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*Working Paper 10/01*

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With Subsistence Consumption**

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# Saving Rates and Portfolio Choice with Subsistence Consumption

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## Abstract

We analytically show that a common across rich/poor individuals Stone-Geary utility function with subsistence consumption in the context of a simple two-asset portfolio-choice model is capable of qualitatively explaining: (i) the higher saving rates of the rich, (ii) the higher fraction of personal wealth held in stocks by the rich, and (iii) the higher volatility of consumption of the wealthier. On the contrary, time-variant “keeping-up with the Joneses” weighted average consumption playing the role of moving benchmark subsistence consumption gives the same portfolio composition and saving rates across the rich and the poor, failing to reconcile the model with what micro data say.

*Keywords:* elasticity of intertemporal substitution, Stone-Geary preferences, two-asset portfolio, household portfolios, wealth inequality, controlled diffusion

*JEL classification:* G11, D91, E21, D81, D14, D11

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## 1. Introduction

A vast literature studying the connection between consumption and portfolio choice has been trying to explain a number of empirical regularities that are considered as stylized facts. First, households with higher lifetime income exhibit higher saving rates.<sup>1</sup> Second, richer households tend to hold stocks while poorer ones either do not hold stocks at all, or hold a lower fraction of their financial wealth in stocks.<sup>2</sup> Third, stockholders' consumption growth is more volatile than that of non-stockholders.<sup>3</sup>

A fourth empirically motivated perception is that high-income households (who are also stockholders) exhibit higher elasticity of intertemporal substitution (EIS) compared to poorer non-stockholding households.<sup>4</sup> This distinction has led some researchers to assume exogenously different EIS across households as a building block for their models (typically, constant-EIS utility functions where the EIS parameter differs).<sup>5</sup> A typical criticism to assuming exogenously heterogeneous EIS is that one can generate any desired result through “(trivially) assuming convenient preferences”. For instance, assuming that some individuals are born with a higher EIS will tend to directly imply a higher saving rate for them, that these individuals will turn out to be richer, that they will tend to hold more stocks, and that their consumption will be more volatile. Apart from the accusation that assumptions and conclusions are too close, another criticism is that the potentially quantifiable heterogeneity of utility functions is difficult to establish empirically. Although we are not against behav-

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<sup>1</sup> See, for example, Dynan, Skinner, and Zeldes (2004). They report a strongly positive correlation between saving rates and lifetime income and a less strong correlation between lifetime income and marginal propensity to save.

<sup>2</sup> See, for example, Poterba and Samwick (1995), Carroll (2002), and Guvenen (2009, p. 1722) for more updated references and stockholding trends in the US.

<sup>3</sup> See, for example, Mankiw and Zeldes (1991).

<sup>4</sup> See, for example, Guvenen (2009, Section 3), for evidence on this statement.

<sup>5</sup> For example, Guvenen (2009) and De Graeve, Dossche, Emiris, Sneessens, and Wouters (2009) use exogenously heterogeneous EIS as a key assumption in their analysis. Notably, Carroll (2002) provides arguments that the pattern of facts is against exogenous variation in risk tolerance, while in Carroll (2000) he proposes an alternative explanation of the high saving rates of the rich based on bequest motives.

ioral approaches to finance and consumer theory, in this paper we explore the possibility of reconciling models with the three empirical regularities about saving, stockholding, and consumption volatility through a single utility function.

The goal of our analysis is precisely set. Using the simple Merton (1969) model as a vehicle for our thought experiment, we confront two alternative concepts of subsistence consumption with each other, in order to examine their promise for simultaneously resolving consumption/savings and portfolio-choice puzzles. The first concept is the standard constant subsistence level of consumption that we examine using a typical Stone-Geary (time-separable) momentary utility function of the form  $u(c(t)) = \{[c(t) - \chi]^{1-1/\eta} - 1\} / (1 - 1/\eta)$ , where  $\chi, \eta > 0$ . The second concept is the “keeping-up-with-the-Joneses” time-variant subsistence level of consumption given by,  $u(c(t)) = \{[c(t) - \gamma\bar{C}(t)]^{1-1/\eta} - 1\} / (1 - 1/\eta)$ , where  $\gamma, \eta > 0$  and  $\bar{C}(t)$  stands for average consumption in a certain community. This second utility function implies a fully external habit with time separability. We focus on the fundamental Merton (1969) framework in order to obtain simple analytical solutions that allow for comparative static analysis which is robust in a way that numerical solutions cannot offer.

What we find is sharp. The Stone-Geary formulation with *time-invariant* subsistence consumption meets all four empirical regularities at the micro level: it generates (i) saving rates, (ii) risky-asset portfolio shares, (iii) consumption-growth volatility, (iv) endogenous consumption-choice dependent EIS, all four positively dependent on initial or current financial wealth. *Time-invariant* subsistence consumption fails in producing a stationary relative wealth distribution with different EIS across the rich and the poor. On the contrary, the Stone-Geary formulation with *time-variant* subsistence consumption produces a stationary relative wealth distribution with different EIS across the rich and the poor throughout the whole equilibrium path, but fails to reconcile the three first empirical regularities appearing

above. It implies that, (i) saving rates, (ii) risky-asset portfolio shares, (iii) consumption-growth volatility, are all the same across the rich and the poor.

We view that the inability of the formulation with time-variant subsistence consumption to reconcile the data is striking and that it sends a pessimistic message to researchers who work on endogenizing stockholding and who want to simultaneously meet empirical regularities of saving and consumption.<sup>6</sup> Although the two-asset Merton (1969) model assumes away any labor income, our qualitative results are likely to be robust in the typical simulation framework employed in the literature (see, for example, Haliassos and Michaelides (2003)), provided that, in the presence of liquidity constraints, corner solutions are not too dominantly frequent in equilibrium. We anticipate that, even in the presence of liquidity constraints, idiosyncratic uncertainty, finite lives, and exogenous participation costs, households will still be affected from the re-balancing effects of the time-variant benchmark consumption along their life cycle, and there will be a tendency among the rich and the poor to exhibit similar saving rates, portfolio choice, and consumption volatility.

Our focus on subsistence consumption using a common across agents utility function has similarities with the approach of Wachter and Yogo (2009) who propose non-homothetic utility but distinguish between basic goods and luxuries, assuming that people are less risk averse about luxuries than about necessities. Wachter and Yogo (2009) also obtain the result that the rich invest more in risky assets, simply because what they are risking is mostly luxury consumption.<sup>7</sup> The role of subsistence consumption is similar: the rich are willing to take more risk, because the chances that any given wealth shortfall will jeopardize their ability to consume the subsistence consumption are much smaller than for the poor,

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<sup>6</sup> For example, Guvenen (2009, p. 1723, and Supplemental Material) recommends a similar preference formulation, assuming exogenous stock-market participation.

<sup>7</sup> Yogo (2006) has used a similar approach to Wachter and Yogo (2009) distinguishing between durable and nondurable consumption in order to analyze the cyclical behavior of stock returns.

whose consumption is hovering around subsistence. We think that subsistence consumption offers more parsimony as our approach allows to work with the consumer basket rather than with micro-level consumer data, facing the additionally tedious task of having to distinguish between luxuries and necessities. Perhaps this aspect is more appealing to macroeconomists who work on idiosyncratic-risk heterogeneous agent models in order to study asset markets among other questions.

Concerning the technical contribution of our paper, we are not the first who have technically investigated the two-asset Merton (1969) model using the Stone-Geary utility function with time-invariant subsistence consumption. Karatzas et al. (1986) and Sethi et al. (1992) mention explicit solutions to this problem among their other results regarding the possibility of investor bankruptcy. Weinbaum (2005) is another case making use of the ability of the model to land an explicit solution in order to apply it to the bond market. Yet, we offer a simpler solution approach based on undetermined coefficients to the case of time-invariant subsistence (our Proposition 1 in Section 2). On the contrary, we are not aware of other studies that analytically work out the time-variant subsistence formulation. For solving the model using “keeping-up-with-the-Joneses” preferences, it is crucial to use an aggregation result that greatly simplifies the problem. Such an aggregation result appears in Koulovatianos (2005, Theorem 3), and we demonstrate how to use this aggregation result in the context of solving a portfolio-choice problem. Most importantly, we are not aware of any study that uses the explicit solution of the two-asset Merton (1969) model with Stone-Geary preferences in order to address empirical consumption/savings and portfolio choice regularities.

We examine the time-invariant subsistence consumption model in Section 2, while in Section 3 we analyze the time-variant subsistence model. We discuss how our results can be advanced and how they can fit, complement, and extend the existing literature in Section 4

and in Section 5 we conclude.

## 2. Time-invariant Subsistence Consumption

Time is continuous, with  $t \in [0, \infty)$ . Consider the Merton (1969) two-asset model, where an investor having initial wealth holdings  $k_0 > 0$  has the opportunity to invest in a risky asset (investing a fraction  $\phi$  of her wealth in the risky asset) and a risk-free asset.<sup>8</sup> So, the investor chooses the consumption path  $(c(t))_{t \geq 0}$  and the path of portfolio composition over time  $(\phi(t))_{t \geq 0}$ , that maximizes her expected utility,

$$E_0 \left\{ \int_0^\infty e^{-\rho t} \frac{[c(t) - \chi]^{1 - \frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} dt \right\} , \quad (1)$$

with  $\rho, \chi, \eta > 0$ . The household's budget constraint is,

$$dk(t) = \{ [\phi(t) R + (1 - \phi(t)) r^f] k(t) - c(t) \} dt + \sigma \phi(t) k(t) dz(t) , \quad (2)$$

where  $R$  is the mean rate of return of the risky asset,  $r^f$  is the risk-free rate ( $R > r^f$ ), and  $dz(t) = \varepsilon(t) \sqrt{dt}$ , where  $\varepsilon(t) \sim N(0, 1)$ , i.e.  $dz(t)$  is a Brownian motion.

### 2.1 Decision Rules and Dynamics of Financial Wealth

The Hamilton-Jacobi-Bellman equation (HJB) is given by,

$$\rho J(k) = \max_{c \geq 0, \phi} \left\{ \frac{(c - \chi)^{1 - \frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} + J'(k) \{ [\phi R + (1 - \phi) r^f] k - c \} + \frac{(\sigma \phi k)^2}{2} J''(k) \right\} . \quad (3)$$

The first-order conditions are,

$$(c - \chi)^{-\frac{1}{\eta}} = J'(k) , \quad (4)$$

<sup>8</sup> If we assume  $n \geq 2$  risky assets, the mutual fund theorem developed in Merton (1971) and also outlined in Karatzas et al. (1986, Section 5), justifies why the model can be equivalently reduced to a single risky investment.



$$\phi = \frac{J'(k)}{-J''(k)} \frac{R - r^f}{k \sigma^2} . \quad (5)$$

We make two assumptions, technical in nature, that enable us to secure that solutions exist and that they are interior. The rationale behind these assumptions becomes obvious in the process of proving Proposition 1 which appears below.<sup>9</sup>

**Assumption 1** *Initial conditions are restricted so that,*

$$k_0 > \frac{\chi}{r^f} .$$

**Assumption 2** *Parameter  $\eta$  is restricted to be strictly below the strictly positive value,  $\bar{\eta}$ , characterized below:*

$$0 < \eta < \bar{\eta} \equiv \frac{1 - \frac{r^f - \rho}{\frac{1}{2} \left( \frac{R - r^f}{\sigma} \right)^2} + \left\{ \left[ \frac{r^f - \rho}{\frac{1}{2} \left( \frac{R - r^f}{\sigma} \right)^2} \right]^2 + 2 \frac{r^f + \rho}{\frac{1}{2} \left( \frac{R - r^f}{\sigma} \right)^2} + 1 \right\}^{\frac{1}{2}}}{2} .$$

It is easy to verify that  $\bar{\eta} > 0$  for any values of  $\sigma, \rho, R, r^f > 0$ .<sup>10</sup> Moreover, placing such a parametric restriction on parameter  $\eta$  may not prevent the quantitative matching of observed/estimated elasticities of intertemporal substitution (EIS), since in this model

<sup>9</sup> The need for some parametric restrictions arises even without the presence of subsistence consumption. Yet, subsistence consumption places a few more constraints which become obvious in our simple model here which offers analytical results. And still, in order to match data it is perhaps necessary to take analysis a step further, matching labor-income time series patterns in a simulated model that is based on micro-data observations. While specifying a more descriptive model and solving it numerically is beyond the scope of this study (here, we want to make a single point about which preference specification with subsistence consumption seems to be most promising), our analytical model helps in becoming familiar with some technical difficulties of subsistence consumption: in this case, interiority involves identifying parametric or other constraints technical in nature, a number of concerns that may be unpleasant. Nevertheless, we claim that the subsistence-consumption model (and, in particular, the version of this section) is worth to be studied in future research through simulation.

<sup>10</sup>Notice that

$$\left[ 1 - \frac{r^f - \rho}{\frac{1}{2} \left( \frac{R - r^f}{\sigma} \right)^2} \right]^2 < \left[ \frac{r^f - \rho}{\frac{1}{2} \left( \frac{R - r^f}{\sigma} \right)^2} \right]^2 + 2 \frac{r^f + \rho}{\frac{1}{2} \left( \frac{R - r^f}{\sigma} \right)^2} + 1 .$$

$EIS = \eta(1 - \chi/c)$ .<sup>11</sup> Proposition 1 provides the analytical solution to the model.

### Proposition 1

*Under Assumptions 1 and 2, the solution to the problem expressed by the HJB equation given by (3) is a decision rule for consumption,*

$$c^* = C(k) = \xi k + \psi ,$$

where

$$\xi = \rho\eta + (1 - \eta)r^f - \frac{\eta(\eta - 1)}{2} \left( \frac{R - r^f}{\sigma} \right)^2 ,$$

and

$$\psi = \eta\chi \frac{r^f - \rho + \frac{\eta-1}{2} \left( \frac{R-r^f}{\sigma} \right)^2}{r^f} ,$$

a decision rule for portfolio choice,

$$\phi^* = \Phi(k) = \eta \frac{R - r^f}{\sigma^2} \left( 1 - \frac{\chi}{k} \right) ,$$

while the value function is given by,

$$J(k) = -\frac{1}{\rho \left( 1 - \frac{1}{\eta} \right)} + \xi^{-\frac{1}{\eta}} \frac{\left( k - \frac{\chi}{r^f} \right)^{1 - \frac{1}{\eta}}}{1 - \frac{1}{\eta}} .$$

**Proof** See the Appendix.  $\square$

<sup>11</sup>Barsky et al. (1997) offer plausible numbers based on a combination of a survey approach and objective observations. Blundell, Browning, and Meghir (1994) have estimated that the  $EIS$  of households in the top income quintile is about three times that of households in the bottom quintile of the distribution. In his calibration exercise for explaining asset prices, Guvenen (2009) uses values 0.1 for (poorer) non-stockholders and 0.3 for (richer) stockholders. With the aid of parameter  $\chi$ , such values can be matched. It can easily be verified that using a parametrization according to US estimates of stock and bond returns (see, for example, Guvenen (2009, Table II, p. 1725)),  $R = 8\%$ ,  $r^f = 2\%$ ,  $\sigma = 20\%$ , and with  $\rho = 1.5\%$ , the implied level of the upper bound  $\bar{\eta}$  is  $\bar{\eta} \cong 1.25$ . If we use  $\rho = 4.5\%$  and the stock/bond returns above (for example, aggregate-economy models with idiosyncratic risk imply that  $r^f < \rho$ ), then the implied value for  $\bar{\eta}$  is  $\bar{\eta} \cong 1.8$ . In any case, the implied value for  $\bar{\eta}$  gives ample space for plausible calibration approaches.

On a technical note, the role of Assumption 2 is to secure that  $\xi$  in Proposition 1 is strictly positive. The role of Assumption 1 is obvious after looking at the functional form of the value function,  $J(k)$ , in Proposition 1, a crucial condition for guaranteeing that the problem is well-defined and that the solution is interior. It is also easy to verify that  $c = \xi (k - \chi/r^f) + \chi$ , which reveals another role of both Assumptions 1 and 2, which is to meet the requirement that  $c \geq 0$ .

Proposition 2 reveals the dynamics of household wealth, which secure that, in equilibrium,  $k^*(t) > \chi/r^f$  for all  $t \geq 0$ .

### Proposition 2

*Under Assumptions 1 and 2, the dynamics of financial wealth,  $k$ , are fully characterized by,*

$$k^*(t) - \frac{\chi}{r^f} = e^{\eta \left[ r^f - \rho + \frac{1}{2} \left( \frac{R - r^f}{\sigma} \right)^2 \right] t + \eta \left( \frac{R - r^f}{\sigma} \right) z(t)} \left( k_0 - \frac{\chi}{r^f} \right) \quad (6)$$

where  $z(t) = \int_0^t dz(s)$  with  $\int$  being the stochastic (Itô) integral.

**Proof** See the Appendix.  $\square$

## 2.2 Characterization of Saving and Portfolio Choices

With Propositions 1 and 2 at hand we proceed to characterizing the savings, consumption, and stock-holding behavior of a price-taking household. Proposition 3 provides necessary and sufficient conditions for having a saving rate which is increasing in wealth.

### Proposition 3

*Under Assumptions 1 and 2, the saving rate of a household is strictly increasing*

in financial wealth if and only if,

$$\eta > -1 - \frac{r^f - \rho}{\frac{1}{2} \left( \frac{R-r^f}{\sigma} \right)^2} . \quad (7)$$

**Proof** Fix any time instant  $t \geq 0$ . Based on Proposition 1, direct substitution of the decision rule for consumption,  $C(k)$ , and also for portfolio choice,  $\Phi(k)$ , into the household's budget constraint given by (2), after some algebra, gives the equilibrium savings level at time  $t$ ,

$$S^*(t) = \zeta \left[ k^*(t) - \frac{\chi}{r^f} \right] ,$$

with

$$\zeta \equiv \eta \left[ \frac{\eta + 1}{2} \left( \frac{R - r^f}{\sigma} \right)^2 + r^f - \rho \right] .$$

The saving rate,  $s^*(t)$ , is

$$s^*(t) = \frac{S^*(t)}{\phi^*(t) (R - r^f) k^*(t) + r^f k^*(t)}$$

and after substituting  $\Phi(k)$  from Proposition 1 it is

$$s^*(t) = \frac{\zeta}{\eta \left( \frac{R-r^f}{\sigma} \right)^2 + \frac{r^f}{1 - \frac{\chi}{k^*(t)}}$$

which implies that  $ds^*(t)/dk^*(t) > 0 \Leftrightarrow \zeta > 0 \Leftrightarrow (7)$ .  $\square$

Proposition 4 is our immediate stock-holding result.

#### **Proposition 4**

*Under Assumptions 1 and 2, the portfolio share of stocks is strictly positive and increasing in wealth.*

**Proof** Immediate from  $\Phi(k)$  of Proposition 1.  $\square$

Proposition 5 examines the relationship between the coefficient of variation of consumption and initial financial wealth and also the relationship between consumption growth and current wealth.

**Proposition 5**

*Under Assumptions 1 and 2, the coefficient of variation of consumption is strictly increasing in initial financial wealth and the variance of the growth rate of consumption is increasing in current financial wealth.*

**Proof** See the Appendix.□

Proposition 5 shows that the same utility function with subsistence consumption for all households leads to a theoretical prediction which is closer to the empirical observation first made by Mankiw and Zeldes (1991), that stockholders' consumption growth is more volatile than that of non-stockholders. Although in our analysis everybody holds stocks, the poorer in our model hold a lower fraction of their wealth in stocks. Our analysis indicates that in a more descriptive simulated model, perhaps one that includes stock-market participation costs, a substantial fraction of poor agents may become non-stockholders.

**2.3 Dynamics of inequality**

A key feature of this simple partial-equilibrium analysis with price-taking households is that inequality in financial wealth increases over time if everybody's financial wealth grows over time. From equation (6) it is easy to verify that the expected value of wealth of any individual will grow at a strictly positive rate if and only if,

$$r^f - \rho + \frac{1 + \eta}{2} \left( \frac{R - r^f}{\sigma} \right)^2 > 0$$

which is equivalent to the condition given by equation (7), the necessary and sufficient condition for having saving rates that increase in wealth. Considering two individuals, a “rich” and a “poor” according to their initial total asset holdings,  $k_{r,0} > k_{p,0}$  (subscripts correspond to “r: rich” and “p: poor”), equation (6) implies that

$$\frac{k_r^*(t) - \frac{\chi}{r^f}}{k_p^*(t) - \frac{\chi}{r^f}} = \frac{k_{r,0} - \frac{\chi}{r^f}}{k_{p,0} - \frac{\chi}{r^f}}$$

which leads to,

$$\frac{k_r^*(t)}{k_p^*(t)} = \frac{k_{r,0} - \frac{\chi}{r^f}}{k_{p,0} - \frac{\chi}{r^f}} - \frac{\chi}{r^f} \cdot \frac{k_{r,0} - k_{p,0}}{k_{p,0} - \frac{\chi}{r^f}} \cdot \frac{1}{k_p^*(t)}. \quad (8)$$

Equation (8) implies that as  $k_p^*(t)$  increases over time, inequality in relative wealth increases over time.

The feature that the model does not lead to a steady-state stationary distribution of relative wealth may be considered as unattractive. Chan and Kogan (2002) using an external-habit model resolve the issue of having a long-run growth model with a stationary relative wealth distribution. Scholars who study simulated models propose the preference formulation of next model’s section (for example, see Guvenen (2009, footnote 7, p. 1723) and his discussion in the paper’s conclusions regarding the extension of his model to long-run growth analysis). We show that in the context of our model’s unit-root aggregate stock market shocks this proposed preference formulation of “keeping-up with the Joneses” benchmark subsistence consumption leads to less promising results: despite that it is capable of producing a stationary relative wealth distribution and EIS that is increasing in wealth, it nevertheless fails to meet all other micro data empirical regularities.

### 3. Time-variant Benchmark Subsistence Consumption

The structure of the model is the same as above, with the sole difference that utility is of the form

$$E_0 \left\{ \int_0^\infty e^{-\rho t} \frac{[c(t) - \gamma \bar{C}(t)]^{1-\frac{1}{\eta}}}{1 - \frac{1}{\eta}} dt \right\}, \quad (9)$$

where  $\gamma \in (0, 1)$  and with  $\bar{C}(t)$  being average consumption in the economy in period  $t$ .<sup>12</sup> Moreover, we would like to restrict  $\gamma$  so that all individuals in the economy have consumption  $c(t) > \gamma \bar{C}(t)$ . This amounts to a restriction on  $\gamma$  which is driven by the distribution of initial asset holdings. Yet, in order to identify such a parametric restriction involving initial conditions and parameters, the model must be first solved under the working assumption of having interior solutions.<sup>13</sup> Preferences given by equation (9) assume "benchmark consumption levels".<sup>14</sup> Such a utility function captures the "keeping-up with the Joneses" idea of having consumption standards influenced by average consumption standards, with the

<sup>12</sup>For this formulation, see, for example, Guvenen (2009, footnote 7, p. 1723). These preferences fall into the category studied in Koulovatianos (2005, Theorem 3) that gives perfect linear aggregation under certainty. This means that, provided that all solutions are interior, also under aggregate uncertainty the aggregate level of consumption,  $\bar{C}(t)$ , corresponds to the choice of a (perhaps fictitious) consumer with wealth holdings equal to the aggregate wealth level of the economy,  $\bar{K}(t)$ . This property will be proved to be true in equilibrium.

<sup>13</sup>Notice in the model with subsistence consumption above that the initial capital stock of the poorest household,  $\underline{k}_0$  is restricted to  $\underline{k}_0 > \chi/r^f$ , which guarantees that all households in the economy have well-defined problems and interior solutions. Notice also that we were able to identify that restriction only after we have solved the problem under the working assumption of interior solutions.

<sup>14</sup>This is also a variant of the preference formulation in Chan and Kogan (2002), which focuses on the "external habit", in contrast to the formulation of Constantinides (1990), which focuses on the "internal habit." The difference in our formulation is that it is not the stock of external habit but the flow of external habit that influences behavior. As it will be shown below, in equilibrium, consumption paths of all agents grow parallelly. Given that in our model prices are exogenous, it would be difficult to empirically distinguish between the external and internal habit from simulated data from our model. A recent empirical study investigating the relative importance of external vs. internal habits is Grishchenko (2009). Notice that the elasticity of intertemporal substitution (EIS) is

$$EIS = \eta \left( 1 - \gamma \frac{\bar{C}}{c} \right),$$

so, for  $c = \bar{C}$ ,  $EIS = (1 - \gamma)$ , and the formula  $\bar{C}/c = (1/\gamma)(1 - EIS/\eta)$  reveals that empirically plausible levels of the  $EIS$  can match observed consumption ratios  $\bar{C}/c$ .

quantitative impact of this influence moderated through parameter  $\gamma$ . Clearly, we assume a community of investors such that each investor is a price taker (for example, members of the community invest in a globalized international market portfolio of stocks) and keeps track of average consumption,  $\bar{C}$ , of his/her (local) community. The path of average consumption,  $\bar{C}$ , over time is generated through the decision-making process of a fictitious household that possesses average initial wealth  $\bar{K}_0$ .<sup>15</sup> This property is due to that perfect linear aggregation holds in our model, and this we confirm once we derive our solution below.

The budget constraint of the individual is,

$$dk(t) = \{[\phi(t)R + (1 - \phi(t))r^f]k(t) - c(t)\}dt + \sigma\phi(t)k(t)dz(t) , \quad (10)$$

however, in this case, keeping track of the dynamics of average consumption,  $\bar{C}(t)$ , is also necessary. Because the (perhaps fictitious) average household possessing  $\bar{K}(t)$  units of wealth also solves an optimal control problem, the household possessing  $k(t)$  units of wealth will form a value function which depends on both the current level of its financial wealth,  $k(t)$ , and the current level of the poorest household's financial wealth,  $\bar{K}(t)$ , i.e. the value function will be of the form  $J(k, \bar{K})$ . In doing so, the household with  $k$  needs to keep track of the dynamics of the average household's budget constraint (with wealth holdings  $\bar{K}$ ), although the household with  $k$  does not control  $\bar{C}(t)$  or  $\bar{K}(t)$ . Below we reconfirm that the decision rules of all individuals imply perfect linear aggregation. This budget constraint is,

$$d\bar{K}(t) = \{[\bar{\Phi}(t)R + (1 - \bar{\Phi}(t))r^f]\bar{K}(t) - \bar{C}(t)\}dt + \sigma\bar{\Phi}(t)\bar{K}(t)dz(t) , \quad (11)$$

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<sup>15</sup>This household is called a “representative consumer (RC)”, a fictitious household who possesses average wealth and who has a utility function composed by the utility functions of all other household types (this holds in the case of preference heterogeneity – in the present paper all agents have the same preferences, so, in our case, RC's utility function is the same as everyone else's), and whose choices coincide with all aggregated choices of the community under any price regime. For further details on the concept of RC see Caselli and Ventura (2000) and Koulovatianos (2005).



and we incorporate it in the HJB equation of household with holdings  $k$  as,

$$\begin{aligned} \rho J(k, \bar{K}) = \max_{c \geq 0, \phi} & \left\{ \frac{(c - \gamma \bar{C})^{1 - \frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} + J_k(k, \bar{K}) \{ [\phi R + (1 - \phi) r^f] k - c \} + \right. \\ & + J_{\bar{K}}(k, \bar{K}) \{ [\bar{\Phi} R + (1 - \bar{\Phi}) r^f] \bar{K} - \bar{C} \} + \\ & \left. + \frac{(\sigma \phi k)^2}{2} J_{kk}(k, \bar{K}) + \frac{(\sigma \bar{\Phi} \bar{K})^2}{2} J_{\bar{K}\bar{K}}(k, \bar{K}) + \sigma^2 \phi \bar{\Phi} k \bar{K} J_{k\bar{K}}(k, \bar{K}) \right\} \quad (12) \end{aligned}$$

where  $J_x$  denotes the first partial derivative with respect to variable  $x \in \{k, \bar{K}\}$ ,  $J_{xx}$  is the second partial derivative with respect to  $x$ , and the notation for the cross-derivative is obvious. For an individual other than the (perhaps fictitious) individual with wealth holdings other than  $\bar{K}$ , the paths  $\bar{C}(t)$  and  $\bar{\Phi}(t)$  are generated through two decision rules,  $\bar{C}(t) = C(\bar{K}(t))$  and  $\bar{\Phi}(t) = \Phi(\bar{K}(t))$  which are consistent with consumer optimization of the individual with wealth holdings  $\bar{K}(t)$  for all  $t \geq 0$ , and also the budget constraint given by (11). First-order conditions are,

$$(c - \gamma \bar{C})^{-\frac{1}{\eta}} = J_k(k, \bar{K}) \quad (13)$$

$$\phi = \frac{R - r^f}{\sigma^2} \frac{J_k(k, \bar{K})}{-J_{kk}(k, \bar{K}) k} + \bar{\Phi} \cdot \frac{J_{k\bar{K}}(k, \bar{K}) \bar{K}}{-J_{kk}(k, \bar{K}) k} \quad (14)$$

Proposition 6 states the solution to the above problem. Yet, interiority of solutions involves making a parametric constraint which involves parameter  $\gamma$ , given by Assumption 3.

**Assumption 3** *Parameter  $\gamma$  and initial conditions are restricted so that,*

$$\gamma < \frac{k_0}{\bar{K}_0},$$

where  $k_0$  is the initial wealth of the poorest household and  $\bar{K}_0$  is the average initial wealth.

Using Assumption 3 we proceed to characterizing the interior solution of the model, which is given by Proposition 6.

**Proposition 6**

*Under Assumptions 2 and 3, the solution to the problem expressed by the HJB equation given by (12) is a decision rule for consumption,*

$$c^* = C(k) = \xi k ,$$

where

$$\xi = \rho\eta + (1 - \eta)r^f - \frac{\eta(\eta - 1)}{2} \left( \frac{R - r^f}{\sigma} \right)^2 ,$$

*a decision rule for portfolio choice,*

$$\phi^* = \Phi(k) = \eta \frac{R - r^f}{\sigma^2} ,$$

*while the value function is given by,*

$$J(k) = -\frac{1}{\rho \left(1 - \frac{1}{\eta}\right)} + \xi^{-\frac{1}{\eta}} \frac{\left(k - \frac{\chi}{r^f}\right)^{1 - \frac{1}{\eta}}}{1 - \frac{1}{\eta}} .$$

**Proof** See the Appendix.□

Corollary 1 characterizes the role of the decision rules implied by Proposition 6.

**Corollary 1**

*Under Assumption 2, the solution to the problem expressed by the HJB equation given by (12) implies that the saving rate, the portfolio composition, the coefficient of variation of personal consumption, and the variance of the growth rate of consumption is the same across richer and poorer individuals.*

**Proof** Immediate after noticing that the dynamics of financial wealth of any household follows a geometric Brownian motion over time with the same coefficients for all households, irrespective of initial conditions.  $\square$

In the absence of idiosyncratic labor-income shocks all households are subject to the same aggregate shocks driven by shocks to stock-market-index returns, and inequality in the distribution of relative wealth will be increasing over time. Moreover, in our analysis these aggregate shocks follow a random walk, and decision makers take into account that each shock realization has a permanent effect on wealth accumulation. Nevertheless, the results of Propositions 3 through 5 point out that all empirical regularities (personal saving rates, risky-asset portfolio shares, and consumption volatility, all being increasing in wealth, in addition to having EIS increasing in wealth) are qualitatively consistent with the model's mechanics. It therefore seems promising to introduce time-invariant subsistence consumption in a typical model of labor-income idiosyncratic risk with liquidity constraints in order to avoid making assumptions about preference heterogeneity.

On the contrary, the exceptionally sharp result of Corollary 1 is certainly due to the aggregation properties inherent in the Stone-Geary formulation. Previous papers on aggregation such as Chatterjee (1994) and Caselli and Ventura (2000) emphasize that time-invariant and time-variant subsistence level of consumption can lead to wealth-dependent saving rates. What we find here is that the “keeping-up-with-the-Joneses” formulation tends to rebalance saving rates and, as it has become clear by this study, it rebalances portfolio choice among the rich and the poor as well. This happens because the benchmark subsistence consumption is subject to aggregation as well (it is proportional to the consumption level of the “representative consumer” – see Caselli and Ventura (2000) or Koulovatianos (2005) for a

definition of the concept). Yet, it was not obvious at all that this feature would survive under portfolio choice, which is a key contribution of this paper. Yet, whenever a time-variant level of benchmark consumption is exogenous, the knife-edge result of Corollary 1 generally fails (on this see Caselli and Ventura (2000) and Koulovatianos (2005, Theorem 3) – the latter also provides necessary conditions for the aggregation result with time-variant subsistence consumption).

#### **4. Discussion**

One non-trivial question about using subsistence consumption which has a time-invariant component in savings and portfolio-choice analysis is how to deal with the identification of subsistence consumption levels. This is an open empirical question and it seems it will take some effort until researchers reach consensus on how to estimate subsistence consumption. Yet, some recent work gives strong empirical support for the existence of subsistence consumption in utility functions and the empirical plausibility of Stone-Geary preferences while it also provides new research directions. In particular, Donaldson and Pendakur (2006) identify family-type subsistence consumption using a restriction on equivalence scales, called “Generalized Absolute Equivalence Scale Exactness (GAESE)” through demand-system analysis. Koulovatianos et al. (2006, 2008) are able to test whether GAESE is a plausible assumption to make, and find very strong evidence in favor of GAESE. Moreover, Koulovatianos et al. (2006, 2008) propose a complementary survey approach for identifying subsistence consumption, finding strongly that in six countries and 49 cases they examine subsistence consumption is always present playing an important role in comparisons of material comfort among individuals living in different family types. These advances provide tools for tackling the demanding empirical question of how to estimate subsistence consumption.

Another question that arises is whether the wealth restriction placed by Assumption 1 is a prohibitively tight constraint for matching micro data on wealth. In order to address this question one needs to represent an equivalence of the wealth analysis of our simplified model to the wealth analysis of a more complex framework that is capable of distinguishing among more household resource variables in available micro data, such lifetime income from both labor and financial assets. In particular, the definition of  $k(t)$  in our model should be seen as a composite form of wealth, able to encompass both income and observable wealth measures that are available in existing micro databases, rather than being seen as a limited measure of financial wealth in the data. A plausible way to interpret  $k(t)$  in our model is to consider equation (2), apply the conditional expectations operator on both sides, solve the resulting equation forward, and apply the transversality condition to see that

$$k_0 = \int_0^{\infty} e^{-\int_0^t r(\tau) d\tau} E(c(t)) dt \equiv PVEC \quad (15)$$

where  $r(t) \equiv \phi(t)R + (1 - \phi(t))r^f$  is the return to investment subject to portfolio choice, and “PVEC” stands for “present value of expected consumption”.<sup>16</sup> Notice that now the constraint given by Assumption 1, necessary for meeting the transversality condition, is

$$PVEC > \frac{\chi}{r^f}, \quad (16)$$

which means that the present value of expected lifetime consumption should exceed the present value of lifetime subsistence consumption discounted by the risk-free rate. A richer version of the budget constraint given by (2), would distinguish between assets,  $a$ , and labor income,  $y$ , having, for example, the form,

$$da(t) = \{ [\phi(t)R + (1 - \phi(t))r^f] a(t) + y(t) - c(t) \} dt + \sigma\phi(t)a(t) dz_a(t), \quad (17)$$

<sup>16</sup>For details on how to solve equation (2) forward after having taken expectations on both sides, see, for example, Barro and Sala-I-Martin (2004, Ch. 2).

and

$$dy(t) = \nu(y, t) dt + \theta(y, t) dz_y(t) , \quad (18)$$

where  $z_y(t) = \rho_{a,y} z_a(t) + \sqrt{1 - \rho_{a,y}^2} z(t)$ , with  $z(t)$  being a standard Brownian motion independent of  $z_a(t)$  ( $z_a(t)$  is also a standard Brownian motion) and with  $\rho_{a,y} \in (-1, 1)$  denoting the correlation coefficient between asset returns and the income process.<sup>17</sup> Applying the conditional expectations operator on both sides of (17), solving the resulting equation forward, and applying the transversality condition gives,

$$PVEC = a_0 + PVEY \quad (19)$$

with

$$PVEY \equiv \int_0^{\infty} e^{-\int_0^t r(\tau) d\tau} E(y(t)) dt ,$$

where  $y(t)$  solves equation (18) and “PVEY” stands for the present value of expected labor income. Equation (19) implies that the requirement given by (16) becomes

$$PVEC = a_0 + PVEY > \frac{\chi}{rf} ,$$

or

$$a_0 > - \left( PVEY - \frac{\chi}{rf} \right) . \quad (20)$$

The definition of financial wealth,  $a$ , in (20) fits realistically micro data of wealth and allows a household to have zero or negative assets (negative net worth), justified by the fact that such a household’s asset allowing for survival is its expected future income (either its time endowment and ability to work and receive income, or its entitlement to external aid, such as social security benefits), which must exceed the present value of subsistence consumption discounted by the risk-free rate. So, our definition of  $k$  should be perceived as a broad

<sup>17</sup>Such a general formulation has been analyzed by Henderson (2005, p. 1241), while more specific versions of equation (18) have been studied by Duffie et al. (1997, p. 755) and Koo (1998).

measure of resources, namely  $k_0 = a_0 + PVEY$ . We emphasize that the expression given by (20) does not constitute a result that corresponds to the solution of the savings and portfolio choice problem which is subject to equations (17) and (18), an analysis that would be a key extension to the present model. The expression given by (20) is only an interpretation of Assumption 1 when a broader concept of lifetime resources is used.

Another issue is that in our model all households hold stocks, an implication standing in strong contrast to the zero stockholding of most households observed in the data. On this we think that a richer (simulated) version of our model with time-invariant subsistence consumption and stock-market participation costs as the type of costs analyzed by Haliassos and Michaelides (2003) is very promising for capturing this aspect of the data. To the extent that our stockholding formula given by Proposition 1,  $\Phi(k) = (\eta/\sigma) \cdot (\text{risk premium}) \cdot (1 - (\chi/r^f)/k)$  plays a role in the process of molding numerical outcomes of such richer simulated models, there is promise for satisfactory simulation results. For example, Heathcote et al. (2010, Fig. 19) who present wealth data indicate that the top 10% of wealth owners possesses about 60% of overall wealth in the US, while the bottom decile holds zero wealth steadily throughout the past 25 years. Setting  $R = 8\%$ ,  $r^f = 2\%$ ,  $\sigma = 20\%$  (see, for example, the calibration exercise in Guvenen (2009, Table II, p. 1725)), and  $\eta = 0.5$ , a group of persons holding 1% of their portfolio in stocks ( $\phi = 1\%$ ) in our model should hold wealth that is about 1.4% higher than  $\chi/r^f$  ( $\chi$  is a free parameter in this exercise). For households holding wealth two times higher than  $\chi/r^f$  the implied  $\phi$  is 37.5% (this means that the ratio of stocks held by agents having wealth twice higher than subsistence is about  $2 \times 37.5$  compared to agents who have wealth holdings 1.4% higher than subsistence who have  $\phi = 1\%$ ), while agents with wealth 20 times higher than subsistence hold about  $20 \times 71.25$  times more stocks than those who have wealth holdings 1.4% higher than subsistence. Such relative numbers

are in accordance with Poterba and Samwick (1995) and close to stockholding population fractions discussed and calibrated by Guvenen (2009, p. 1722). We believe this simple exercise shows that, perhaps after introducing stock-market participation costs, to the extent that some mechanics implied by equation  $\Phi(k) = (\eta/\sigma) \cdot (\text{risk premium}) \cdot (1 - (\chi/r^f)/k)$  are still at work, time-invariant subsistence consumption is a very promising element to introduce in a model in order to match the relationship between wealth holdings and stockholding seen in the data.

## 5. Concluding remarks

Researchers who work on endogenizing stockholding and who want to simultaneously meet empirical regularities of saving and consumption tend to dislike the idea of assuming different preferences between rich vs. poor, or between stockholders vs. non-stockholders. For many researchers it would be desirable not to pursue a behavioral explanation beyond standard utility/choice theory, and to be able to achieve three stylized facts through a single utility function, namely that the rich have: (i) higher saving rates, (ii) a larger fraction of their wealth held in stocks, and (iii) higher consumption-growth volatility. Since it is a broadly accepted stylized fact that the rich (and stockholders) exhibit higher EIS, it seems promising to achieve the above goal through assuming a single utility function with subsistence consumption. Here, we have distinguished and studied two types of introducing subsistence consumption: (a) a time-invariant level of subsistence consumption, and (b) a time-variant level of subsistence consumption,  $\gamma \bar{C}(t)$  ( $\bar{C}(t)$  is average consumption in a community at time  $t$ , and  $\gamma > 0$ ), with the latter formulation capturing the idea of “keeping-up-with-the-Joneses” preferences. We have analytically studied savings and portfolio choice in the simple two-asset Merton (1969) model, and have shown that the formulation in (a) above is quite



promising for achieving the goal of meeting all three stylized facts above, while the formulation in (b) is not promising at all. Despite that the preference formulation in (b) generates higher EIS for the rich in equilibrium, this heterogeneity in EIS is not capable of generating any difference in saving rates and portfolio choice. The reason is, as future plans take into account the evolution of  $\bar{C}(t)$ , marginal utility implied by the indirect utility function tends to be re-balanced across the rich and the poor.<sup>18</sup>

Conveying this negative message for the role of Stone-Geary preferences with subsistence consumption of the form  $\gamma\bar{C}(t)$  is the central contribution of this paper. Another way of interpreting our results is that assuming some time-invariant component of subsistence consumption seems very promising for reconciling the implications of a savings/portfolio-choice model with micro evidence if one wants to retain rational expectations without assuming preference heterogeneity. Our study raises plausible questions for future research. A key extension is the empirical identification of subsistence consumption. Recent studies following different empirical approaches show promise in this regard.<sup>19</sup> Perhaps the next two most plausible questions to ask in future research should be addressed through a model such as this of Haliassos and Michaelides (2003) with labor income, liquidity constraints, exogenous stock-market participation costs, and even finite lives. First, is a time-invariant component of subsistence consumption capable of resolving both the qualitative and quantitative properties of the data expressed through points (i)-(iii) above? Second, does assuming “keeping-up-with-the-Joneses” benchmark subsistence consumption (of the form  $\gamma\bar{C}(t)$ ) tend to make saving rates and portfolio choice the same among the rich and the poor in an environment as different as in Haliassos and Michaelides (2003)? We believe our analytical work in this

<sup>18</sup>See, for example, the role of equation (14) above.

<sup>19</sup>See, for example, Donaldson and Pendakur (2006) for a demand-system approach for identifying subsistence consumption, and Koulovatianos et al. (2006, 2008) for doing so through a survey approach for different family types.

study has clarified what is most likely to expect and why, and that it can help researchers make more promising preference assumptions (avoid less promising assumptions) for their models, saving research time and effort.

## 6. Appendix - Proofs

### Proof of Proposition 1

We make a guess on the functional form of the value function, namely,

$$J(k) = a + b \frac{(k - \omega)^{1 - \frac{1}{\eta}}}{1 - \frac{1}{\eta}}, \quad (21)$$

which implies,

$$J'(k) = b(k - \omega)^{-\frac{1}{\eta}}, \quad (22)$$

and

$$J''(k) = -\frac{1}{\eta} b(k - \omega)^{-\frac{1}{\eta} - 1}. \quad (23)$$

From (22) and (4) it is,

$$c = b^{-\eta} k + \chi - b^{-\eta} \omega. \quad (24)$$

Moreover, substituting (22) and (23) into (5) gives,

$$\phi = \eta \frac{R - r^f}{\sigma^2} \left(1 - \frac{\omega}{k}\right). \quad (25)$$

Substituting (21), (24), (22), (25), and (23) into the HJB given by (3) becomes,

$$\begin{aligned} \rho a + \rho b \frac{(k - \omega)^{1 - \frac{1}{\eta}}}{1 - \frac{1}{\eta}} &= \frac{b^{1 - \eta} (k - \omega)^{1 - \frac{1}{\eta}}}{1 - \frac{1}{\eta}} - \frac{1}{1 - \frac{1}{\eta}} + \\ &+ b(k - \omega)^{-\frac{1}{\eta}} \left[ \eta \frac{(R - r^f)^2}{\sigma^2} \left(1 - \frac{\omega}{k}\right) k + r^f k - b^{-\eta} k - \chi + b^{-\eta} \omega \right] - \\ &- \frac{1}{2\eta} \left( \eta \frac{R - r^f}{\sigma} \right)^2 b (k - \omega)^{1 - \frac{1}{\eta}} \end{aligned} \quad (26)$$

Setting

$$a = -\frac{1}{\rho \left(1 - \frac{1}{\eta}\right)}, \quad (27)$$

dividing both sides of (26) by  $b(k - \omega)^{1 - \frac{1}{\eta}}$  and re-arranging terms, it is,

$$\begin{aligned} \left[ \frac{\rho - b^{-\eta}}{1 - \frac{1}{\eta}} - \frac{\eta}{2} \left( \frac{R - r^f}{\sigma} \right)^2 \right] k - \left[ \frac{\rho - b^{-\eta}}{1 - \frac{1}{\eta}} - \frac{\eta}{2} \left( \frac{R - r^f}{\sigma} \right)^2 \right] \omega = \\ = (r^f - b^{-\eta}) k + b^{-\eta} \omega - \chi . \end{aligned} \quad (28)$$

In order that the guess we made for  $J(k)$  be operative, it must be that we can find  $b$  and  $\omega$  such that both the coefficient of  $k$  in equation (28) and the constant part must both be equal to zero. So, (28) implies

$$\frac{\rho - b^{-\eta}}{1 - \frac{1}{\eta}} - \frac{\eta}{2} \left( \frac{R - r^f}{\sigma} \right)^2 = r^f - b^{-\eta} , \quad (29)$$

which leads to,

$$b^{-\eta} = \rho\eta + (1 - \eta)r^f - \frac{\eta(\eta - 1)}{2} \left( \frac{R - r^f}{\sigma} \right)^2 . \quad (30)$$

Moreover, the constant terms of (28) should sum up to zero, so

$$\left[ \frac{\rho - b^{-\eta}}{1 - \frac{1}{\eta}} - \frac{\eta}{2} \left( \frac{R - r^f}{\sigma} \right)^2 \right] \omega = \chi - b^{-\eta} \omega$$

and combining it with (29) becomes,

$$(r^f - b^{-\eta}) \omega = \chi - b^{-\eta} \omega ,$$

or

$$\omega = \frac{\chi}{r^f} . \quad (31)$$

So, after substituting (31) into (24), the decision rule for consumption becomes,

$$c = b^{-\eta} k + \chi \left( 1 - \frac{b^{-\eta}}{r^f} \right)$$

and after substituting (30) it is,

$$c = \left[ \rho\eta + (1 - \eta)r^f - \frac{\eta(\eta - 1)}{2} \left( \frac{R - r^f}{\sigma} \right)^2 \right] k + \eta\chi \frac{r^f - \rho + \frac{\eta - 1}{2} \left( \frac{R - r^f}{\sigma} \right)^2}{r^f} , \quad (32)$$

confirming the statement of the Proposition. It remains to verify that Assumption 2 guarantees that  $\xi > 0$ . From (32) the requirement that  $\xi > 0$  implies,

$$\eta^2 - \left[ 1 - \frac{r^f - \rho}{\frac{1}{2} \left( \frac{R-r^f}{\sigma} \right)^2} \right] \eta - \frac{r^f}{\frac{1}{2} \left( \frac{R-r^f}{\sigma} \right)^2} < 0 . \quad (33)$$

The discriminant of the quadratic polynomial with respect to  $\eta$  given by (33) is

$$\left[ \frac{r^f - \rho}{\frac{1}{2} \left( \frac{R-r^f}{\sigma} \right)^2} \right]^2 + 2 \frac{r^f + \rho}{\frac{1}{2} \left( \frac{R-r^f}{\sigma} \right)^2} + 1 > 0 ,$$

which means that real roots exist, while the constant term of (33) reveals that both roots are different from zero and that they have opposite signs. Since there is a parametric restriction that  $\eta > 0$ , the formula for  $\bar{\eta}$  appearing in the statement of the proposition is the positive root of the quadratic polynomial with respect to  $\eta$  given by (33).  $\square$

## Proof of Proposition 2

Direct substitution of the decision rule for consumption,  $C(k)$ , and also for portfolio choice,  $\Phi(k)$ , into the household's budget constraint given by (2), after some algebra, gives,

$$dk = \eta \left[ \frac{\eta + 1}{2} \left( \frac{R - r^f}{\sigma} \right)^2 + r^f - \rho \right] \left( k - \frac{\chi}{r^f} \right) dt + \eta \left( \frac{R - r^f}{\sigma} \right) \left( k - \frac{\chi}{r^f} \right) dz \quad (34)$$

Applying Itô's lemma on (34) yields,

$$d \ln \left( k - \frac{\chi}{r^f} \right) = \eta \left[ \frac{1}{2} \left( \frac{R - r^f}{\sigma} \right)^2 + r^f - \rho \right] dt + \eta \left( \frac{R - r^f}{\sigma} \right) dz \quad (35)$$

and after integrating (35) using Itô's stochastic integral (while setting  $z(0) = 0$  by convention), proves the proposition.  $\square$

## Proof of Proposition 5

Fix any  $t > 0$ . From Proposition 1 it is  $C(k^*(t)) = \xi k^*(t) + \psi$ . From Proposition 2 we can see that

$$E[C(k^*(t))] = \xi E[\zeta(t)] \left(k_0 - \frac{\chi}{r^f}\right) + \xi \frac{\chi}{r^f} + \psi \quad (36)$$

where

$$\zeta(t) \equiv e^{\eta \left[ r^f - \rho + \frac{1}{2} \left( \frac{R-r^f}{\sigma} \right)^2 \right] t + \eta \left( \frac{R-r^f}{\sigma} \right) z(t)} .$$

Since  $\psi = \chi(1 - \xi/r^f)$ , (36) implies,

$$E[C(k^*(t))] = \xi E[\zeta(t)] \left(k_0 - \frac{\chi}{r^f}\right) + \chi . \quad (37)$$

So, apparently, the coefficient of variation is,

$$CoeffVar(C(k^*(t))) = \frac{\xi \{Var[\zeta(t)]\}^{\frac{1}{2}}}{\xi E[\zeta(t)] + \frac{\chi}{k_0 - \frac{\chi}{r^f}}}$$

and since both  $E[\zeta(t)]$  and  $\{Var[\zeta(t)]\}^{1/2}$  are strictly positive,<sup>20</sup> this last equation implies that  $dCoeffVar(C(k^*(t)))/dk_0 > 0$ . Regarding the volatility of consumption growth, direct substitution of the decision rule for consumption,  $C(k)$ , and also for portfolio choice,  $\Phi(k)$ , into the household's budget constraint given by (2), after some algebra, gives,

$$dk = \theta \left(k - \frac{\chi}{r^f}\right) dt + \eta \left(\frac{R-r^f}{\sigma}\right) \left(k - \frac{\chi}{r^f}\right) dz \quad (38)$$

where

$$\theta = \eta \left[ \frac{\eta + 1}{2} \left(\frac{R-r^f}{\sigma}\right)^2 + r^f - \rho \right] .$$

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<sup>20</sup>Notice that

$$E[\zeta(t)] = e^{\eta \left[ r^f - \rho + \frac{1}{2} \eta \left( \frac{R-r^f}{\sigma} \right)^2 \right] t} ,$$

and

$$Var[\zeta(t)] = e^{2\eta \left[ r^f - \rho + \frac{1}{2} \eta \left( \frac{R-r^f}{\sigma} \right)^2 \right] t} \left\{ e^{\eta^2 \left( \frac{R-r^f}{\sigma} \right)^2 t} - 1 \right\} .$$

Using  $C(k^*(t)) = \xi k^*(t) + \psi = \xi [k^*(t) - \chi/r^f] + \chi$ , and applying Itô's lemma on (38) gives,

$$d \ln [C(k^*)] = \left\{ \xi \theta \frac{\hat{k}^*}{\xi \hat{k}^* + \chi} - \frac{1}{2} \left[ \xi \eta \left( \frac{R - r^f}{\sigma} \right) \right]^2 \left( \frac{\hat{k}^*}{\xi \hat{k}^* + \chi} \right)^2 \right\} dt + \xi \eta \left( \frac{R - r^f}{\sigma} \right) \frac{\hat{k}^*}{\xi \hat{k}^* + \chi} dz$$

where  $\hat{k}^* = k^* - \chi/r^f$ , and which implies that

$$Var(d \ln [C(k^*)]) = \left[ \xi \eta \left( \frac{R - r^f}{\sigma} \right) \right]^2 \left( \frac{\hat{k}^*}{\xi \hat{k}^* + \chi} \right)^2 dt ,$$

an increasing function of  $k^*$ .  $\square$

### Proof of Proposition 6

Let's assume that indeed such interior solutions are guaranteed. Our guess for the value function is,

$$J(k, \bar{K}) = a + b \frac{(k - \gamma \bar{K})^{1 - \frac{1}{\eta}}}{1 - \frac{1}{\eta}} . \quad (39)$$

So, (39) implies,

$$J_k(k, \bar{K}) = b (k - \gamma \bar{K})^{-\frac{1}{\eta}} , \quad (40)$$

$$J_{kk}(k, \bar{K}) = -\frac{1}{\eta} b (k - \gamma \bar{K})^{-\frac{1}{\eta} - 1} , \quad (41)$$

$$J_{\bar{K}}(k, \bar{K}) = -\gamma b (k - \gamma \bar{K})^{-\frac{1}{\eta}} , \quad (42)$$

$$J_{\bar{K}\bar{K}}(k, \bar{K}) = -\frac{\gamma^2}{\eta} b (k - \gamma \bar{K})^{-\frac{1}{\eta} - 1} , \quad (43)$$

$$J_{k\bar{K}}(k, \bar{K}) = \frac{\gamma}{\eta} b (k - \gamma \bar{K})^{-\frac{1}{\eta} - 1} . \quad (44)$$

Combining (13) with (40) gives,

$$c = b^{-\eta} (k - \gamma \bar{K}) + \gamma \bar{C} , \quad (45)$$

while combining (14) with (40), (41) and (44) implies,

$$\phi = \eta \frac{R - r^f}{\sigma^2} \left( 1 - \gamma \frac{\bar{K}}{k} \right) + \bar{\Phi} \gamma \frac{\bar{K}}{k} . \quad (46)$$

When equations (45) and (46) are substituted into the individual's budget constraint, equation (10), the result is,

$$dk = \left\{ \left[ \eta \left( \frac{R - r^f}{\sigma} \right)^2 - b^{-\eta} + r^f \right] k + \gamma \left[ b^{-\eta} + (R - r^f) \bar{\Phi} - \eta \left( \frac{R - r^f}{\sigma} \right)^2 \right] \bar{K} - \gamma \bar{C} \right\} dt \\ + \left[ \eta \frac{R - r^f}{\sigma} k + \gamma \left( \sigma \bar{\Phi} - \eta \frac{R - r^f}{\sigma} \right) \bar{K} \right] dz .$$

This last equation implies exact linear aggregation of the equilibrium law of motion of financial wealth among rich and poor. This means that the guess we have made is consistent with the way we have set up the problem, and it remains to see whether there exists a constant term  $b$  that validates the solution. Linear aggregation allows us to substitute for  $\bar{K}$  and  $\bar{\Phi}$  in equation (46) in order to characterize the portfolio choice of the household with average wealth. Doing so leads to,

$$\bar{\Phi} = \eta \frac{R - r^f}{\sigma^2} , \quad (47)$$

and substituting (47) into (46) implies

$$\phi = \bar{\Phi} = \eta \frac{R - r^f}{\sigma^2} , \quad \text{for all } k > 0 . \quad (48)$$

Moreover, by linearly aggregating equation (45) we obtain

$$c = b^{-\eta} k , \quad \text{for all } k > 0 , \quad (49)$$

which includes  $\bar{K}$ . Substituting equations (39) through (46) into the HJB equation given by (12), and imposing  $a = -1/[\rho(1 - 1/\eta)]$ , we arrive, after some algebra to,

$$\frac{\rho - \frac{1}{\eta} b^{-\eta}}{1 - \frac{1}{\eta}} - r^f = (R - r^f) \frac{\phi k - \gamma \bar{\Phi} \bar{K}}{k - \gamma \bar{K}} - \frac{\sigma^2}{2\eta} \left( \frac{\phi k - \gamma \bar{\Phi} \bar{K}}{k - \gamma \bar{K}} \right)^2 . \quad (50)$$



Substituting that  $\phi = \bar{\Phi}$  from (48) into (50) leads to the expression stated in the proposition. Given this interior solution, the role of Assumption 3 is reconfirmed in a straightforward manner.  $\square$

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