the consequences of a fire-sale externality from one bank's liquidation decision on the liquidity choice of banks. My main results hold for alternative assumptions about the interim payment.

[12]This objective arises as an equilibrium outcome in a generalized model with competition for deposits. The competition between banks for investors implies that banks offers the best possible liquidity holding to investors. In a related paper, Gale (2010) shows that a bank's optimal behaviour under free entry

Banks serve any withdrawals by using liquidity first. Let $w_n \in [0, 1-\lambda]$ denote the amount of withdrawals by late investors of bank n. If withdrawals from late investors are sufficiently high ($w_n > y_n - \lambda$), the bank partially liquidates its investment project, where the liquidation amount is given by $l_n \equiv \max\{0, w_n + \lambda - y_n\} \in [0, 1 - y_n]$. The liquidation amount decreases in a bank's liquidity holding y_n and increases in the amount of withdrawals by late investors w_n.

To prevent costly liquidation, the bank may hold excess liquidity ($y_n > \lambda$) - more liquidity than required to serve withdrawals from early investors. Holding excess liquidity drives a wedge between the proportion of prematurely withdrawing late investors w_n and the liquidation volume of the investment project l_n. As it is never optimal to face certain liquidation, the lower bound of a bank's liquidity level is the share of early investors ($y_n \geq \lambda$).

Payoffs Early investors always withdraw at the interim date. Late investors that withdraw at the interim date receive the same payoff as early investors since the liquidity preference of investors is unobserved by banks. To shed more light on a late investor's payoff, consider the cases of no liquidation and positive liquidation in turn.

No liquidation ($w_n \leq y_n - \lambda$) If few late investors withdraw at the interim date, excess liquidity holdings $y_n - \lambda$ suffice to serve them. There is no liquidation ($l_n = 0$), and some excess liquidity is carried over to the final date. The payoff to a late investor at the final date is:

$$c_{2n} = \frac{[y_n - \lambda - w_n] + (1 - y_n)r}{1 - \lambda - w_n} \tag{3}$$

where the asset payments available to investors at the final date consist of remaining liquidity ($y_n - \lambda - w_n$) and proceeds from investment in the project (numerator), all of which to be shared with the proportion of investors that wait for the final date (denominator). The realisation of the stochastic investment project r enters this expression, while the

and subject to the investors' participation constraint can be expressed as the solution to a contracting problem in which the welfare of investors is maximised subject to the zero-profit constraint of the bank. If any given bank were not to choose this investment plan, it would fail to attract any deposits. Given the alignment of interest between a bank and its investors as well as the bank's enhanced access to projects, all depositors deposit in full.

amount of liquidation by the other bank (l_{-n}) has no effect on the payoff of late investors of bank n in the absence of liquidation ($l_n = 0$).

Positive amount of liquidation ($w_n > y_n - \lambda$) If many late investors withdraw at the interim date, the excess liquidity holding $y_n - \lambda$ is drawn down, and some amount of the project is liquidated ($l_n = w_n + \lambda - y_n$) to serve withdrawing investors. The payoff to a late investor at the final date is:

$$c_{2n} = \frac{(1 - y_n - l_n)[r - \chi(l_n, l_{-n})]}{1 - \lambda - w_n} = r - \chi(l_n, l_{-n}) \tag{4}$$

A fire-sale externality, which is a negative liquidation externality, is present if and only if there are systemic liquidation costs ($d = 1$) and the other bank liquidates a positive amount ($l_{-n} > 0$).

Remark 1. *Conditional on a positive liquidation ($l_n > 0$), there is a strategic complementarity between the withdrawal decisions of late investors of the same bank ($\frac{\partial c_{2n}}{\partial w_n} < 0$). If there is also positive liquidation by the other bank ($l_{-n} > 0$) and systemic liquidation costs are present ($d = 1$), there is also strategic complementarity between the withdrawal decisions of late investors across banks ($\frac{\partial c_{2n}}{\partial w_{-n}} < 0$).*

There are two dimensions to the strategic behaviour of a late investor. The first dimension is the strategic complementarity between the withdrawal decisions of late investors of a given bank. More withdrawals by other late investors have two effects. First, the bank draws down its excess liquidity and then liquidates a larger share of the project. This effect is detrimental to a late investor who keeps his funds for the final date. Second, there are fewer late investors to share the remaining resources with at the final date. This effect is beneficial for a late investor who keeps his funds. In the positive-liquidation case, the first effect unambiguously dominates and the incentives to withdraw increase in the proportion of withdrawing late investors ($\frac{\partial c_{2n}}{\partial w_n} = -\chi < 0$). By contrast, the incentives to withdraw decrease in the proportion of withdrawing late investors if no liquidation takes place and the project return is sufficiently high ($r \geq 1$).

The second dimension is a strategic complementarity between the withdrawal decisions of late investors across banks in the presence of systemic liquidation costs. The more late investors in the other bank $-n$ withdraw, the more of the investment project of bank $-n$ is liquidated, the lower the final-date payoff to investors at bank n due to the fire-sale externality. This increases the incentive for a late investor of bank n to withdraw as well, conditional on positive liquidation at bank n ($\frac{\partial c_{2n}}{\partial w_{-n}} = -\chi < 0$).

Timeline The following timeline summarizes the model:

Initial date $t = 0$

- Investors receive their endowment and deposit at banks.
- Each bank holds liquidity y_n and invests the remainder $1 - y_n$.

Interim date $t = 1$

- Each investor privately observes his consumption preferences θ_i (early or late).
- Each investor receives a private signal x_i about the investment return and updates his forecast about the return and the proportion of withdrawing investors.
- Investors may withdraw, and the mass of late investors that withdraw is w_n.
- Banks serve withdrawals using liquidity first. If necessary, a bank (partially) liquidates the investment project (liquidation amount l_n).
- Early investors consume and withdrawing late investors store their withdrawals.

Final date $t = 2$

- The investment project matures.
- Remaining late investors receive an equal share of the investment proceeds.
- Late investors consume.

3 Equilibrium

There are two stages: a perfect-information portfolio choice stage between banks at the initial date and an imperfect-information withdrawal stage between investors at the interim date. As the portfolio choices of banks are observed by investors, the equilibrium is best characterised by working backwards, starting with the equilibrium in the withdrawal subgame.

An investor's *strategy* is a plan of action for each private signal x_i. A profile of strategies is a Bayesian Nash equilibrium in the subgame at the interim date if the actions described by investor i's strategy maximize his expected utility conditional on x_i, taking as given the strategies followed by all other investors. *Threshold strategies* are considered by which a late investor withdraws if and only if his private signal falls short of a bank-specific threshold (to be determined): $x_i < x_n^*$. These thresholds depend on the liquidity choices of banks at the initial date: $x_n^* = x_n^*(y_n, y_{-n})$.

Next, consider the game between banks who choose a liquidity level y_n. Each bank takes the effect of its liquidity choice on the bank-specific withdrawal threshold $x_n^*(y_n, y_{-n})$ into account. I will determine the Nash equilibrium (y_A, y_B) in the game between banks at the interim date, where each y_n maximizes the bank's objective function subject to the effect on the withdrawal threshold, taking as given the level of liquidity held by the other bank.

Each investor uses his private information x_i to update his forecasts about the investment return and the proportion of withdrawing late investors at either bank. The posterior distributions are derived in Appendix A.1. Let $\boldsymbol{R_i} \equiv r|x_i$ denote the posterior distribution of the investment return as formed by an investor who receives the private signal x_i. The posterior mean R_i is equivalent to the signal x_i because there is a bijective mapping between them. The equilibrium posterior mean R_n^*, which is computationally more convenient than the equilibrium signal x_n^*, is used to describe the equilibrium conditions.[13] Likewise, $\boldsymbol{W_{i,n}^n} \equiv w_n|x_i$ and $\boldsymbol{W_{i,-n}^n} \equiv w_{-n}|x_i$ denote the posterior distributions of the proportions of withdrawing late investors at bank the investor's bank and the other bank, respectively. Similarly, the expected amount of liquidation by bank n is given by $\boldsymbol{L}_{i,n}^n$.

Consider the equilibrium withdrawal behaviour of investors at the interim date. Early investors always withdraw, while late investors may withdraw. The bank-specific threshold R_n^* is defined as the mean of the posterior return that makes a late investor indifferent between withdrawing and not withdrawing his funds:

$$1 = c_{2n}(R_n^*, R_{-n}^*) \tag{5}$$

[13]Note that both converge as the private noise vanishes ($x_n^* - R_n^* \to 0$ as $\gamma \to \infty$).

where the left-hand side is the payoff from withdrawing and the right-hand side is the expected payoff from not withdrawing conditional on the threshold signal x_n^*. Equation (5) implicitly defines the best response function $R_n^*(R_{-n}^*)$, where investors take the other bank's threshold R_{-n}^* as given. The withdrawal threshold of investors of one bank depends on the threshold of investors in the other bank in case of positive liquidation ($l_n > 0$) and systemic liquidation costs ($d = 1$).

The subsequent subsections construct a complete description of equilibrium in the subgame by analysing the role of liquidity for equilibrium multiplicity and the effect of systemic liquidation costs. In line with the global games literature (e.g. Morris and Shin (2003)), I shall assume vanishing private noise ($\gamma \to \infty$) throughout.

3.1 No systemic liquidation costs

First consider the case without systemic liquidation costs ($d = 0$).

3.1.1 No expected liquidation

Suppose the marginal investor expects no liquidation in equilibrium ($L_n(x_n^*) = 0$). Then the indifference condition yields $(1 - y_n)(R_n^* - 1)$. If there is no intermediation ($y_n = 1$ or "narrow banking"), then the withdrawal decision is irrelevant since the bank's assets are always unity and the investor receives unity irrespective of his withdrawal decision. If there is intermediation ($y_n < 1$), the withdrawal threshold is unity ($R_n^* = 1$). To be consistent with zero liquidation *as expected by the marginal investor*, the bank's liquidity level must be sufficiently high. The marginal investor expects half of the late investors of mass $1 - \lambda$ to withdraw as private noise vanishes ($W_n^n \to \frac{1-\lambda}{2}$). Therefore, liquidity must be abundant to serve withdrawals from early and late investors:

$$y_n \geq \lambda + \frac{1 - \lambda}{2} \tag{6}$$

Zero *actual* liquidation arises if and only if the number of withdrawals does not exceed excess liquidity ($w_n \leq y_n - \lambda$), requiring a sufficiently high realisation of the investment return ($r \geq \tilde{r}_0$). As the distribution of signals conditional on the economy-wide return is $\mathcal{N}\left(r, \frac{1}{\gamma}\right)$, the *lower* bound on the realised investment return is:

$$\tilde{r}_0 \equiv 1 - \frac{\alpha}{\gamma}(\bar{r} - 1) - \frac{1}{\sqrt{\gamma}}\Phi^{-1}\left(\frac{y_n - \lambda}{1 - \lambda}\right) \to 1$$

where $\Phi^{-1}(\cdot)$ is the inverse of the cumulative probability function of the standard normal distribution and $\tilde{r}_0 < 1$. More liquidity ensures that more withdrawals are consistent with zero actual liquidation. Hence, the lower bound on the investment return decreases in the liquidity holding ($\frac{\partial \tilde{r}_0}{\partial y_n} < 0$). As private noise vanishes, the lower bound converges to unity ($\tilde{r}_0 \to 1$). Lemma 1 summarizes:

Lemma 1. *Consider the withdrawal subgame without systemic liquidation costs ($d = 0$), vanishing private noise ($\gamma \to \infty$), and abundant liquidity $y_n \in [\frac{(1+\lambda)}{2}, 1]$. Then, any threshold equilibrium has the following features:*

- $L_n^* = 0$*: the marginal investor expects no liquidation;*
- $x_n^* \to 1$*: a late investor withdraws if and only if his signal falls short of unity;*
- $l_n^* = 0 \Leftrightarrow r \geq 1$*: no actual liquidation occurs if and only if the economic condition is sufficiently good.*

In sum, the level of liquidity determines whether the marginal investor expects positive liquidation to occur in equilibrium, while the realised economic condition determines whether liquidation actually occurs.

3.1.2 Positive expected liquidation

For the marginal investor to expect a positive amount of liquidation in equilibrium ($L_n(x_n^*) > 0$), the bank's liquidity level must be scarce:

$$y_n < \frac{1+\lambda}{2} \tag{7}$$

As I show in section 3.3, liquidity is scarce in equilibrium if it has a high opportunity cost in terms of a high expected investment return ($\bar{r} \geq \bar{r}_L$). With positive expected liquidation the indifference condition of the marginal investor reduces to:

$$R_n^* = 1 + \chi[(1-\lambda)\Phi(\sqrt{\delta}[R_n^* - \bar{r}] + \lambda - y_n)]$$

where $\delta \equiv \frac{\alpha^2(\alpha+\gamma)}{\gamma(\alpha+2\gamma)}$ collects precision parameters. As in Morris and Shin (2000), uniqueness requires the slope of the left-hand side to exceed the slope of the right-hand side and vanishing private noise is sufficient for this requirement.[14] A closed-form expression for

[14]The requirements is $1 > D_n^* \equiv \chi(1-\lambda)\sqrt{\delta}\phi\left(\sqrt{\delta}[R_n^* - \bar{r}]\right) > 0$, where $\phi(\cdot)$ is the probability distribution function of the standard normal distribution. This condition is satisfied as the private noise vanishes since $\delta \to 0$.

the threshold is obtained for vanishing private noise:

$$R_n^* \to 1 + \chi[\frac{1-\lambda}{2} + \lambda - y_n] \tag{8}$$

Coordination failure between investors induces the threshold to exceed unity ($R_n^* > 1$), the efficient liquidation level. If there is no cost of premature liquidation ($\chi \to 0$), then the strategic complementarity between investors of the same bank is absent and the efficient allocation obtains in the withdrawal game. For a positive liquidation cost ($\chi > 0$), however, there is coordination failure between investors that pushes the threshold above the efficient level. Fearing that other late investors withdraw prematurely and thereby cause costly liquidation, another late investor has an incentive to withdraw prematurely – even if the nominal investment return exceeds the payoff from withdrawing prematurely. Furthermore, the threshold is below the expected return of the project ($R_n^* < \bar{r}$) if the (individual) liquidation cost is sufficiently low relative to the expected return ($\bar{r} > 1 + \frac{\chi(1-\lambda)}{2}$).

Actual liquidation occurs in equilibrium if the realised investment return is sufficiently low ($r < \tilde{r}_1$). Finding the equilibrium proportion of withdrawals for a given investment return as in the previous case, the *upper* bound on the investment return is:

$$\tilde{r}_1 \equiv R_n^* - \frac{\alpha}{\gamma}(\bar{r} - R_n^*) - \frac{1}{\sqrt{\gamma}}\Phi^{-1}\left(\frac{y_n - \lambda}{1-\lambda}\right) \to R_n^*$$

Holding more liquidity has two effects. First, it allows to serve a larger proportion of withdrawing investors without liquidating the project. As liquidation is costly, this reduces the amount of coordination failure between late investors for a given investment return and thus the withdrawal threshold:

$$\frac{\partial R_n^*}{\partial y_n} = -\frac{\chi}{1 - D_n^*} < 0 \tag{9}$$

Second, more liquidity reduces the upper bound on the investment return for which the equilibrium with positive liquidation exists ($\frac{\partial \tilde{r}_1}{\partial y_n} < 0$). More liquidity implies more available resources for withdrawing investors and therefore requires a worse economic condition to sustain positive liquidation as supposed. Lemma 2 summarizes:

Lemma 2. *Consider the withdrawal subgame without systemic liquidation costs* ($d = 0$)*, vanishing private noise* ($\gamma \to \infty$)*, and scarce liquidity* $y_n \in \left(\lambda, \frac{(1+\lambda)}{2}\right)$*. Then any threshold equilibrium has the following features:*

- *$L_n^* > 0$: the marginal investor expects liquidation;*

- *$x_n^* \to 1 + \chi[\frac{1+\lambda}{2} - y_n] > 1$: a late investor withdraws if and only if his signal falls short of this threshold;*

- *$l_n^* > 0$ if and only if $r < x_n^*$: actual liquidation occurs if the economic condition is sufficiently bad.*

Taking the previous lemmas together, the overall threshold equilibrium in the withdrawal subgame, which depends on the realised economic condition r and the amount of liquidity y_n held by the bank, is described in proposition 1:

Proposition 1. *Consider the withdrawal subgame without systemic liquidation costs ($d = 0$) and vanishing private noise ($\gamma \to \infty$).*

- *If liquidity is abundant ($y_n \in [\frac{(1+\lambda)}{2}, 1]$), then there exists a unique threshold equilibrium in the subgame. The marginal investor expects no liquidation to take place ($L_n(x_n^*) = 0$), and the implied withdrawal threshold is $x_n^* = 1$. No liquidation occurs if the economic condition is good, while some liquidation occurs if it is bad ($l_n^* = 0 \Leftrightarrow r \geq 1$).*

- *If liquidity is scarce ($y_n \in \left[\lambda, \frac{(1+\lambda)}{2}\right)$), however, then there exist multiple equilibria in the subgame. The marginal investor expects liquidation to take place ($L_n(x_n^*) > 0$) and the implied withdrawal threshold is $x_n^* \to 1+\chi[\frac{1+\lambda}{2} - y_n] > 1$. The no-liquidation equilibrium occurs for a good economic condition ($r \geq 1$), while the equilibrium with liquidation occurs for a bad economic condition ($r < x_n^*$). Therefore, multiple equilibria exist for a range of economic conditions $[1, x_n^*]$.*

- *The range of economic conditions that support multiple equilibria shrinks as the bank's liquidity increases ($\frac{\partial x_n^*}{\partial y_n} < 0$).*

How does the multiplicity result relate to bank run models that obtain a unique equilibrium with positive liquidation (Goldstein and Pauzner (2005), Morris and Shin (2000))? If there is no excess liquidity ($y_n \to \lambda$), as in these papers, the no-liquidation equilibrium disappears. In fact, the lower bound on the economic condition consistent with no liquidation becomes arbitrarily high ($\tilde{r}_0 \to \infty$) for bounded private noise ($\underline{\gamma} < \gamma < \infty$).

Thus, any equilibrium features a positive amount of liquidation in these papers. By contrast, liquidity drives a wedge between the amount of withdrawals and the liquidation volume in the present paper. If liquidity is scarce and the economic condition good, this supports an equilibrium without liquidation apart from the usual equilibrium with liquidation.

Corollary 1. *If there is no liquidity to serve late investors ($y_n \to \lambda$), the equilibrium without liquidation vanishes ($\tilde{r}_0 \to \infty$). Therefore, models without liquidity and unique equilibria, such as Morris and Shin (2000), are a special case of my model with vanishing liquidity for late investors.*

Finally, consider the marginal benefits of liquidity on the threshold equilibrium in the withdrawal subgame. There is no marginal benefit of liquidity in the no-liquidation equilibrium since the lower bound of the economic condition is unaffected by liquidity. By contrast, the marginal benefit from liquidity is positive in the equilibrium with liquidation. On the one hand, liquidity reduces the range for which the equilibrium with liquidation exists (also for bounded private noise). On the other hand, more liquidity reduces the amount of withdrawals and therefore costly liquidation for a given investment return. The marginal cost of liquidity is the reduction in the payoff to a late investors conditional on no-liquidation.

3.2 Systemic liquidation costs

I now consider the case with systemic liquidation costs ($d = 1$). Suppose that the marginal investor expects liquidation ($L_n^n(x_n^*) > 0$) and liquidation by the other bank ($L_{-n}^n(x_n^*) > 0$). If the marginal investor expects no liquidation by the other bank, then systemic liquidation costs have no impact and the equilibrium threshold is determined as in the previous case. The indifference condition of the marginal investor becomes:

$$\begin{aligned} R_n^*(R_{-n}^*; y_n, y_{-n}) \quad = \quad & 1 + \chi \left[(1-\lambda)\Phi\left(\sqrt{\delta}[R_n^* - \bar{r}]\right) + \lambda - y_n \right] + \cdots \\ & \cdots + \chi \left[(1-\lambda)\Phi\left(\sqrt{\delta}(1+\frac{\gamma}{\delta})[R_{-n}^* - \bar{r}] - \frac{\gamma}{\delta}\sqrt{\delta}[R_n^* - \bar{r}]\right)\right] \end{aligned} \tag{10}$$

As the marginal investor takes the withdrawal threshold of investors in the other bank R_{-n}^* as given, equation 10 specifies a best-response function since there exists a unique solution R_n^* for any given R_{-n}^* as shown below.

Following Morris and Shin (2003) and Goldstein (2005), the uniqueness proof is in two steps. First, a unique solution R^*_n must be obtained for any R^*_{-n}, requiring that the slope of the left-hand side of the best response function exceeds the slope of the right-hand side. Second, there is a unique intersection of best response functions, requiring that the best response function is bounded and that its slope is strictly within zero and one. Since the cumulative distribution function lies within zero and one, these conditions are all satisfied if the private noise is sufficiently small, yielding a unique solution $R_A(y_A, y_B), R_B(y_B, y_A)$.[15]

Coordination failure again induces an inefficiently large withdrawal threshold ($R^*_n > 1$). Coordination failure now takes place not only between investors of a given bank, but also between investors of different banks. Fearing that late investors of another bank withdraw, thereby increasing the liquidation volume of the other bank and therefore the liquidation costs of a given bank, a late investor of the given bank has a higher incentive to withdraw at the interim date as well.

Furthermore, the threshold is below the expected return of the project ($R^*_n < \bar{r}$) if the (total) liquidation cost is sufficiently low relative to the expected return ($\bar{r} > \bar{r}_L \equiv 1 + \chi(1 - \lambda)$). The behaviour of investors is consistent with a liquidation in equilibrium if the realised investment return is sufficiently bad ($r < \tilde{r}_1$). As the private noise vanishes, the symmetric thresholds converge to:[16] [17]

$$R^*_A = R^*_B = R^* \equiv 1 + \chi(1 + \lambda - y_A - y_B) \in (1, \bar{r}) \tag{11}$$

The banks' liquidity choices affects the withdrawal thresholds. More liquidity allows

[15] The slope of the right-hand side is now:

$$D_n \equiv \chi(1-\lambda)\sqrt{\delta}\{\phi\left(\sqrt{\delta}[R^*_n - \bar{r}]\right) - \frac{\gamma}{\alpha}\phi\left(\sqrt{\delta}(\frac{\alpha+\gamma}{\gamma})[R^*_{-n} - \bar{r}] - \frac{\gamma}{\alpha}\sqrt{\delta}[R^*_n - \bar{r}]\right)\}$$

[16] This can be proved by contradiction. Suppose that $R^*_A > R^*_B$. Then $W^A_B \to 0$ and $W^B_A \to (1-\lambda)$. The implied expressions for the thresholds can never satisfy the supposed inequality $R^*_A > R^*_B$. The argument applies for $R^*_A < R^*_B$ as well. Therefore, $R^*_A = R^*_B$ as claimed.

[17] The symmetry in the liquidation cost function implies an equal weight of liquidity choices in the withdrawal threshold expression. This can be relaxed, for example by putting a larger weight on the own liquidation volume or with a convex liquidation specification. Either specification implies a larger weight of a bank's withdrawal threshold on its own liquidity. For example, a liquidation cost function that is linear in both the own and the total liquidation volume $x(l_n, l_{-n}) = \chi l_n(l_n + dl_{-n})$ yields $R^*_n \to 1 + \chi(1 + \lambda - y_A - y_B)(\frac{1+\lambda}{2} - y_n)$ in the case of systemic liquidation costs.

to serve more withdrawing investors and thus reduces the coordination failure among investors of a given bank. Thus, more liquidity held by bank n reduces the withdrawal threshold R_n^*. Because of systemic liquidation costs, there is also coordination failure among investors of different banks. More liquidity held by a given bank reduces the degree of this coordination failure and therefore the other bank's threshold R_{-n}^*:

$$\frac{\partial R_n^*}{\partial y_n} = \frac{\partial R_n^*}{\partial y_{-n}} = -\chi < 0$$

Lemma 3 summarizes the new results in the case of systemic liquidation costs.

Lemma 3. *Consider the withdrawal subgame with systemic liquidation costs ($d = 1$), vanishing private noise ($\gamma \to \infty$), and scarce liquidity ($y_n \in (\lambda, \frac{(1+\lambda)}{2})$). Then, the marginal investor expects liquidation to occur ($L_n(x_n^*) > 0$), and the withdrawal threshold is $x_n^* \to 1+\chi[1+\lambda-y_A-y_B] \in (1, \bar{r})$ if $\bar{r} > \bar{r}_L = 1+\chi(1-\lambda)$. There is actual liquidation ($l_n^* > 0$) if and only if the economic condition is bad ($r \leq \tilde{r}_1 \to x_n^*$). The equilibrium threshold highlights the system-wide effects of liquidity because more liquidity held at either bank reduces the withdrawal threshold of a given bank.*

3.3 Optimal portfolio choice

I complete the characterisation of the equilibrium by studying the banks' privately optimal liquidity choice at the initial date. To generate macro-prudential implications, I consider the setup with systemic liquidation costs and scarce liquidity. A lower bound on the expected investment return derived below suffices to generate scarce liquidity in equilibrium. As multiple equilibria occur for scarce liquidity (see proposition 1), some equilibrium selection is required. Since liquidity has a beneficial effect in the equilibrium with liquidation, I assume that this equilibrium in the subgame is selected whenever it exists.[18] A bank's objective function is the expected utility of its investors derived in Appendix A.2 and given by:

$$EU_n(y_n, y_{-n}) = y_n + (1-y_n)\left[F(R^*)\cdot 1 + (1-F(R^*))\cdot\left(\bar{r} + \frac{f(R^*)}{\alpha(1-F(R^*))}\right)\right] \quad (12)$$

where $f(r) = \phi(\sqrt{\alpha}[r-\bar{r}])$ is the probability distribution function of the normally distributed investment return and $F(r)$ the associated cumulative distribution function. The expected utility has two terms. The first term is the amount of liquidity, and the second

[18]This which can be generalized to any fraction $p \in [0, 1]$.

term is the payoff from the investment $(1 - y_n)$. If the investment return falls short of the threshold R^*, which occurs with probability $F(R^*)$, the project is liquidated. Otherwise, the project is continued, which occurs with probability $1-F(R^*)$, and the expected investment return conditional on continuation is $\mathrm{E}[r|r > R^*] = \bar{r} + \frac{f(R^*)}{\alpha(1-F(R^*))}$.

A lower withdrawal threshold improves expected utility as it implies a smaller area of inefficient withdrawals by reducing the extent of coordination failure ($\partial EU_n/\partial R^* < 0$), as derived in Appendix A.2. This highlights the beneficial role of liquidity in the equilibrium with liquidation: more liquidity reduces coordination failure and therefore the withdrawal threshold, thereby indirectly improving the expected utility of an investor:

$$\frac{\partial EU_n}{\partial R^*}\frac{\partial R^*}{\partial y_n} > 0$$

There is also a detrimental role of liquidity. As the ex-ante opportunity cost of liquidity is the foregone higher expected investment return, holding more liquidity is costly. This is further exacerbated by optimal liquidation, shielding the investor from particularly adverse outcomes of the project. The direct effect of liquidity is:

$$\frac{\partial EU_n}{\partial y_n} = -(1 - F(R^*))\left(\mathrm{E}[r|r \geq R^*] - 1\right) < 0$$

In case of continuation, which occurs with probability $1 - F(R^*)$, the expected investment return conditional on continuation exceeds the unit return to liquidity. In case of liquidation, which occurs with probability $F(R^*)$, the project and liquidity both yield a unit return.

The bank balances the beneficial and detrimental effects of liquidity. It takes the response of investors at the interim date $R^*(y_n, y_{-n})$ into account and the other bank's choice of liquidity y_{-n} as given. The optimal liquidity choice of bank n solves the following problem:

$$y_n^*(y_{-n}) \equiv \arg\max_{y_n} EU_n(y_n, y_{-n}) \text{ s.t. } R^* = R^*(y_n, y_{-n}) \tag{13}$$

where the best response function $y_n^*(y_{-n})$ is determined by the first-order condition:

$$\begin{aligned} \frac{dEU_n}{dy_n} = \frac{\partial EU_n}{\partial y_n} + \frac{\partial EU_n}{\partial R_n^*}\frac{\partial R_n^*}{\partial y_n} &= 0 \\ \underbrace{\chi(1-y_n)(R^*-1)f(R^*)}_{\text{marginal benefits from liquidity}} &= \underbrace{[1-F(R^*)]\left(\frac{1}{\alpha}\frac{f(R^*)}{1-F(R^*)} + \bar{r} - 1\right)}_{\text{marginal (opportunity) costs of liquidity}} \end{aligned} \tag{14}$$

I derive conditions on the expected investment return to ensure the existence of a unique best response function in Appendix A.3. First, an upper bound on the expected investment return $\bar{r}_H$ ensures that the first-order condition has a solution for any feasible liquidity choice of the other bank. Second, a lower bound on the investment return $\bar{r}_L$ ensures that liquidity is indeed scarce as supposed, again for any feasible liquidity choice of the other bank. Finally, I show that the objective function EU_n is globally concave in the level of liquidity y_n. Therefore, a unique solution $y_n^*(y_{-n})$ exists for any level of liquidity held by the other bank.

There is strategic substitutability in liquidity holdings. If the other bank holds more liquidity, the liquidation cost of a given bank is reduced for any given level of liquidity. As holding liquidity is costly, the bank optimally reduces its liquidity level, free-riding on the other bank's liquidity. The other bank's liquidity holding is only useful for partially deterring a run since a potential liquidation cost is reduced, but not for serving investors when they do withdraw. Thus, the reduction in liquidity is less than one-for-one:

$$\frac{dy_n^*}{dy_{-n}} = -\frac{(R^*-1)+\chi(1-y_n)[1+\alpha(R^*-1)(\bar{r}-R^*)]}{2(R^*-1)+\chi(1-y_n)[1+\alpha(R^*-1)(\bar{r}-R^*)]} \in (-1,0) \tag{15}$$

Since the slope of the best-response function lies strictly within the unit circle, is bounded and symmetric, there exists a unique and symmetric level of liquidity held at each bank: $y_n^* \equiv y^*$. It is implicitly given by $\frac{dEU_n}{dy_n}(y^*, y^*) = 0$. Proposition 2 summarizes.

Proposition 2. *Consider the overall game with systemic liquidation costs* $(d = 1)$*, vanishing private noise* $(\gamma \to \infty)$*, and an expected investment return within the range* $(\bar{r}_L, \bar{r}_H)$*. Suppose that the equilibrium with liquidation is selected if multiple equilibria exist in the withdrawal subgame. Then, there exists a unique and symmetric equilibrium in threshold strategies. It is characterized by a bank's liquidity choice* $y_A^* = y_B^* \equiv y^* \in \left(\lambda, \frac{1+\lambda}{2}\right)$ *at the initial date and withdrawal threshold of investors in the subgame that are implicitly given by:*

$$R^* = 1+\chi(1+\lambda-2y^*) \in (1,\bar{r}) \tag{16}$$

$$\chi(1-y^*)(R^*-1) = \frac{1}{\alpha}+(\bar{r}-1)\frac{1-F(R^*)}{f(R^*)} \tag{17}$$

The boundaries on the expected investment return are $\bar{r}_L \equiv 1+\chi(1-\lambda)$ *and* $\bar{r}_H \equiv 1+\frac{f(1+0.5\chi(1-\lambda))}{1-F(1+0.5\chi(1-\lambda))}[0.5\chi^2(1-\lambda)^2-1/\alpha]$.

The equilibrium is characterised by partial free-riding on the respective other bank's liquidity.

4 Welfare

This section derives the liquidity choice of a social planner and compares it to the bank's optimal portfolio choice. As in Lorenzoni (2008), I adopt the notion of constrained efficiency: the social planner chooses the levels of liquidity but takes the optimal withdrawal decision of investors at the interim date as given. A direct choice of the threshold would achieve the first-best allocation ($R_n^* = 1$). In contrast to a bank, the planner internalizes the beneficial effects of liquidity for another bank's investors (system-wide effects of liquidity). Therefore, the constrained planner can be thought of as a macro-prudential authority.

The constrained *socially efficient* levels of liquidity (y_A^{SP}, y_B^{SP}) solve the planner's portfolio choice problem at the initial date, taking investors' responses at the interim date $R_n^*(y_A, y_B)$ into account:

$$(y_A^{SP}, y_B^{SP}) \equiv \arg\max_{y_A, y_B} SWF \equiv EU_A + EU_B \text{ s.t. } R_A^*(y_A, y_B) = R_B^*(y_A, y_B) \tag{18}$$

The first-order condition for the social planner's problem is:

$$\begin{aligned} 0 = \frac{dSWF}{dy_n} &= \frac{\partial EU_n}{\partial y_n} + \frac{\partial EU_n}{\partial R_n^*}\frac{\partial R_n^*}{\partial y_n} + \frac{\partial EU_{-n}}{\partial R_{-n}^*}\frac{\partial R_{-n}^*}{\partial y_n} \\ \underbrace{\chi(1 - y_A + 1 - y_B)(R^* - 1)f(R^*)}_{\text{social marginal benefits from liquidity}} &= \underbrace{[1 - F(R_n^*)]\left(\frac{1}{\alpha}\frac{f(R_n^*)}{1 - F(R^*)} + \bar{r} - 1\right)}_{\text{social marginal costs of liquidity}} \end{aligned} \tag{19}$$

The planner balances the social marginal cost of liquidity in terms of foregone investment return conditional on continuation ($\partial EU_n/\partial y_n < 0$) with the social marginal benefits from liquidity in terms of lower withdrawal thresholds. The private and social marginal costs of liquidity coincide, while the social marginal benefits from liquidity exceed the private marginal benefit. Apart from the beneficial effect of liquidity on the investors of one bank ($\partial R_n^*/\partial y_n < 0$), which is identical to the private benefit from liquidity, the planner also considers the beneficial effect of liquidity on the other bank's investors ($\partial R_{-n}^*/\partial y_n < 0$). Recall that more liquidity allows to serve more withdrawing investors and therefore avoids costly liquidation for a given number of withdrawals, thereby mitigating the coordination failure between investors.

The optimization problem is fully symmetric. There is full substitutability between liquidity held at one bank and that held at another to reduce the withdrawal threshold $R_n^{SP} = R^{SP} = 1 + \chi[1 + \lambda - y_A^{SP} - y_B^{SP}]$. Furthermore, both first-order conditions yield the same condition (equation 19). Therefore, only the total amount of liquidity is determined $y_{total}^{SP} \equiv y_A^{SP} + y_B^{SP}$.

I derive conditions on the expected investment return to ensure the existence of a unique constraint efficient liquidity level. First, the upper bound on the expected investment return changes relative to the bank's portfolio choice, and the following upper bound on the investment return is now required:

$$\bar{r} < \bar{r}_H^{SP} \equiv 1 + \left(2\chi^2(1-\lambda)^2 - \frac{1}{\alpha}\right) \frac{f(1+\chi(1-\lambda))}{1-F(1+\chi(1-\lambda))} \tag{20}$$

which is strictly below the upper bound of $1 + 2\chi$ implied by no-dominance. Second, the upper bound $y_{total} \to 2\bar{y}$ is never optimal. Finally, the global concavity of the social welfare function in the total amount of liquidity is established in Appendix (A.4) for which the no-dominance bound on the expected investment return suffices. Taking these points together, there exists a unique level of total liquidity y_{total}^{SP} that maximizes social welfare and is implicitly given by $\frac{dSWF}{dy_n}(y_{total}^{SP}) = 0$.

Proposition 3 summarizes and compares the total amount of liquidity held by a planner with the total amount of liquidity held by banks:

Proposition 3. *Consider the overall game with systemic liquidation costs ($d = 1$), vanishing private noise ($\gamma \to \infty$), and an expected investment return $\bar{r}_H < \bar{r}_H^{SP} \equiv 1 + \left(2\chi^2(1-\lambda)^2 - \frac{1}{\alpha}\right) \frac{f(1+\chi(1-\lambda))}{1-F(1+\chi(1-\lambda))}$. A macro-prudential authority, the constrained social planner, chooses the liquidity level and investors respond optimally as before. Then, there exists a unique level of total liquidity y_{total}^{SP} that maximizes social welfare and is implicitly given by:*

$$\chi(2 - y_{total}^{SP})(R^{SP} - 1) = \left(\frac{1}{\alpha} + [\bar{r} - 1]\frac{1 - F(R^{SP})}{f(R^{SP})}\right) \tag{21}$$

where $R^{SP} = 1 + \chi[1 + \lambda - y_{total}^{SP}]$ is the withdrawal threshold of investors at either bank.

A macro-prudential authority holds more liquidity than the private banking system:

$$y_{total}^{SP} > y_A^* + y_B^* \tag{22}$$

Proof. The higher level of liquidity held by a macro-prudential authority remains to be proven. Relative to the bank's first-order condition, the right-hand side of the social planner's first-order condition (19) has an additional positive term, the positive externality of liquidity in terms of reducing the other bank's withdrawal threshold. Thus, the social benefits from liquidity exceed the social cost of liquidity when evaluated at the optimal level $y_A^* = y_b^* = y^*$:

$$\left.\frac{dSWF}{dy_n}\right|_{y_n=y_{-n}=y^*} > 0 \tag{23}$$

Given the strict global concavity of the objective function in the total amount of liquidity y_{total}, the planner's total amount of liquidity must be higher ($y_{total}^{SP} > y_A^* + y_B^*$), thereby internalising the positive system-wide externality of liquidity. □

The difference between the constraint efficient and the optimal level of liquidity is interpreted as a *macro-prudential liquidity buffer.* A constrained planner, such as a macro-prudential authority, takes all economy-wide effects into account by holding more liquidity, internalising the social costs of liquidation that arise in the presence of systemic liquidation cost.

5 Comparative Statics

This sections studies how the equilibrium allocation y^* and the planner's allocation y_{total}^{SP} vary with the exogenous parameters of the model. Parameters of interest are the liquidation cost parameter χ, the expected investment return $\bar{r}$, and the proportion of early investors λ. Proposition 4 summarizes the results.

Proposition 4. *The private and social levels of liquidity vary according to*

(a) $\frac{\partial y^*}{\partial \chi} > 0$ *and* $\frac{\partial y_{total}^{SP}}{\partial \chi} > 0$ *such that a higher liquidation cost raises the liquidity held privately and socially;*

(b) $\frac{\partial y^*}{\partial \bar{r}} < 0$ *and* $\frac{\partial y_{total}^{SP}}{\partial \bar{r}} < 0$ *such that a higher investment return lowers the private and social levels of liquidity;*

(a) $\frac{\partial y^*}{\partial \lambda} > 0$ *and* $\frac{\partial y_{total}^{SP}}{\partial \lambda} > 0$ *such that a larger proportion of early investors induces higher liquidity holdings.*

See Appendix A.5 for a proof. The intuition underlying these results is as follows. First, a larger proportion of early investors increases the liquidity held privately and socially. Since early investors wish to consume at the interim date and always withdraw, more liquidity is held to serve them.

Second, the strength of the liquidation cost is captured by the liquidation cost parameter χ. It affects the benefits from holding liquidity in terms of avoiding costly liquidation in case of elevated withdrawals, thereby reducing the withdrawal threshold. Thus, if liquidation is more costly, such as in times of financial distress, then liquidity is particularly valuable and more liquidity is held both privately and socially.

Third, a higher expected investment return $\bar{r}$ affects the ex-ante opportunity cost of holding liquidity. Therefore, both banks and the planner hold more liquidity when the project pays a better return on average. Note that there is no effect of the expected investment return on the withdrawal threshold R_n^* as private noise vanishes. However, if the private noise is bounded ($\underline{\gamma} < \gamma < \infty$), a higher investment return also reduces the run threshold. This second effect would further reduce the level of liquidity held.

6 Conclusion

This paper examined the role of liquidity in an economy with many banks subject to runs and systemic liquidation costs. I showed that the presence of liquidity, which drives a wedge between the amount of withdrawals and the liquidation volume, restores multiple equilibria – even if a global game refinement is used. Apart from the usual equilibrium with liquidation (Morris and Shin (2000); Goldstein and Pauzner (2005)), a no-liquidation equilibrium exists for a range of economic conditions. Furthermore, systemic liquidation costs imply that one bank's liquidity holding reduces the liquidation costs of other banks. The positive implication is the partial substitutability of private liquidity holdings as banks free-ride on the liquidity holdings of other banks. The normative implication is that banks hold insufficient liquidity relative to the average liquidity holding of a constrained planner. Since a planner internalizes the system-wide effects of liquidity, I interpret the planner as a macro-prudential authority.

This framework provides a natural laboratory for studying macro-prudential policies in a micro-founded setting more generally. I abstracted from capital requirements, diversification, and taxes on withdrawals in this paper, but analyze some of these in other work. There are other elements relevant to the conduct of macro-prudential regulation omitted in this framework, such as limited liability, 'too big to fail', and perverse incentives arising from incentive schemes. These are all exciting avenues for subsequent research.

A Appendix

A.1 Posterior distributions

Investment return The posterior mean of the investment project return is a weighted average of the mean of the prior distribution and the private signal, in which the relative weights are given by the respective precisions. The precision of the posterior distribution is the sum of the precisions of the prior and the signal. Normality is preserved:

$$\boldsymbol{R}_i^n \sim \mathcal{N}\left(\frac{\alpha\bar{r}+\gamma x_i}{\alpha+\gamma}, \frac{1}{\alpha+\gamma}\right) \tag{24}$$

The ratio of the precision of the prior (public signal) relative to the private signal, $\frac{\alpha}{\gamma}$, determines the extent to which the posterior mean depends on the private signal. The more precise the private signal relatively to the prior, the more the posterior is determined by the private signal. In the limit of vanishing private noise ($\frac{\alpha}{\gamma} \to 0$ as $\gamma \to \infty$), the posterior mean converges to the private signal.

Proportion of prematurely withdrawing late investors at bank n Using the definition of the proportion of withdrawing investors, the posterior distribution of the mean, and a law of large numbers, the posterior proportion of withdrawing late investors at a given investor's bank $W_{i,n}^n$ can be written as:

$$W_{i,n}^n = (1-\lambda)\Phi\left(\sqrt{\delta}\left[R_n^* - \bar{r}\right] + \sqrt{\frac{\gamma(\alpha+\gamma)}{\alpha+2\gamma}}\left[R_n^* - R_i^n\right]\right) \tag{25}$$

$$\delta \equiv \frac{\alpha^2(\alpha+\gamma)}{\gamma(\alpha+2\gamma)} \tag{26}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution and δ summarizes precision parameters. A late investor that receives the threshold signal $x_i = x_n^*$ thus forms the following posterior mean of the proportion of withdrawing late investors at his bank:

$$(W_n^n)^* \equiv W_{i,n}^n\Big|_{x_i=x_n^*} = (1-\lambda)\Phi(z_{1n}) \tag{27}$$

$$z_{1n} \equiv \sqrt{\delta}\left[R_n^* - \bar{r}\right] \tag{28}$$

If the private noise vanishes ($\gamma \to \infty$), then $\delta \to 0$ and $(W_n^n)^* \to \frac{1-\lambda}{2}$.

Proportion of prematurely withdrawing late investors at bank $-n$ Withdrawal thresholds may differ across banks. Depending on the other bank's threshold R^*_{-n}, an investors at bank n expects the following proportion of withdrawing late investors at bank $-n$:

$$W^n_{i,-n} = (1-\lambda)\Phi\left(\sqrt{\delta}\left[R^*_{-n} - \bar{r}\right] + \sqrt{\frac{\gamma(\alpha+\gamma)}{\alpha+2\gamma}}\left[R^*_{-n} - R^n_i\right]\right) \tag{29}$$

$$\left(W^n_{-n}\right)^* \equiv W^n_{i,-n}\big|_{x_i = x^*_n} = (1-\lambda)\Phi(z_{2n}) \tag{30}$$

$$z_{2n} \equiv \sqrt{\delta}\left[R^*_{-n} - \bar{r}\right] + \sqrt{\delta}\frac{\gamma}{\alpha}\left[R^*_{-n} - R^*_n\right] \tag{31}$$

A.2 Derivation of expected utility EU_n

When private noise vanishes ($\gamma \to \infty$), equilibrium withdrawals by late investors at the interim date are:

$$w^*_n(r) = (1-\lambda)\Phi\left(\frac{\alpha}{\sqrt{\gamma}}[R^*_n - \bar{r}] + \sqrt{\gamma}[R^*_n - r]\right) \to \begin{cases} 0 & r > R^*_n \\ \frac{1-\lambda}{2} \quad \text{if} & r = R^*_n \\ 1-\lambda & r < R^*_n \end{cases} \tag{32}$$

Therefore, there is no liquidation if the project return is above the threshold R^*_n, while the investment project is completely liquidated if the investment return is below the threshold. Late investors receive the continuation payoff c_{2n} in the former case and unity in the latter. Early investors always receive unity as promised. Adding these components up, the expected utility is:

$$EU_n(y_n, y_{-n}) = \int_{-\infty}^{R^*_n} 1 \cdot 1 dF(r) + \int_{R^*_n}^{\infty} \lambda \cdot 1 + (1-\lambda) \cdot \frac{y_n - \lambda + (1-y_n)r}{1-\lambda} dF(r) \tag{33}$$

$$= y_n + (1-y_n)\left[F(R^*_n)\cdot 1 + (1 - F(R^*_n)) \cdot \left(\bar{r} + \frac{f(R^*_n)}{\alpha(1-F(R^*_n))}\right)\right] \tag{34}$$

The partial derivatives are:

$$\frac{\partial EU_n}{\partial y_n} = -(1-F(R^*_n))\left(\mathrm{E}[r|r \geq R^*_n] - 1\right) < 0 \tag{35}$$

$$\frac{\partial EU_n}{\partial R^*_n} = -(1-y_n)(R^*_n - 1)f(R^*_n) < 0 \tag{36}$$

$$\frac{\partial^2 EU_n}{\partial y_n^2} = \frac{\partial EU_n}{\partial y_{-n}} = 0 \tag{37}$$

$$\frac{\partial^2 EU_n}{\partial y_n \partial R^*_n} = (R^*_n - 1)f(R^*_n) > 0 \tag{38}$$

$$\frac{\partial^2 EU_n}{\partial (R^*_n)^2} = -(1-y_n)f(R^*_n)[1 + \alpha(R^*_n - 1)(\bar{r} - R^*_n)] < 0 \tag{39}$$

where the signs are implied by the ordering $1 < R^*_n < \bar{r}$ (Lemma 3).

A.3 Unique best response $y_n^*(y_{-n})$

Let $\Lambda(R^*) \equiv \frac{1-F(R^*)}{f(R^*)} > 0$ and therefore $\Lambda'(R^*) = -\sqrt{\alpha} - \alpha(\bar{r} - R^*)\Lambda(R^*) < 0$. The first-order condition becomes:

$$\chi^2(1 - y_n^*)(1 + \lambda - y_n^* - y_{-n}) = \frac{1}{\alpha} + (\bar{r} - 1)\Lambda(R_n^*) \tag{40}$$

where $R_n^* = 1 + \chi(1 + \lambda - y_n^* - y_{-n})$. Note that the left-hand side (LHS) of equation (40) is decreasing in the liquidity level y_n^*, while the right-hand side (RHS) is increasing in it.

First, existence of equilibrium requires that the LHS exceeds the RHS when evaluated at the lower bound $y_n^* \to \lambda$ for any liquidity level y_{-n}. This inequality is hardest to satisfy for $y_{-n} \to \bar{y} \equiv \frac{1+\lambda}{2}$. Rewriting yields an upper bound on the expected investment return:

$$\bar{r} < \bar{r}_H \equiv 1 + \frac{\frac{\chi^2(1-\lambda)^2}{2} - \frac{1}{\alpha}}{\Lambda(1 + 0.5\chi(1 - \lambda))} \tag{41}$$

This upper bound is strictly below the level of $1 + 2\chi$ as implied by no dominance and replaces this upper bound.

Second, the supposed scarcity of liquidity requires that the marginal cost of liquidity just exceeds its marginal benefit as the liquidity level converges its upper bound $\bar{y}$. Therefore: $LHS(y_n^* \to \bar{y}) < RHS(y_n^* \to \bar{y})$ for any liquidity level y_{-n}. This inequality is hardest to satisfy for $y_{-n} \to \lambda$. Rewriting yields a lower bound on the expected investment return:

$$\bar{r} > \bar{r}'_L \equiv 1 + \frac{\frac{\chi^2(1-\lambda)^2}{4} - \frac{1}{\alpha}}{\Lambda(1 + 0.5\chi(1 - \lambda))} < \bar{r}_H \tag{42}$$

This lower bound is strictly below the level of $\bar{r}_L = 1+\chi(1-\lambda)$ as implied by no constraint that ensures $R_n^* < \bar{r}$. Therefore, the lower bound of $\bar{r}_L$ is maintained. Note that $y_n^* \neq \bar{y}$ implies $y_n^* < \bar{y}$ by the global concavity of the objective function, which can be seen by the sign of the second-order derivative of the objective function:

$$\frac{d^2 EU_n}{dy_n^2} = \frac{\partial R_n^*}{\partial y_n}\left[2\frac{\partial^2 EU_n}{\partial y_n \partial R_n^*} + \frac{\partial^2 EU_n}{\partial (R_n^*)^2}\frac{\partial R_n^*}{\partial y_n}\right] + \frac{\partial EU_n}{\partial R_n^*}\frac{\partial^2 R_n^*}{\partial y_n^2} < 0$$

where $\frac{\partial^2 R_n^*}{\partial y_n^2} = 0$ and the sign follows directly from the previously established signs on the partial derivatives of the withdrawal threshold R_n^* and the expected utility EU_n.

A.4 Global concavity of SWF

Consider the second derivative of the social welfare function:

$$\frac{d^2 SWF}{d(y^{SP})^2} = -\chi f(R^*)[\sqrt{\alpha}(\bar{r}-1)+\chi(3+\lambda-y_{total})-(\bar{r}-R^*)(1-\alpha\chi^2(1+\lambda-y_{total}(2-y_{total})))] < 0$$

The highest possible values is reached when $\alpha \to 0$ and $y_{total} \to 2\bar{y}$. Then, the second-order derivative is still negative as $1 + 2\chi > \bar{r}$ by no-dominance. Therefore, the second-order derivative is always negative, establishing global concavity of the social welfare function.

A.5 Comparative statics

A.5.1 Privately optimal liquidity level y^*

Parameters of interest are $\chi, \bar{r}, \lambda$. The effect of parameters on the withdrawal threshold $R^* = 1 + \chi(1 + \lambda - 2y^*)$ is:

$$\frac{\partial R^*}{\partial \chi} = (1 + \lambda - 2y^*) > 0 \quad (43)$$

$$\frac{\partial R^*}{\partial \bar{r}} = 0 \quad (44)$$

$$\frac{\partial R^*}{\partial \lambda} = \chi > 0 \quad (45)$$

The first-order condition for the private level of liquidity y^* can be written as $G(a, R^*, y^*) = 0$, where $a \in \{\chi, \bar{r}, \lambda\}$ is a parameter:

$$G(a, R^*, y^*) = (r - 1)\frac{1 - F(R^*)}{f(R^*)} + \frac{1}{\alpha} - \chi^2(1 - y^*)(1 + \lambda - 2y^*) \quad (46)$$

Then, the effect of a parameter on the equilibrium liquidity level is given by $\frac{dy^*}{da} = -\frac{\partial G/\partial a}{\partial G/\partial y}$. Note that:

$$\frac{\partial G}{\partial y^*} = -2\chi(\bar{r} - 1)\Lambda'(R^*) + \chi^2(3 + \lambda - 4y^*) > 0 \quad (47)$$

$$\frac{\partial G}{\partial \bar{r}} = (\bar{r} - 1)[\sqrt{\alpha} + (\bar{r} - R^*)\Lambda(R^*)] + \Lambda(R^*) > 0 \quad (48)$$

$$\frac{\partial G}{\partial \chi} = (\bar{r} - 1)(1 + \lambda - 2y^*)\Lambda'(R^*) - 2\chi(1 - y^*)(1 + \lambda - 2y^*) < 0 \quad (49)$$

$$\frac{\partial G}{\partial \lambda} = -\chi^2(1 - y^*) + \chi(\bar{r} - 1)\Lambda'(R^*) < 0 \quad (50)$$

Therefore, the partial derivatives of the privately held liquidity levels have the signs as claimed.

A.5.2 Socially efficient liquidity level y_{total}^{SP}

The effect of parameters on the withdrawal threshold $R^{SP} = 1 + \chi(1 + \lambda - y_{total}^{SP})$ is:

$$\frac{\partial R^{SP}}{\partial \chi} = (1 + \lambda - y_{total}^{SP}) > 0 \tag{51}$$

$$\frac{\partial R^{SP}}{\partial \bar{r}} = 0 \tag{52}$$

$$\frac{\partial R^{SP}}{\partial \lambda} = \chi > 0 \tag{53}$$

The first-order condition for the social level of liquidity y_{total}^{SP} can be written as $\tilde{G}(a, R^{SP}, y^{SP}) = 0$, where $a \in \{\chi, \bar{r}, \lambda\}$ is a parameter:

$$\tilde{G}(a, R^{SP}, y_{total}^{SP}) = (r-1)\frac{1 - F(R^*)}{f(R^*)} + \frac{1}{\alpha} - \chi^2(2 - y_{total}^{SP})(1 + \lambda - y_{total}^{SP}) \tag{54}$$

Then, the effect of a parameter on the equilibrium liquidity level is given by $\frac{dy_{total}^{SP}}{da} = -\frac{\partial \tilde{G}/\partial a}{\partial \tilde{G}/\partial y}$. As above, partial differentiation of $\tilde{G}$ proves the signs on the comparative statics of the total level of liquidity held by the planner as claimed.

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