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The horizontal stability of a ball bouncing upon a vertically vibrated concave surface

H. S. WRIGHT, M. R. SWIFT and P. J. KING

School of Physics and Astronomy, University of Nottingham - Nottingham, NG7 2RD, UK

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Abstract – In experiments a ball will not bounce repeatedly in the same place upon a vertically vibrating horizontal surface due to imperfections in the ball and the surface. Consequently, a concave surface is often used to restrain the horizontal movement of the ball while measurements are made of its vertical motion. Here we use two numerical-simulation models to study the horizontal motion of an inelastic sphere bouncing chaotically under gravity upon a vertically vibrated parabolic surface. Both models predict the same generic features. For almost flat surfaces the ball makes wandering horizontal excursions and as the surface curvature is increased the horizontal motion of the ball becomes more and more constrained. However, the horizontal motion may exhibit intermittent, abrupt and erratic bursts of very large amplitude. These bursts occur with a probability which increases extremely rapidly with surface curvature such that there is an effective threshold curvature for their observation within a finite sequence of bounces. We study the behaviour of this intermittency as a function of the vibration amplitude, of the surface curvature and of the normal coefficient of restitution.

Introduction. – The behaviour of a ball or a rod bouncing upon a *flat* and smooth horizontal surface which is undergoing sinusoidal vertical vibration has been studied extensively through analysis and numerical simulation [1–5]. In such studies it is customary to describe the vertical amplitude of the surface motion in terms of the dimensionless parameter $\Gamma = A\omega^2/g$. Here A is the amplitude of vibration, $\omega = 2\pi f$ is the angular frequency of vibration and g is the acceleration due to Earth's gravity. Such systems are examples of "impact oscillators" [6–8] for which the vertical motion is generally chaotic provided that Γ appreciably exceeds unity.

In experimental studies of restitution or of the chaotic motion of a bouncing ball it is often inconvenient to use a vertically vibrating horizontal plane since imperfections in the system will cause the ball to wander horizontally across the surface. A slightly dished surface such as that provided by a long focal length concave glass lens or mirror is often used to restrain the horizontal movement without greatly modifying the motion in the vertical direction [5,9,10]. The following questions are then raised. Is the horizontal motion increasingly restrained as the curvature is increased, as might be expected? Is there an optimum curvature of the surface? In experimental observations in our laboratory using a concave lens it has been observed that a long sequence of horizontally restrained bounces may be interrupted by a large horizontal excursion, the ball flying from the surface. Can this behaviour be described by a simple model, and if so how may it be avoided?

Here we report numerical simulations of an inelastic sphere bouncing in two dimensions upon a vertically vibrating parabolic surface, carried out to investigate the nature of the horizontal motion of the ball as a function of the surface curvature, the amplitude parameter Γ, and the normal coefficient of restitution, e_N. We consider two models, one with and one without rotation, and find that they both exhibit the same generic dynamical behaviour. In these simulations the horizontal stability of the motion is tested by introducing small perturbations which mimic those which might occur in a real system through imperfections in the ball or surface, or through the influence of small particulates should the experiments not be carried out in dust-free conditions.

For almost flat surfaces the ball makes substantial wandering horizontal excursions; as the surface curvature is increased the horizontal motion of the ball is increasingly constrained. However, we find that the ball may indeed make abrupt, erratic and very large horizontal excursions across the surface. The probability of such an intermittent burst increases extremely rapidly with surface curvature and it is thus possible to define an effective threshold curvature for the observation of such an event within a finite sequence of bounces. The conditions under which this behaviour occurs are studied in order to determine the vibratory and system parameters for bouncing a ball which yield variations in the horizontal coordinate which are optimally constrained.

Model with rotation. – We consider the dynamics of a ball moving in two dimensions under gravity g. The ball has a non-zero radius R and is free to rotate. It bounces upon a vibrating parabolic surface defined, at time t, by $z_s = A\sin(\omega t) + Qx_s^2$, where z_s and x_s are the vertical and horizontal coordinates of the surface. The focal length, F, of such a surface is given by $F = 1/(4Q)$.

In flight the ball moves under gravity according to the following equations:

$$z = z_b + \dot{z}_b(t - t_b) - \frac{1}{2}g(t - t_b)^2, \tag{1}$$

$$x = x_b + \dot{x}_b(t - t_b), \tag{2}$$

$$\dot{z} = \dot{z}_b - g(t - t_b). \tag{3}$$

Here x and z are the coordinates of the ball and $\dot{x}$ and $\dot{z}$ are the horizontal and vertical velocities. The suffix b refers to the values of these properties immediately following the last bounce. The horizontal velocity, $\dot{x}$, and angular velocity $\dot{\theta}$ are assumed to be constant during flight. The collisions with the surface are dealt with solely through the use of a normal coefficient of restitution, e_N, and a tangential coefficient of restitution, e_T.

Within the simulation programme a "bounce detection" procedure provides an estimate of the horizontal coordinate of the surface contact point, x_s, at a collision. From this the local slope of the surface may be calculated from Q and x_s. It is only here that Q enters the calculation. The horizontal and vertical velocities of the ball are transformed into those in a frame aligned and travelling with the local surface, the velocities of the point of contact parallel and perpendicular to the surface being $v_\parallel$ and $v_\perp$, respectively. The collision is conducted in this frame to produce "post-bounce" values of $v'_\parallel$ and $v'_\perp$ through the equations,

$$v'_\perp = -e_N v_\perp, \tag{4}$$

$$v'_\parallel = -e_T v_\parallel. \tag{5}$$

The post-bounce velocities, $v'_\perp$ and $v'_\parallel$, calculated using eqs. (4) and (5), are transformed back into the velocities of the x, z coordinate system. The corresponding post-bounce angular velocity is determined from conservation of angular momentum. A multiplicative noise term, v_{noise}, is added to the ball to mimic the effects of surface roughness. The strength of this noise is taken to be a small fraction, δ, of the post-bounce normal velocity and added to the post bounce parallel velocity, all in the frame of reference of the moving surface. This perturbation would appear in a real system through imperfections in the form of the ball, of the surface upon which it bounces, or through particulates upon the surface. It is through the response of the system to these perturbations that we examine the horizontal stability of the motion.

The motion of the ball is followed for a very large number of bounces, and the nature of the motion and statistical information on the bounces collected. In collecting this information early bounces are ignored so that the starting position of the ball does not influence the behaviour studied.

The parameters of this problem are A and ω, e_N, e_T, g, the surface curvature parameter Q and the particle size R. However, eqs. (1) to (5) could be transformed to dimensionless forms through the use of a time scale $1/\omega$ and a length scale A. When transformed in this way, the flight eqs. (1) to (3) retain the same general form but with g replaced by $1/\Gamma$ and eqs. (4) and (5) are unchanged in form. It is sufficient to study the motion at a single frequency since changing frequency is equivalent to a corresponding change in Γ and a change of time scale. The problem may be simplified further by noting that $0.4 < e_T < 0.8$ for very many common hard materials for impacts close to normal incidence [11–13]. We initially set e_N to a typical value within this range.

In general the motion is chaotic and therefore extremely sensitive to the initial conditions; it is not possible to predict extended series of bounces in detail. However, the model implementation is able to capture the general nature of the motion which we now describe.

The behaviour under vibration. – Since it is difficult to present a description of the behaviour as a function of all the independent variables we restrict ourselves to reporting simulations which correspond to typical practical situations. Initially, we consider a ball of radius 0.5 mm and set the vibration frequency to 20 Hz, set $g = 9.81\,\text{ms}^{-2}$, and set $e_T = 0.60$ and $e_N = 0.975$. We consider noise with $\delta = 0.001$. The behaviour of the motion is then studied as a function of Γ and Q.

Figure 1 shows typical sequences of the horizontal positions of the ball at each of 4000 successive bounces for $e_N = 0.975$ and $\Gamma = 5.0$ and for five different values of Q. In each example the data show the sequence following the first 1000 bounces.

For $Q = 0.001\,\text{m}^{-1}$, fig. 1a, the horizontal motion consists, as is to be expected, of extensive wandering backwards and forwards across the surface. As Q is increased, the amplitude of the horizontal wandering reduces; the motion is more constrained by the increased surface curvature. Figure 1b shows this behaviour for $Q = 0.05\,\text{m}^{-1}$. Note the change in vertical scale. However, as Q is further increased a dramatic change in the nature of the horizontal motion is observed, as fig. 1c shows for $Q = 0.15\,\text{m}^{-1}$. The ball now spends most of its time in small-scale horizontal fluctuations but from time to time exhibits erratic bursts consisting of extremely

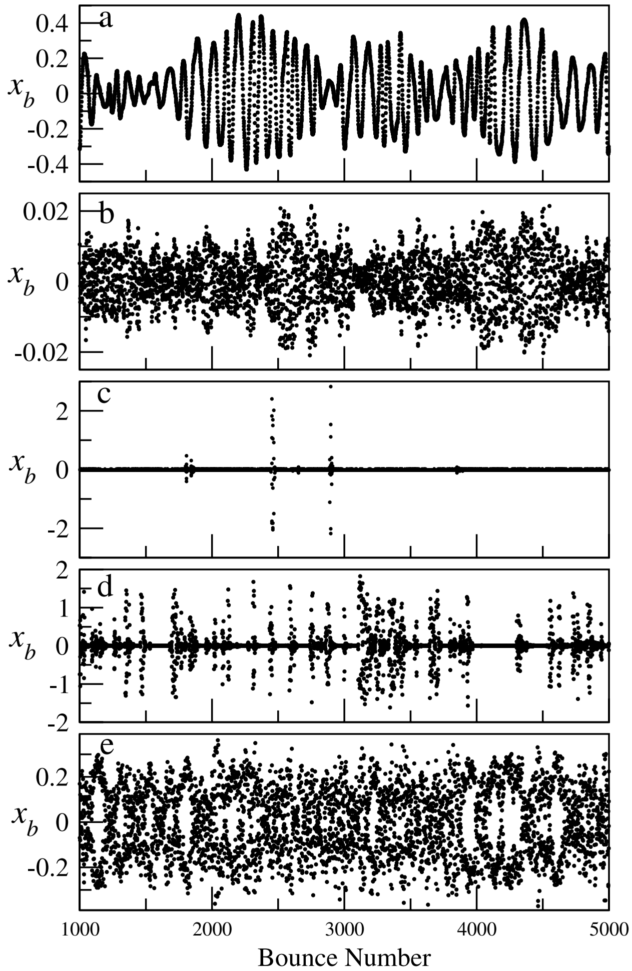

Fig. 1: The horizontal positions x_b (m) for a typical sequence of 4000 successive bounces for 20 Hz, $\Gamma = 5.0$, $e_N = 0.975$, $\delta = 0.001$ and for (a) $Q = 0.001$, (b) $Q = 0.05$, (c) $Q = 0.15$, (d) $Q = 0.4$ and (e) $Q = 5.0\,\mathrm{m}^{-1}$. Note the changes of scale of the vertical axes. In each case data following the first 1000 bounces are shown.

large-amplitude horizontal excursions across the surface. Details of these bursts are discussed below. Each burst involves 20–60 bounces before low-amplitude fluctuations are resumed. As Q is further increased these intermittent bursts become more frequent but of smaller amplitude. Figure 1d shows data for $Q = 0.4\,\mathrm{m}^{-1}$. If Q is increased further these bursts may merge and eventually the nature of the bouncing changes. Figure 1e shows data for $Q = 5.0\,\mathrm{m}^{-1}$. Here the ball is less likely to bounce with x_b close to zero than at values of $|x_b|$ between 0.1 m and 0.25 m. The ball spends much of its time in chaotic arching motion over the central region of the surface, x_b exhibiting positive and negative values alternately.

Model without rotation. – For a wide range of curvatures it is found that the radius of the ball, R, and ball rotation do not substantially influence the nature of motion provided that $R \ll F$. We therefore consider a simpler, two-variable model in which the ball is assumed to be a point object without spin and that its motion consists solely of periods of flight between bounces. This constrains the values of Γ and e_N which we may use, but these constraints are not too severe for the present purposes. As eq. (5) is now assumed to apply to the centre of mass, the effective value of e_T is now negative [14].

Two forms of noise are also considered in this simplified model, in which we add a term v_{noise} to the post-collisional vertical velocity. In the first, v_{noise} lies with uniform probability in the range $-\epsilon$ to $+\epsilon$, where ϵ is a small constant. In the second, multiplicative noise is used, v_{noise} lying in the range $-\delta\dot{z}_b$ to $+\delta\dot{z}_b$ with uniform probability. Again, δ is a small constant. For simplicity, we have used $\dot{z}$ rather than the relative vertical velocity to set the scale of the fluctuations. However, we note that under our vibratory conditions, $\dot{z}$ is large compared to the velocity of the surface. We find that the resulting dynamics is rather insensitive to the choice of noise used.

Figure 2 shows a typical sequence of bounces using the simplified model for the same driving conditions and surface curvatures shown in fig. 1. At low curvatures, the character of the motion is somewhat different for the two models. With rotation, the particle wanders backwards and forwards across the surface, whereas, without rotation, the trajectory looks more like a random walk. However, as the curvature is increased, the qualitative agreement between fig. 1 and fig. 2 is very good. The simulations show that the development of bursts and arching behaviour are generic and not heavily dependent on model details, including the presence or absence of rotation. We will therefore use the simplified two-variable model to investigate this behaviour in more detail.

Details of the dynamics. – Since they are such a curious feature of the behaviour, the form of the large-amplitude horizontal bursts has been examined in detail. If a suitable value of Q is selected these bursts occur individually between long periods of quiescent behaviour and each one may be examined in isolation. The great majority of these bursts have a form similar to that shown in fig. 3, which shows on an expanded time scale part of fig. 2c. A typical burst begins, as here, with the sudden build-up of very large-amplitude horizontal motion consisting of alternate impacts with the surface at positive and negative values of x_b, the motion we have described as "arching". After a number of impacts of this type the motion changes in form to an oscillatory behaviour of lower amplitude with a number of impacts at positive x followed by a number of impacts at negative x and so on. The amplitude dies down to the background level in typically 20–40 impacts for the value of e_T which we have used here.

This behaviour has been investigated for a wide range of values of Γ, e_N and the other system parameters including the noise. The same generic behaviour is found, that shown schematically in fig. 4, in which the peak excursions are

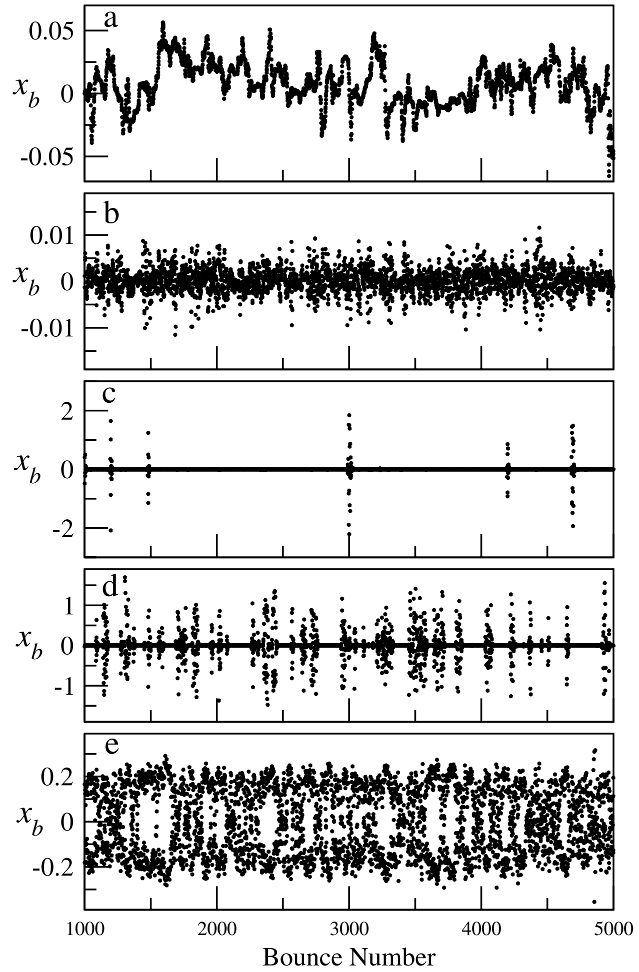

Fig. 2: The horizontal positions x_b (m) for a typical sequence of 4000 successive bounces for 20 Hz, $\Gamma = 5.0$, $e_N = 0.975$, $\delta = 0.001$ and for (a) $Q = 0.001$, (b) $Q = 0.05$, (c) $Q = 0.15$, (d) $Q = 0.4$ and (e) $Q = 5.0\,\text{m}^{-1}$. Note the changes of scale of the vertical axes. In each case data following the first 1000 bounces are shown.

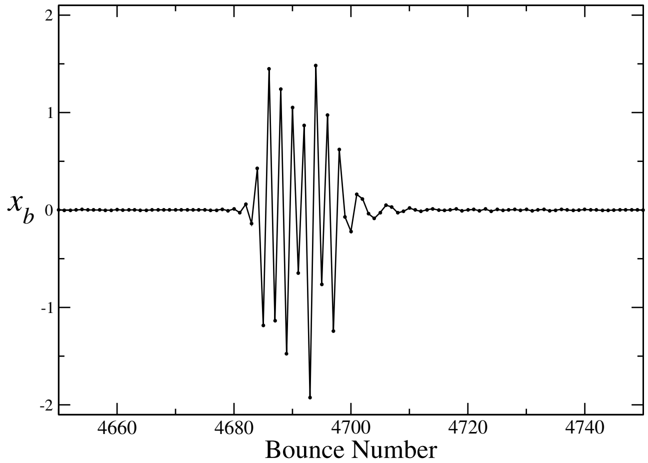

Fig. 3: An expanded view of one of the high-amplitude bursts shown in fig. 2c for $Q = 0.15\,\text{m}^{-1}$. The points are joined as a guide to the eye.

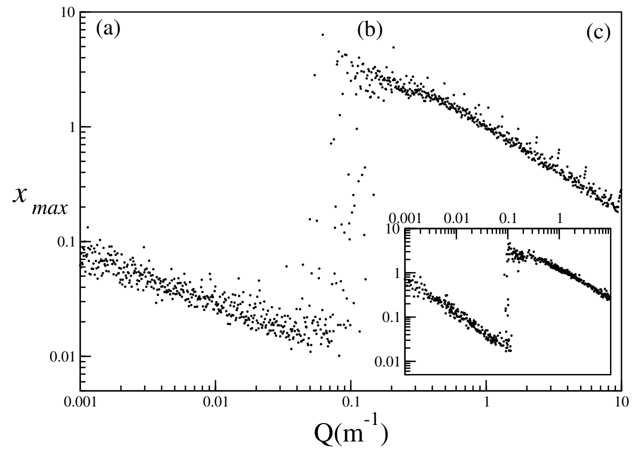

Fig. 4: The generic behaviour of the motion as a function of Q, the other system parameters being those of fig. 2. x_{max} is an estimate of the maximum horizontal excursion (in meters) found in a sequence of 4000 bounces. (a) indicates the region where the ball wanders backwards and forwards across the surface, (b) a region of sporadic bursts out of low-amplitude fluctuations, while (c) indicates a region of arching. The character of the motion in these three regions may be seen in the corresponding figs. 2. The inset shows the equivalent data for a sphere with rotation, with the parameters used in fig. 1.

plotted against Q, the other system parameters being those of fig. 2. Here we take the peak excursion to be the maximum amplitude of x_b, x_{max}, found in a sequence of 4000 bounces. As Q is raised the behaviour passes from a wandering behaviour across the surface to lower-amplitude fluctuations. In this region x_{max} varies approximately as $Q^{-1/2}$. Over a narrow range of Q, intermittent large-amplitude bursts of the type shown in fig. 3 appear out of the fluctuating background. These bursts have amplitudes more than two orders of magnitude greater than those of the background. The simulations show that there is no substantial change in the nature of the vertical motion as this transition region is crossed. Upon further increases in Q the sporadic bursts become more frequent but of lower amplitude, x_{max} varying approximately as $Q^{-2/3}$. At very large values of Q, the bursts merge and the motion has the principal character of arching. The inset to fig. 4 shows the corresponding data for a sphere with rotation. It can be seen that the same general features are present. However, at low Q, x_{max} varies approximately as Q^{-2}, while the high-Q behaviour is very similar for the two models.

The nature of the transition from low-amplitude to high-amplitude horizontal motion has been studied by considering the mean number of bounces between bursts, $\langle \Delta N \rangle$, as a function of Q, for fixed values of Γ and the other system parameters. Figure 5 shows data for a frequency of 20 Hz, $\Gamma = 5.0$ and for $e_N = 0.975$. Results are shown for noise with $\epsilon = 10^{-15}\,\text{ms}^{-1}$, and $\epsilon = 10^{-3}\,\text{ms}^{-1}$ and for multiplicative noise with $\delta = 10^{-3}$. It is seen that

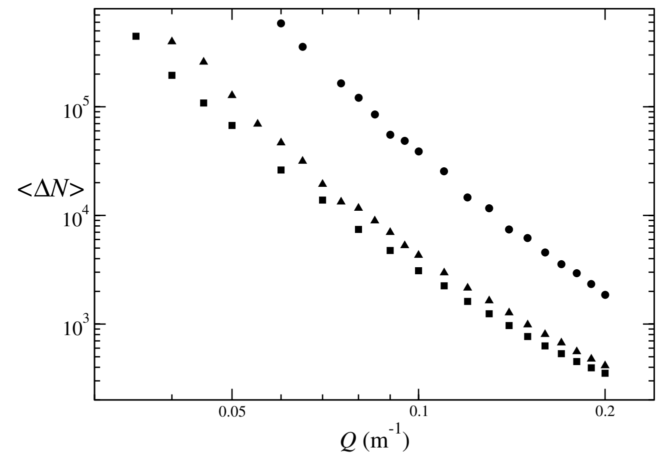

Fig. 5: The mean number of bounces between bursts, $\langle \Delta N \rangle$, plotted against Q for 20 Hz, $\Gamma = 5.0$ and $e_N = 0.975$. Data is shown for noise with $\epsilon = 10^{-15}\,\mathrm{ms}^{-1}$ (circles) and $\epsilon = 10^{-3}\,\mathrm{ms}^{-1}$ (triangles) and for multiplicative noise with $\delta = 10^{-3}$ (squares).

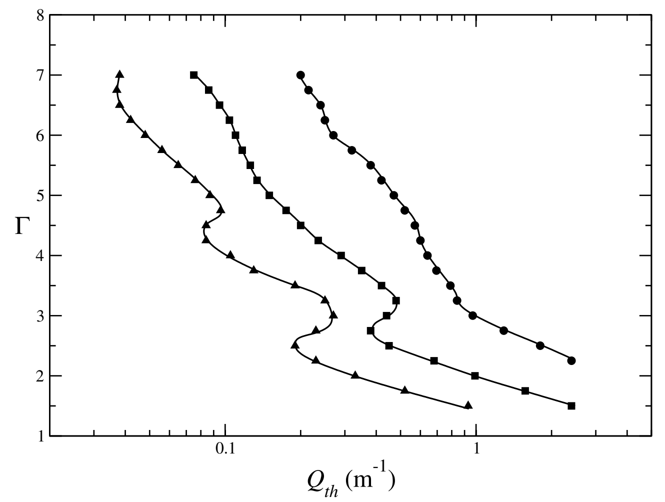

Fig. 6: The values of Q_{th}, based on an average of 4000 bounces between large-amplitude horizontal events, plotted against Γ for $e_N = 0.975$ (triangles), 0.95 (squares) and 0.90 (circles). A frequency of 20 Hz and a multiplicative noise parameter $\delta = 0.001$ have been used. The lines through the data points are a guide to the eye.

in each case the mean number of bounces between bursts is a smooth and very rapidly varying function of Q. However, it is not clear that $\langle \Delta N \rangle$ diverges at a non-zero value of Q for any of the sets of data shown here, and it is not clear therefore that there is a true critical onset. However, the swift variation of $\langle \Delta N \rangle$ with Q does enables us to define an effective "threshold", Q_{th}, for the onset of the observation of busts within any finite sequence of bounces since, as Q is varied, there is a rapid transition between it being very likely that one or more bursts will be observed within the sequence and it being very unlikely that any burst will be observed. As an example, it may be seen from fig. 5 that for multiplicative noise with $\delta = 10^{-3}$ the mean number of bounces between bursts is equal to 4000 at $Q = 0.09\,\mathrm{m}^{-1}$. A threshold value close to $0.09\,\mathrm{m}^{-1}$ is just that seen in fig. 4 which shows data for sequences of 4000 bounces.

It is noteworthy that the data are somewhat sensitive to noise; the points being slightly lower in Q as the noise is increased and depending a little on its type. We note from our simulations that the amplitude of the low-level horizontal wanderings found at low values of Q increase with the level of noise for high levels of noise, while the amplitude of the bursts found at higher values of Q are less sensitive to noise.

The data used to obtain fig. 5 have been examined to determine the distribution of intervals between bursts. In each of the cases which have been examined, the probability, $P(\Delta N)$, of observing a gap of ΔN bounces follows the distribution $P(\Delta N) \sim \exp(-\Delta N/n)$. Here n is a constant which is a function of Q and Γ. As Q is reduced at fixed Γ, n tends smoothly and rapidly to very large values.

Such a distribution is consistent with the following explanation for the build-up of an erratic burst. From the present studies and from the work of previous authors it is known that that phase of the vertical vibration at which bounces occur is close to random [3]. However, from the present simulations it is clear that the growth of a burst occurs in the unlikely event that the phase at a bounce lies within a restricted range on each of a large number of consecutive bounces. Such a situation leads to the exponential probability distribution which has been found. As Q is reduced, a larger number of sequential bounces within the correct range of phase are needed, and this is increasingly improbable, leading to the rapidly increasing values of $\langle \Delta N \rangle$ described above.

We have studied how the Q values for the onset of the large-amplitude bursts, Q_{th}, vary with Γ for the three principal values of e_N used, 0.975, 0.95 and 0.9. We use as criterion for the onset an average of 4000 bounces between events, the average being taken using a sequence of 100000 bounces. For a particular e_N, a set of values of Γ is chosen and the value of Q_{th} for an average separation of 4000 bounces identified for each. The results are shown in fig. 6. It is found that, for each value of e_N, the Q_{th} *vs.* Γ dependence is complicated in form, with structure superimposed upon the general trend. It is seen that the general trend is to lower Γ at higher Q, and to higher Γ as e_N is reduced.

The effect of a number of the approximations made in the simple model without rotation has also been studied. The validity of not allowing slip between the ball and the parabolic surface may be examined by using the changes in normal and tangential velocities during a bounce to calculate the coefficient of Coulombic friction, μ, sufficient to prevent slip. Using the parameters of fig. 2, it is found that below Q_{th} extremely low values of μ would prevent slip. Just above the transition, a value lower than that of most real surfaces would be sufficient to completely prevent slip. However, the value of μ required to avoid slip rises steadily with Q and by $Q = 5.0$ slip would occur in a small fraction of bounces for any realistic value of μ. Thus at high values of Q a more sophisticated model may be required to accurately capture the motion.

It is found that the principal effect of varying e_T is to alter the length of the high amplitude bursts. If $|e_T|$ lies less than ≈ 0.7 the form of the bursts is not particularly sensitive to e_T but as $|e_T|$ approaches unity the oscillatory tail to each burst becomes increasingly long.

Discussion. – We have considered the way in which changes in the curvature of a vertically vibrated parabolic surface affect the horizontal confinement of a spherical ball bounced upon it. We have considered two different models, one with rotation and one without, and have shown that they both exhibit the same generic dynamical behaviour. Both models predict that as the curvature is increased the ball is increasingly confined but that there is an increasing probability of intermittent bursts of high-amplitude horizontal excursions. While it is not clear from our data if there is a true critical onset, the probability of observing these bursts increases very rapidly with the parameter Q offering an effective "onset threshold" for observing a burst within a specified sequence of bounces.

The results which we have presented may be used to consider the effect of slowly increasing Γ while bouncing a ball upon the surface. Figure 6 suggests that for sufficiently small Γ the ball will bounce vertically in a stable fashion with a very low probability of observing large horizontal excursions. As Γ is increased the mean vertical amplitude will increase. As the threshold curve is approached, there is a rapidly increasing probability that the ball will undergo a large horizontal excursion. Since in practice the surface is of finite horizontal extent, the ball may well leave the surface.

In preliminary laboratory experiments in which a stainless-steel ball is bounced upon a vertically vibrating glass telescope mirror, we find that there is indeed a range of lower values of Γ for which the motion is predominantly vertical, the ball bouncing "on the spot". However, as Γ is increased the motion of the ball becomes abruptly unstable, after a sequence of horizontally localised bounces, through it making large amplitude horizontal excursions and leaving the surface. This is just the behaviour which is predicted by the simulations which have been described.

It is also interesting to note that the behaviour found in this system is an example of on-off intermittency [15]. The bursts of horizontal motion result from a coupling between the vertical chaotic motion and the horizontal variable, induced by the noise. The system switches between laminar behaviour, in which the motion is predominantly vertical, and chaotic arching motion involving the ball bouncing alternately at positive and negative values of x. The exponential distribution of bounces between bursts is a characteristic of on-off intermittency away from the onset [16]. It would therefore be interesting to carry out further studies of a ball bouncing on a curved surface to see whether there is a critical value of the curvature below which no intermittency is observed.

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