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After VaR: The Theory, Estimation, and Insurance Applications of Quantile-Based Risk Measures

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After VaR: The Theory, Estimation, and Insurance Applications of Quantile-Based Risk Measures

By

Kevin Dowd and David Blake*

We discuss a number of quantile-based risk measures (QBRMs) that have recently been developed in the financial risk and actuarial/insurance literatures. The measures considered include the Value-at-Risk (VaR), coherent risk measures, spectral risk measures, and distortion risk measures. We discuss and compare the properties of these different measures, and point out that the VaR is seriously flawed. We then discuss how QBRMs can be estimated, and discuss some of the many ways they might be applied to insurance risk problems. These applications are typically very complex, and this complexity means that the most appropriate estimation method will often be some form of stochastic simulation.

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1. INTRODUCTION

The measurement of financial risk has been one of the main preoccupations of actuaries and insurance practitioners for a very long time. Measures of financial risk manifest themselves explicitly in many different types of insurance problem, including the determination of reserves or capital, the setting of premiums and thresholds (e.g., for deductibles and reinsurance cedance levels), and the estimation of magnitudes such as expected claims, expected losses and probable maximum losses; they also manifest themselves implicitly in problems involving shortfall and ruin probabilities. In each of these cases, we are interested, explicitly or implicitly, in quantiles of some loss function or, more generally, in quantile-based risk measures (QBRMs).

Interest in these measures has also come from more recent developments, most particularly from the emergence of Value-at-Risk (VaR) in the mainstream financial risk management (FRM) area, and from the development of a number of newer risk measures, of which the best-known are coherent and distortion risk measures. Increased interest in risk measurement also arises from deeper background developments, such as: the impact of financial engineering in insurance, most particularly in the emerging area of alternative risk transfer (ART); the increasing securitization of insurance-related risks; the increasing use of risk measures in regulatory capital and solvency requirements; the trend toward convergence between insurance, banking and securities markets, and the related efforts to harmonize their regulatory treatment; and the growth of enterprise-wide risk management (ERM).

This paper provides an overview of the theory and estimation of these measures, and of their applications to insurance problems. We focus on three key issues: the different *types* of QBRMs and their relative merits; the *estimation* of these risk measures; and the many ways in which they can be *applied* to insurance problems.[1] We draw on both the mainstream FRM literature and the actuarial/insurance literature. Both literatures have witnessed important developments in this area, but the amount of cross-fertilization between them has also been curiously imbalanced, as the actuarial/insurance community has tended to pick up on developments in financial risk management much more quickly than financial risk managers have picked up on developments in actuarial science. Indeed, important developments in the actuarial field – such as the theory of distortion risk measures – are still relatively little known outside actuarial circles.[2]

In comparing the various risk measures and discussing how they might be estimated and applied, we wish to make three main arguments, which will become clearer as we proceed. (1) There are many QBRMs that have respectable properties and are demonstrably superior to the VaR, but the choice of 'best' risk measure(s) is a subjective one that can also depend on the context. (2) The estimation of any QBRM is a relatively simple matter, once we have a good VaR estimation system. This is because the VaR is itself a quantile, and any calculation engine that can estimate a single quantile can also easily estimate a set of them, and thence estimate any function of them. This implies, in turn, that it should be relatively straightforward for institutions to upgrade from VaR to more sophisticated risk measures. (3) Insurance risk measurement problems are often extremely complex. This complexity is due to many

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[1] For reasons of space we restrict ourselves to QBRMs and ignore other types of risk measure (e.g., the variance, semi-variance, mean absolute deviation, entropy, etc.). We also have relatively little to say on closely related risk measures that are well covered in the actuarial literature, such as premium principles, stop-loss measures and stochastic ordering. For more on these, see, e.g., Denuit *et alia* (2005).

[2] It is also sometimes the case that important contributions can take a long time to become widely accepted. A good case in point here is the slowness with which axiomatic theories of financial risk measurement – of which the theory of coherent risk measures is the most notable example – have been accepted across the FRM community, despite highly persuasive arguments that coherent measures are superior to the VaR. This slowness to adopt superior risk measures seems to be due to the fact that many FRM practitioners still do not understand the axiomatic theories of financial risk measures, and has led to the patently unsustainable situation that the VaR continues to be the most widely used risk measure despite the fact that it is now effectively discredited as a risk measure

different factors (which we will address in due course), and implies that the overwhelming majority of insurance risk measurement problems need to be handled using stochastic simulation methods.

This paper is organized as follows. Section 2 discusses and compares the main types of QBRMs, focusing mainly on the VaR, coherent measures (including those familiar to actuaries as CTE or tail VaR), spectral measures and distortion measures. Section 3 looks at the estimation of QBRMs, i.e., it shows *how* to estimate our risk measure, once we have decided *which* risk measure we wish to estimate: this section reviews the standard 'VaR trinity' of parametric methods, nonparametric methods and stochastic simulation methods. Section 4 investigates some of the complicating features of insurance risk-measurement problems: these include valuation problems, 'badly behaved' and heterogeneous risk factors, nonlinearity, optionality, parameter and model risk, and long forecast horizons. Section 5 then discusses some example applications and seeks to illustrate the thinking behind these applications. After this, section 6 briefly addresses some further issues that often arise in insurance risk measurement problems: these include issues of capital allocation and risk budgeting, risk-expected return analysis and performance evaluation, long-run issues, problems of model evaluation, the issues raised by enterprise-wide risk management, and regulatory issues (including regulatory uses of QBRMs). Section 7 concludes.

2. QUANTILE-BASED MEASURES OF RISK

2.1 Value at Risk (VaR)

For practical purposes, we can trace the origins of VaR back to the late 1970s and 1980s, when a number of major financial institutions started work on internal risk-forecasting models to measure and aggregate risks across the institution as a whole.[3] They started work on these models for their own risk management purposes – as firms became more complex, it was becoming increasingly difficult, but yet also increasingly important, to be able aggregate their risks taking account of how they interact with each other, and institutions lacked the methodology to do so. The best known of these systems was that developed by JP Morgan, which was operational by around 1990. This system was based on standard portfolio theory using estimates of the standard deviations and correlations between the returns to different traded instruments, which it aggregated across the whole institution into a single firm-wide risk measure. The measure used was the hitherto almost unknown notion of daily Value at Risk (or VaR) – the maximum likely loss over the next trading day. The term 'likely' was interpreted in terms of a 95% level of confidence, so the VaR was the maximum loss that the firm could expect to experience on the 'best' 95 days out of a 100. However, different VaR models differed in terms of the horizon periods and confidence levels used, and also in terms of their estimation methodologies: some were based on portfolio theory, some were based on historical simulation methods, and others were based on stochastic simulation methods.

Once they were operational, VaR models spread very rapidly, first among securities houses and investment banks, then among commercial banks, pension funds, insurance companies, and non-financial corporates. The VaR concept also became more familiar as the models proliferated, and by the mid-1990s, the VaR had already established itself as *the* dominant measure of

[3] The roots of the measure go further back. One can argue that the VaR measure was at least implicit in the initial reserve measure that appears in the classical probability of ruin problem that actuaries have been familiar with since the early twentieth century. The VaR can also be attributed to Baumol (1963, p. 174), who suggested a risk measure equal to $\mu - k\sigma$, where μ and σ are the mean and standard deviation of the distribution concerned, and k is a subjective confidence-level parameter that reflects the user's attitude to risk. This risk measure is equivalent to the VaR under the assumption that losses are elliptically distributed. However, the term 'value at risk' did not come into general use until the early 1990s. For more on the history of VaR, see Guldimann (2000) or Holton (2002, pp. 13-19).

financial risk in the mainstream financial risk area. Since then, VaR models have become much more sophisticated, and VaR methods have been extended beyond market risks to measure other risks such as credit, liquidity (or cashflow), and operational risks.

To consider the VaR measure more formally, suppose we have a portfolio that generates a random loss over a chosen horizon period.[4] Let α be a chosen probability and q_α be the α-quantile of the loss density function. The VaR of the portfolio at the α confidence level is then simply the q_α quantile of the loss distribution, i.e.:

(1) $$VaR_\alpha = q_\alpha .$$ [5]

The rapid rise of VaR was due in large part to the VaR having certain characteristics, which gave it an edge over the more traditional risk assessment methods used in capital markets contexts:

- The VaR provides a *common* measure of risk across different positions and risk factors. It can be applied to any type of portfolio, and enables us to compare the risks across different (e.g., fixed-income and equity) portfolios. Traditional methods are more limited: duration measures apply only to

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[4] We use the term 'portfolio' as a convenient label. However, it could in practice be any financial position or collection of positions: it could be a single position, a book or collection of positions, and it could refer to assets, liabilities, or some net position (e.g., as in asset-liability management).

[5] The VaR is thus predicated on the choice of two parameters, the holding or horizon period and the confidence level. The values of these parameters are (usually) chosen arbitrarily, but guided by the context. For example, if we are operating in a standard trading environment with marking-to-market, then the natural horizon period is a trading day; if we are dealing with less liquid assets, a natural horizon might be the period it takes to liquidate a position in an orderly way. However, an insurance company will sometimes want a much longer horizon. The other parameter, the confidence level, would usually be fairly high, and banks and securities firms often operate with confidence levels of around 95% to 98%. But if we are concerned with extreme (i.e., low-probability, high impact) risks, we might operate with confidence levels well above 99%.

fixed-income positions, Greek measures apply only to derivatives positions, portfolio-theory measures apply only to equity and similar (e.g., commodity) positions, and so forth.

- VaR enables us to *aggregate* the risks of positions taking account of the ways in which risk factors correlate with each other, whereas most traditional risk measures do not allow for the 'sensible' aggregation of component risks.
- VaR is *holistic* in that it takes full account of all driving risk factors, whereas many traditional measures only look at risk factors one at a time (e.g., Greek measures) or resort to simplifications that collapse multiple risk factors into one (e.g., duration-convexity and CAPM measures). VaR is also holistic in that it focuses assessment on a complete portfolio, and not just on individual positions in it.
- VaR is *probabilistic*, and gives a risk manager useful information on the probabilities associated with specified loss amounts. Many traditional measures (e.g., duration-convexity, Greeks, etc.) only give answers to 'what if?' questions and don't give an indication of loss likelihoods.
- VaR is expressed in the simplest and most easily understood *unit of measure*, namely, 'lost money'. Many other measures are expressed in less transparent units (e.g., average period to cashflow, etc.).

These are very significant attractions.

However, the VaR also suffers from some serious limitations. One limitation is that the VaR only tells us the most we can lose in good states where a tail event does *not* occur; it tells us nothing about what we can lose in 'bad' states where a tail event does occur (i.e., where we make a loss in excess of the VaR). VaR's failure to consider tail losses can then create some perverse outcomes. For instance, if a prospective investment has a high expected return but also involves the possibility of a very high loss, a VaR-based decision calculus might suggest that the investor should go ahead with the investment if the higher loss does not affect (and therefore exceeds) the VaR, regardless of the sizes of the higher expected return and possible higher losses. This undermines ‘sensible’ risk-return analysis, and can leave the investor exposed to very high losses.

The VaR can also create moral hazard problems when traders or asset managers work to VaR-defined risk targets or remuneration packages. Traders who face a VaR-defined risk target might have an incentive to sell out-of-the-money options that lead to higher income in most states of the world and the occasional large hit when the firm is unlucky. If the options are suitably chosen, the bad outcomes will have probabilities low enough to ensure that there is no effect on the VaR, and the trader will benefit from the higher income (and hence higher bonuses) earned in ‘normal’ times when the options expire out of the money. The fact that VaR does not take account of what happens in ‘bad’ states can distort incentives and encourage traders to ‘game’ a VaR target (and/or a VaR-defined remuneration package) to promote their own interests at the expense of the institutions that employ them.[6]

2.2. The theory of coherent risk measures

More light was shed on the limits of VaR by some important theoretical work by Artzner, Delbaen, Eber and Heath in the 1990s (Artzner *et alia* (1997, 1999)). Their starting point is that although we all have an intuitive *sense* of what financial risk entails, it is difficult to give a good assessment of financial risk unless we specify what a *measure* of financial risk actually means. For example, the notion of temperature is difficult to conceptualize without a clear notion of a thermometer, which tells us how temperature should be measured. Similarly, the notion of risk itself is hard to appreciate without a clear idea of what we mean by a measure of risk. To clarify these issues, Artzner *et alia* proposed to do for risk what Euclid and others had done for geometry: they postulated a set of risk-measure axioms – the axioms of coherence – and began to work out their implications.

[6] Some further, related, problems with the VaR risk measure are discussed in Artzner *et alia* (1999, pp. 215-218) and Acerbi (2004).

Suppose we have a risky position *X* and a risk measure $\rho(X)$ defined on *X*.[7] We now define the notion of an acceptance set as the set of all positions acceptable to some stakeholder (e.g., a financial regulator). We then interpret the risk measure $\rho(.)$ as the minimum extra cash that has to be added to a risky position and invested prudently in some reference asset to make the risky position acceptable. If $\rho(.)$ is positive, then a positive amount must be added to make the position acceptable; and if $\rho(.)$ is negative, its absolute value can be interpreted as the maximum amount that can be withdrawn and still leave the position acceptable. An example might be the minimum amount of regulatory capital specified by (i.e., 'acceptable to') a financial regulator for a firm to be allowed to set up a fund management business.

Now consider *any* two risky positions *X* and *Y*, with values given by *V*(*X*) and *V*(*Y*). The risk measure $\rho(.)$ is then said to be coherent if it satisfies the following properties:

- *Monotonicity*: $V(Y) \geq V(X) \Rightarrow \rho(Y) \leq \rho(X)$.
- *Subadditivity*: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.
- *Positive homogeneity*: $\rho(hX) = h\rho(X)$ for $h > 0$.
- *Translational invariance*: $\rho(X+n) = \rho(X) - n$ for some certain amount *n*.

The first, third and fourth properties can be regarded as 'well-behavedness' conditions. Monotonicity means that if *Y* has a greater value than *X*, then *Y* should have lower risk: this makes sense, because it means that less has to be added to *Y* than to *X* to make it acceptable, and the amount to be added *is* the risk measure. Positive homogeneity implies that the risk of a position is proportional to its scale or size, and makes sense if we are dealing with liquid positions in marketable instruments. Translational invariance requires that the addition of a sure amount reduces *pari passu* the cash still needed to make our position acceptable, and its validity is obvious.

[7] *X* itself can be interpreted in various other ways, e.g., as the random future value of the position or as its random cashflow, but its interpretation as the portfolio itself is the most straightforward.

The key property is the second, subadditivity. This tells us that a portfolio made up of sub-portfolios will risk an amount which is no more than, and in some cases less than, the sum of the risks of the constituent sub-portfolios. Subadditivity is the most important criterion we would expect a 'respectable' risk measure to satisfy. It reflects our expectation that aggregating individual risks should not increase overall risk, and this is a basic requirement of any 'respectable' risk measure, coherent or otherwise.[8]

It then follows that the VaR cannot be a 'respectable' measure in this sense, because VaR is not subadditive.[9] In fact, VaR is only subadditive in the restrictive case where the loss distribution is elliptically distributed, and this is of limited consolation because most real-world loss distributions are not elliptical ones. The failure of VaR to be subadditive is a fundamental problem because it means, in essence, that VaR has no claim to be regarded as a 'true' risk measure at all. The VaR is merely a quantile. There is also a deeper problem:

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[8] Although we strongly agree with the argument that subadditivity is a highly desirable property in a risk measure, we also acknowledge that it can sometimes be problematic. For example, Goovaerts *et alia* (2003b) suggest that we can sometimes get situations where the 'best' risk measure will violate subadditivity (see the last bullet point in section 2.7 below): we therefore have to be careful to ensure that any risk measure we use makes sense in the context in which it is to be used. There can also be problems in the presence of liquidity risk. If an investor holds a position that is 'large' relative to the market, then doubling the size of this position can more than double the risk of the position, because bid prices will depend on the position size. This raises the possibility of liquidity-driven violations of homogeneity and subadditivity. A way to resolve this difficulty is to replace coherent risk measures with convex ones. An alternative, suggested by Acerbi (2004, p. 150), is to add a liquidity charge to a (strongly) coherent risk measure. This charge would take account of relative size effects, but also have the property of going to zero as size/illiquidity effects become negligible.

[9] The non-subadditivity of the VaR is most easily shown by a counter-example. Suppose we have two identical bonds, A and B. Each defaults with probability 4%, and we get a loss of 100 if default occurs, and 0 if no default occurs. The 95%-VaR of each bond is therefore 0, so $\mathrm{VaR}_{0.95}(A) = \mathrm{VaR}_{0.95}(B) = \mathrm{VaR}_{0.95}(A)+\mathrm{VaR}_{0.95}(B)=0$. Now suppose that defaults are independent. Elementary calculations then establish that we get a loss of 0 with probability $0.96^2=0.9216$, a loss of 200 with probability $0.04^2=0.0016$, and a loss of 100 with probability $1-0.9216-0.0016=0.0768$. Hence $\mathrm{VaR}_{0.95}(A+B)=100$. Thus, $\mathrm{VaR}_{0.95}(A+B)=100>0=\mathrm{VaR}_{0.95}(A)+\mathrm{VaR}_{0.95}(B)$, and the VaR violates sub-additivity.

> the main problem with VaR is not its lack of subadditivity, but rather the very fact that no set of axioms for a risk measure and therefore no unambiguous definition of financial risk has ever been associated with this statistic. So, despite the fact that some VaR supporters still claim that subadditivity is not a necessary axiom, none of them, to the best of our knowledge, has ever tried to write an alternative meaningful and consistent set of axioms for a risk measure which are fulfilled also by VaR. (Acerbi (2004, p. 150))

Given these problems, we seek alternative risk measures that retain the benefits of VaR – in terms of globality, universality, probabilistic content, etc. – whilst avoiding its drawbacks.[10] Furthermore, if it is to retain the benefits of the VaR, it is reasonable to suppose that any such risk measures will be 'VaR-like' in the sense that they will reflect the quantiles of the loss distribution, but will be non-trivial functions of those quantiles rather than a single quantile on its own.

2.3. Expected Shortfall (ES)

One promising candidate is the Expected Shortfall (ES), which is the average of the worst $1-\alpha$ losses. In the case of a continuous loss distribution, the ES is given by:

(2) $$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_p dp$$

If the distribution is discrete, then the ES is the discrete equivalent of (2):

[10] In this context, it is also worth noting that coherent risk measures also have another important advantage over VaR: the risk surface of a coherent risk measure is convex (i.e., any line drawn between two coherent risk measures lies above the coherent risk surface), whereas that of a VaR might not be. This is a very important advantage in optimization routines, because it ensures that a risk minimum is a unique global one. By contrast, if the risk surface is not guaranteed to be convex (as with a VaR surface), then we face the problem of having potentially multiple local mimina, and it can be very difficult to establish which of these is the global one. For optimization purposes, a convex risk surface is therefore a distinct advantage. For more on this issue, see, e.g., Rockafellar and Uryasev (2002) or Acerbi (2004, pp. 186-197).

(3) $$ES_{\alpha} = \frac{1}{1-\alpha}\sum_{p=\alpha}^{1}[p^{\text{th}} \text{ worst outcome}] \times [\text{probability of } p^{\text{th}} \text{ worst outcome}]$$

This ES risk measure is very familiar to actuaries, although it is usually known in actuarial circles as the Conditional Tail Expectation (in North America) or the Tail VaR (in Europe).[11] In mainstream financial risk circles, it has been variously labelled Expected Tail Loss, Tail Conditional Expectation, Conditional VaR, Tail Conditional VaR, and Worst Conditional Expectation. Thus, there is no consistency of terminology in either actuarial or financial risk management literatures. However, the substantive point here is that this measure (whatever we call it) belongs to a family of risk measures that has two key members. The first is the measure we have labelled the ES, which is defined in terms of a probability threshold. The other is its quantile-delimited cousin, the average of losses exceeding VaR, i.e., $E[X \mid X > q_{\alpha}(X)]$. The two measures will always coincide when the loss distribution is continuous. However, this latter measure can be ambiguous and incoherent when the loss distribution is discrete (see Acerbi (2004, p. 158)), whereas the ES is always unique and coherent. As for terminology, we prefer the term 'expected shortfall' because it is clearer than alternatives, because there is no consensus alternative, and because the term is now gaining ascendancy in the financial risk area.

It is easy to establish the coherence of ES. If we have N equal-probability quantiles in a discrete distribution, then

(4) $$\begin{aligned} & ES_{\alpha}(X) + ES_{\alpha}(Y) \\ &= \text{Mean of } N\alpha \text{ worst cases of } X + \text{Mean of } N\alpha \text{ worst cases of } Y \\ &\geq \text{Mean of } N\alpha \text{ worst cases of } (X+Y) \end{aligned}$$

[11] This measure has also been used by actuaries for a very long period of time. For example, Artzner *et alia* (1999, pp. 219-220) discuss its antecedents in German actuarial literature in the second third of the 19th century. Measures similar to the ES have been long prominent in areas of actuarial science such as reserving theory.

$= ES_\alpha(X+Y)$.

A continuous distribution can be regarded as the limiting case as N gets large. In general, the mean of the $N\alpha$ worst cases of X plus the mean of the $N\alpha$ worst cases of Y will be bigger than the mean of the $N\alpha$ worst cases of (X+Y), except in the special case where the worst X and Y occur in the same $N\alpha$ events, and in this case the sum of the mean will equal the mean of the sum. This shows that ES is subadditive. It is also easy to show that the ES also satisfies the other properties of coherence, and is therefore coherent (Acerbi (2004, proposition 2.16)).

The ES is an attractive risk measure for a variety of reasons besides its coherence. It has some very natural applications in insurance (e.g., it is an obvious measure to use when we wish to estimate the cover needed for an excess-of-loss reinsurance treaty, or more generally, when we are concerned with the expected sizes of losses exceeding a threshold). It also has the attraction that it is very easy to estimate: the actuary simply generates a large number of loss scenarios and takes the ES as the average of the 100(1-α)% of largest losses.

2.4. Scenarios and generalized scenarios

The theory of coherent risk measures has some radical (and sometimes surprising) implications. For example, it turns out that the results of scenario analyses (or stress tests) can be interpreted as coherent risk measures. Suppose we consider a set of loss outcomes combined with a set of associated probabilities. The losses can be regarded as tail drawings from the relevant distribution function, and their expected (or average) value is the ES associated with this distribution function. Since the ES is a coherent risk measure, this means that the outcomes of scenario analyses are also coherent risk measures. The theory of coherent risk measures therefore provides a risk-theoretical justification for the practice of stress testing.

This argument can also be generalized in some interesting ways. Consider a set of 'generalized scenarios' – a set of n loss outcomes and a *family* of distribution functions from which the losses are drawn. Take any one of these distributions and obtain the associated ES. Now do the same again with another distribution function, leading to an alternative ES. Now do the same again and again. It turns out that the maximum of these ESs is itself a coherent risk measure: if we have a set of m comparable ESs, each of which corresponds to a different loss distribution function, then the maximum of these ESs is a coherent risk measure.[12] Furthermore, if we set n=1, then there is only one tail loss in each scenario and each ES is the same as the probable maximum loss or likely worst-case scenario outcome. If we also set m=1, then it immediately follows that the highest expected loss from a single scenario analysis is a coherent risk measure; and if m>1, then the highest expected of m worst case outcomes is also a coherent risk measure. In short, the ES, the highest expected loss from a set of possible outcomes (or loss estimates from scenario analyses), the highest ES from a set of comparable ESs based on different distribution functions, and the highest expected loss from a set of highest losses, are all coherent risk measures.

The foregoing shows that the outcomes of (simple or generalized) scenarios can be interpreted as coherent risk measures. However, the reverse is also true, and coherent risk measures can be interpreted as the outcomes of scenarios. This is useful, because it means that we can always estimate coherent risk measures by specifying the relevant scenarios and then taking (as relevant) their (perhaps probability-weighted) averages or maxima: in principle, all we need to know are the loss outcomes (which are quantiles from the loss

[12] A good example of a standard stress testing framework whose outcomes qualify as coherent risk measures is the SPAN system used by the Chicago Mercantile Exchange to calculate margin requirements. As explained by Artzner *et alia* (1999, p. 212), this system considers 16 specific scenarios, consisting of standardised movements in underlying risk factors. 14 of these are fairly moderate scenarios, and 2 are extreme. The measure of risk is the maximum loss incurred across all scenarios, using the full loss from the first 14 scenarios and 35% of the loss from the two extreme ones. (Taking 35% of the losses on the extreme scenarios can be regarded as an ad hoc adjustment allowing for the extreme losses to be less probable than the others.) The calculations involved can be interpreted as producing the maximum expected loss under 16 distributions. The SPAN risk measures are coherent because the margin requirement is equal to the shortfall from this maximum expected loss.

distribution), the density functions to be used (which give us our probabilities), and the type of coherent risk measure we are seeking. However, in practice, implementation is even more straightforward: we would often work with a (typically stochastic) scenario generation program, take each generated scenario as equally likely (which allows us to avoid any explicit treatment of probabilities) and then apply the weighing function of our chosen risk measure to the relevant set of loss scenarios.

2.5. Spectral risk measures

If we are prepared to 'buy into' risk-aversion theory,[13] we can go on to relate coherent risk measures to a user's risk aversion. This leads us to the spectral risk measures proposed by Carlo Acerbi (2002, 2004). Let us define more general risk measures M_ϕ that are weighted averages of the quantiles of our loss distribution:

$$M_\phi = \int_0^1 \phi(p) q_p dp \tag{5}$$

where the weighting function, $\phi(p)$, also known as the risk spectrum or risk-aversion function, remains to be determined.

The ES is a special case of M_ϕ obtained by setting $\phi(p)$ to the following:

[13] Risk-aversion theory requires us to specify a user risk-aversion function, and this can provide considerable insights (as shown in the following text) but can also be controversial. Amongst the potential problems it might encounter are: (1) the notion of a risk-aversion function can be hard to motivate when the user is a firm, or an employee working for a firm, rather than, say, an individual investor working on their own behalf; (2) one might argue with the type of risk-aversion function chosen; and (3) one might have difficulty specifying the value that the risk aversion parameter should take.

(6) $$\phi(p) = \begin{cases} 1/(1-\alpha) & \text{if } p > \alpha \\ 0 & \phantom{\text{if }} p \le \alpha \end{cases}$$

As the name suggests, the ES gives tail-loss quantiles an equal weight of $1/(1-\alpha)$, and other quantiles a weight of 0.

However, we are interested here in the broader class of coherent risk measures. In particular, we want to know what conditions $\phi(p)$ must satisfy in order to make M_ϕ coherent. The answer is the class of (non-singular) spectral risk measures, in which $\phi(p)$ takes the following properties:[14]

- *Non-negativity:* $\phi(p) \ge 0$ for all p belong in the range [0,1].
- *Normalization*: $\int_0^1 \phi(p)dp = 1$.
- *Increasingness*: $\phi(p_1) \le \phi(p_2)$ for all $0 \le p_1 \le p_2 \le 1$.

The first condition requires that the weights are non-negative, and the second requires that the probability-weighted weights should sum to 1. Both are obvious. The third condition is more interesting. This condition is a direct reflection of risk-aversion, and requires that the weights attached to higher losses should be bigger than, or certainly no less than, the weights attached to lower losses. The message is clear: *the key to coherence is that a risk measure must give higher losses at least the same weight as lower losses*. This explains

[14] See Acerbi (2004, proposition 3.4). Strictly speaking, the set of spectral risk measures is the convex hull (or set of all convex combinations) of ES_α for all α belonging to [0,1]. There is also an 'if and only if' connection here: a risk measure M_ϕ is coherent if and only if M_ϕ is spectral and $\phi(p)$ satisfies the conditions indicated in the text. There is also a good argument that the spectral measures so defined are the only really interesting coherent risk measures: Kusuoka (2001) and Acerbi (2004, pp. 180-182) show that all coherent risk measures that satisfy the two additional properties of comonotonic additivity and law invariance are also spectral measures. The former condition is that if two random variables X and Y are comonotonic (i.e., always move in the same direction), then $\rho(X+Y) = \rho(X) + \rho(Y)$; comonotonic additivity is an important aspect of subadditivity, and represents the limiting case where diversification has no effect. Law-invariance is equivalent to the (for practical purposes essential) requirement that a measure be estimable from empirical data.

why the VaR is not coherent and the ES is; it also suggests that the VaR's most prominent inadequacies are closely related to its failure to satisfy the increasingness property.

It is important to appreciate that the weights attached to higher losses in spectral risk measures are a direct reflection of the user's risk aversion. If a user has a 'well-behaved' risk-aversion function, then the weights will rise smoothly, and the more risk-averse the user, the more rapidly the weights will rise.

To obtain a spectral risk measure, we must specify the user's risk-aversion function. This decision is subjective, but can be guided by the economic literature on risk-aversion theory. For example, we might choose an exponential risk-aversion function which would lead to the following weighting function:

$$\phi(p) = \frac{ke^{-k(1-p)}}{1-e^{-k}} \tag{7}$$

where $k>0$ is the user's coefficient of absolute risk aversion. This function satisfies the conditions of a spectral risk measure, but is also attractive because it is a simple well-behaved function of a single parameter k. To obtain our risk measure, we then specify the value of k and plug (7) into (5).

The connection between the $\phi(p)$ weights and user risk-aversion sheds further light on our earlier risk measures. We saw earlier that the ES is characterized by all losses in the tail region (i.e., the $100(1-\alpha)\%$ largest losses) having the same weight. If we interpret the weights as reflecting the user's attitude toward risk, this can only be interpreted as the user being risk-neutral, at least between tail-region outcomes. So the ES is appropriate if the user is risk-neutral at the margin in this region. Since we usually assume that agents are risk-averse, this would suggest that the ES might not always be such a good risk measure, notwithstanding its coherence. If we believe that a particular user is risk-averse, we should have a weighting function that rises as *p* gets bigger, and this rules out the ES.[15]

The implications for the VaR are much worse. With the VaR, we give a large weight to the loss associated with a *p*-value equal to α, and we give a lower (indeed, zero) weight to any *greater* loss. The implication is that the user is actually risk-*loving* (i.e., has *negative* risk-aversion) in the tail loss region, and this is highly uncomfortable.[16] To make matters worse, since the weight drops to zero, we are also talking about risk-loving of a rather extreme sort. If the ES is an inappropriate measure for a risk-averse user, then the VaR is much more so.

2.6. Distortion risk measures

Distortion risk measures are closely related to coherent measures. They were introduced by Denneberg (1990) and Wang (1996) and have been applied to a wide variety of insurance problems, most particularly to the determination of insurance premiums.[17]

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[15] The downside risk literature also suggests that the use of the ES as the preferred risk measure indicates risk-neutrality (see, e.g., Bawa (1975) and Fishburn (1977)). Coming from within an expected utility framework, these papers suggest that we can think of downside risk in terms of lower-partial moments (LPMs), which are probability-weighted deviations of returns r from some below-target return r^*: more specifically, the LPM of order $k \geq 0$ around r^* is equal to $\mathrm{E}[\max(0, r^* - r)^k]$. The parameter k reflects the degree of risk aversion, and the user is risk-averse if $k > 1$, risk-neutral if $k = 1$, and risk-loving if $0 < k < 1$. However, we would only choose the ES as our preferred risk measure if $k = 1$ (Grootveld and Hallerbach (2004, p. 36)). Thus the use of the ES implies that we are risk-neutral.

[16] Following on from the last footnote, the expected utility-downside risk literature also indicates that we obtain the VaR as the preferred risk measure if $k = 0$. From the perspective of this framework, $k = 0$ indicates an extreme form of risk-loving. Thus, two very different approaches both give the same conclusion that VaR is only an appropriate risk measure if preferences exhibit extreme degrees of risk-loving.

[17] The roots of distortion theory can be traced further back to Yaari's dual theory of risk (Yaari (1987)), and in particular the notion that risk measures could be constructed by transforming the probabilities of specified events.

A distortion risk measure is the expected loss under a transformation of the cumulative density function known as a distortion function, and the choice of distortion function determines the risk measure. More formally, if $F(x)$ is some cdf, the transformation $F^*(x) = g(F(x))$ is a distortion function if g:[0,1]→[0,1] is an increasing function with $g(0)=0$ and $g(1)=1$. The distortion risk measure is then the expectation of the random loss X using probabilities obtained from $F^*(x)$ rather than $F(x)$. Like coherent risk measures, distortion risk measures have the properties of monotonicity, positive homogeneity, and translational invariance; they also share with spectral risk measures the property of comonotonic additivity. To make good use of distorted measures, we would choose a 'good' distortion function, and there are many distortion functions to choose from. The properties we might look for in a 'good' distortion function include continuity, concavity, and differentiability; of these, continuity is necessary and sufficient for the distortion risk measure to be coherent, and concavity is sufficient (Wang *et alia* (1997); Darkiewicz *et alia* (2003)).

The theory of distortion risk measures also sheds further light on the limitations of VaR and ES. The VaR can be shown to be a distortion risk measure obtained using the binary distortion function:

(8) $$\begin{cases} g(u) = 1 \\ g(u) = 0 \end{cases} \text{for } \begin{cases} u \geq \alpha \\ u < \alpha \end{cases}$$

This is a poor function because it is not continuous, due to the jump at $u = \alpha$; and since it is not continuous, it is not coherent. Thus, from the perspective of distortion theory, the VaR is a poor risk measure because it is based on a 'badly behaved' distortion function. For its part, the ES is a distortion risk measure based on the distortion function:

Going further back, it also has antecedents in the risk neutral density functions used since the 1970s to price derivatives in complete markets settings.

(9) $$\begin{cases} g(u) \geq (u-\alpha)/(1-\alpha) \\ g(u) = 0 \end{cases} \text{for} \begin{cases} u \geq \alpha \\ u < \alpha \end{cases}$$

This distortion function is continuous, which implies that the ES is coherent. However, this distortion function is still flawed: it throws away potentially valuable information, because it maps all percentiles below α to a single point *u*; and it does not take full account of the severity of extremes, because it focuses on the mean shortfall. As a result of these weaknesses, the ES can fail to allow for the mitigation of losses below VaR, can give implausible rankings of relative riskiness, and can fail to take full account of the impact of extreme losses (Wirch and Hardy (1999); Wang (2002)).

Various distortion functions have been proposed to remedy these sorts of problems, but the best-known of these is the following, the famous Wang Transform (Wang (2000)):

(10) $$g(u) = \Phi[\Phi^{-1}(u) - \lambda]$$

where $\Phi(.)$ is the standard normal distribution function and λ is a market price of risk term that might be proxied by something like the Sharpe ratio. The Wang Transform has some attractive features: for example, it recovers CAPM and Black-Scholes under normal asset returns, and it has proven to be very useful for determining insurance premiums. However, for present purposes what we are most interested in here is that this distortion function is everywhere continuous and differentiable. The continuity of this distortion function means that it produces coherent risk measures, but these measures are superior to the ES because they take account of the losses below VaR, and also take better account of extreme losses (Wang (2002a)).

Wang (2002b) also suggests a useful generalization of the Wang Transform:

(11) $$g(u) = \Phi[b\Phi^{-1}(u) - \lambda]$$

where $0<b<1$. This second transform provides for the volatility to be distorted as well, and Wang suggests that this is good for dealing with extreme or tail risks (e.g., those associated with catastrophe losses). Another possible transformation is the following, also due to Wang (2002b):

$$g(u) = Q[\Phi^{-1}(G(u)) - \lambda] \tag{12}$$

where Q is a Student-t distribution degrees of freedom equal to our sample size minus 2, and $G(u)$ is our estimate of the distribution function of u. He suggests that this transformation would be good for dealing with the impact of parameter uncertainty on premium pricing or risk measurement.[18]

2.7. Other risk measures

There are also many other types of QBRM (and related risk measures) that we have not had space to discuss at any length. These include:

- Convex risk measures (e.g., Heath (2001), Fritelli and Gianin (2002)): These risk measures are based on an alternative set of axioms to the coherent risk measures, in which the axioms of subadditivity and linear homogeneity are replaced by the weaker requirement of convexity.
- Dynamic risk measures (e.g., Wang (1999), Pflug and Ruszczyński (2004)): These are multi-period axiomatic risk measures and that are able to take account of interim cash flows, which most coherent measures are not. These risk measures are therefore potentially more useful for longer-term applications where interim income issues might be more important.
- Comonotonicity approaches (e.g., Dhaene *et alia* (2003a,b)): These apply to situations where we are interested in the sums of random variables and cannot plausibly assume that these random variables are

[18] Many other distortion function have also been proposed, and a useful summary of these is provided by Denuit *et alia* (2005, pp. 84-95)).

independent. An example might be insurance claims that are driven off the same underlying risk factors (e.g., such as earthquakes). In such cases, the dependence structure between the random variables might be cumbersome or otherwise difficult to model, but we can often work with comonotonic approximations that are more tractable.

- Markov bounds approaches (e.g., Goovaerts *et alia* (2003a)): These approaches derive risk measures based on the minimization of the Markov bound for a tail probability. This leads to a risk measure π that satisfies $E[\phi(S,\pi) = \alpha E[v(S)]$, where S is a random variable, $\phi(S)$ and $v(S)$ are functions of that random variable, and $\alpha \leq 1$ is some exogenous parameter. These approaches provide a unified framework that permits the derivation of well-known premium principles and other risk measures that arise as special cases by appropriate specifications of ϕ and v.[19]
- 'Best practices' risk measures (Goovaerts *et alia* (2003b)): These are based on the argument that there are no sets of axioms generally applicable to all risk problems. The most appropriate risk measure sometimes depends on the economics of the problem at hand and the use to which the risk measure is to be put. They give as an example the case of the insurance premium for two buildings in the same earthquake zone, where good practice would suggest that the insurer charge more than twice what it would have charged for insuring either building on its own. In such a case, the 'best' premium is not even subadditive. Their work suggests that actuaries might need to pay more attention to the context of

[19] These include the mean value, Swiss, zero-utility, mixture of Esscher premium principles, as well as Yaari's dual theory of risk (Yaari (1987)) and the ES. For more on these premium principles, see, e.g., Bühlmann (1970), Gerber (1974), Gerber and Goovaerts (1981) and Goovaerts *et alia* (1984).

a problem, and not just focus on the theoretical properties of risk measures considered a priori.[20]

2.8. Some tentative conclusions

All these measures are indicative of the wide variety of risk measures now available, but there is as yet little agreement on any major issues other than that the VaR is a poor risk measure. Various (in comparison, minor) problems have also been pointed out regarding the ES (i.e., that it is not consistent with risk-aversion, and that it is inferior to the Wang Transformation). Going beyond these, the broader families of risk measures – in particular the families of coherent, spectral, and distortion risk measures – give us many possible risk measures to choose from. However, in some respects we are spoilt for choice and it is generally not easy to identify which particular one might be best. Nor is there any guarantee that an arbitrarily chosen member of one of these families would necessarily be a 'good' risk measure: for example, the outcome of a badly designed stress test would be a coherent risk measure, but it would not be a good risk measure. We therefore need further criteria to narrow the field down and (hopefully) eliminate possible bad choices, but any criteria we choose are inevitably somewhat ad hoc. At a deeper level, there is also no straightforward way of determining which family of risk measures might be best: all three families have different epistemological foundations, even though they have many members in common, and there is no clear way of comparing one family with another.

In the circumstances the only solid advice we can offer at the moment is: in general, avoid the VaR as a risk measure, and try to pick a risk measure that has good theoretical properties and seems to fit in well with the context at hand.

[20] And this list is by no mean exhaustive. For example, there are additional approaches based on one-sided moments (e.g., Fischer 2003), Goovaerts *et alia* (2003b)), Bayesian Esscher scenarios (Siu *et alia* (2001a)), imprecise prevision approaches (Pelessoni and Vicig (2001)), entropy based approaches (McLeish and Reesor (2003)), consistent risk measures (Goovaerts *et alia* (2004)), etc.

3. ESTIMATION METHODS

We now turn to the estimation of our risk measures. This requires that we estimate all or part of the loss distribution function. In doing so, we can think of a set of cumulative probabilities p as given, and we seek to estimate the set of quantiles q_p associated with them. The distribution function might be continuous, in which case we would have a function giving q_p in terms of a continuously valued p, or it might be discrete, in which case we would have N different values of q_p for each p equal to, say, $1/N$, $2/N$, etc.

Once we have estimated the quantile(s) we need, obtaining estimates of the risk measures is straightforward:

- If our risk measure is the VaR, our estimated risk measure is the estimated quantile (2).
- If our risk measure is a coherent or spectral one, we postulate a weighting function $\phi(p)$, discretize (5), estimate the relevant quantiles, and take our coherent risk estimate as the suitably weighted average of the quantile estimates. The easiest way to implement such a procedure is to break up the cumulative probability range into small, equal, increments (e.g., we consider p=0.001, p=0.002, etc.). For each p, we estimate the corresponding quantile, q_p, and our risk estimate is their $\phi(p)$-weighted average.[21]
- If our risk measure is a distortion one, we first discretize the 'original' probabilities (to get p=0.001, p=0.002, etc.) and estimate their matching quantiles, the q_p. We then distort the probabilities by running them through the chosen distortion function, and our risk measure is the weighted average

[21] In cases where the risk measure formula involves an integral, we also have to solve the relevant integral, and might do so using analytical methods (where they can be applied) or numerical methods (e.g., quadrature methods such as the trapezoidal rule or Simpson's rule, Gauss-Legendre, pseudo- or quasi-random number integration methods, etc.).

of the quantile estimates, where the weights are equal to the increments in the distorted (cumulative) probabilities.

From a practical point of view, there is very little difference in the work needed to estimate these different types of risk measure. This is very helpful as all the building blocks that go into quantile or VaR estimation – risk drivers, databases, calculation routines, etc. – are exactly what we need for the estimation of the other types of risk measure as well. Thus, if an institution already has a VaR engine, then that engine needs only small adjustments to produce estimates of more sophisticated risk measures: indeed, in many cases, all that needs changing is the last few lines of code in a long data processing system. This means that the costs of upgrading from VaR to more sophisticated risk measures are very low.

We can now focus on the remaining task of quantile (or equivalently, density) estimation. However, this is not a trivial matter, and the literature on quantile/VaR/density estimation is vast. Broadly speaking, there are three classes of approach we can take:

- Parametric methods.
- Non-parametric methods.
- Monte Carlo methods.

We now briefly consider each of these in turn.[22]

3.1. Parametric methods

Parametric approaches estimate quantiles based on the assumption that a loss distribution takes a particular parametric form, and the first task is to determine what this might be. The choice of distribution would be guided by informal diagnostics (e.g., use of quantile-quantile plots, mean excess function plots, etc.) in which we informally check the goodness-of-fit of a variety of possible distributions. The choice of distribution might also be guided by theoretical

[22] The foregoing summary is inevitably brief. More detailed treatments of quantile estimation are to be found, e.g., in Duffie and Pan (1997), Crouhy *et alia* (2001a) or Dowd (2005a).

considerations if there are reasons to think that the distribution might take a particular form: for example, if we were dealing with extremes, we would use an extreme-value distribution.[23] It would also be guided by past experience (e.g., knowledge that particular distributions tend to provide good fits for similar datasets). In choosing a distribution, we would also have to take account of any conditionality in the loss process: losses might follow a temporal pattern (i.e., depend on past losses), or might be driven off some other random variables (e.g., losses might depend on the processes driving, say, earthquakes). In such cases, we would fit the loss distribution conditional on the relevant driving factors rather than unconditionally. The key is to ensure (as best we can!) that we choose the 'right' parametric (conditional or unconditional) distribution – the one that best fits the characteristics (e.g., the sample moments) of the distribution we are trying to model. Depending on the problem, the distribution chosen might be any of a large number, including: normal, lognormal, *t*, log-*t*, stable Paretian, elliptical, hyperbolic, Pareto, normal-mixture, jump-diffusion, Pearson-family, Johnson-family, skew-*t*, extreme-value, etc.

Having identified the distribution, we can then look up that distribution's quantile formula. However, the quantile formula will involve parameters that need to be estimated, and we would estimate these parameters using a method suitable to the selected distribution: this method might be maximum-likelihood, least-squares, method-of-moments, semi-parametric, etc. We then plug the parameter estimates into our quantile equation to obtain our quantile estimates.

Where we are dealing with multiple distributions (e.g., a collection of loss distributions applying to different positions), we would want to model a

[23] If we are dealing with extreme events (e.g., catastrophes, large claims, ruin probabilities for solvent institutions, extreme mortality risks, etc.), then it is important to use an extreme value (EV) method. Typically, this would be based on some version of Generalised EV theory (which models extremes using Weibull, Gumbel or Fréchet distributions) or peaks-over-threshold theory (which models exceedances over a high threshold using a Generalised Pareto distribution). For more on EV theory and its implementation and/or some illustrative discussions of its insurance applications, see Embrechts *et alia* (1997), Embrechts *et alia* (1999), Reiss and Thomas (1997), Cotter (2001) and Cebrián *et alia* (2003).

multivariate distribution.[24] Multivariate approaches require us to either specify a particular multivariate distribution (e.g., multivariate normal, multivariate *t*, etc.) with the dependence structure between the random variables modelled by a correlation matrix, or to specify a copula function, in which case we would choose marginal distributions for each random variable and a suitable copula function to model their dependence structure. Correlation approaches are more familiar and easier to work with, but can (usually) only be applied to elliptical distributions; on the other hand, copula approaches are much more flexible because they allow us to fit different marginal distributions to different risk factors and also allow us a much wider range of possible dependence structure.[25] However, they are also harder to work with and, in practice, usually require stochastic simulation.

Parametric methods are suited to risk measurement problems where the distributions concerned are known or reliably estimated. However, this condition is often not met in practice, especially when we have small sample sizes, and in such circumstances they can be very unreliable. Furthermore, they are generally only appropriate for relatively 'simple' risk measurement problems, which few insurance problems are.

3.2. Non-parametric methods

Non-parametric methods seek to estimate risks without making strong assumptions about the distribution under consideration. Instead of imposing some parametric distribution on the data, they let the data speak for themselves as much as possible, and estimate risk measures from the *empirical* distribution. Relative to parametric approaches, non-parametric approaches have the major

[24] We might also want to work with multivariate distributions because they are more flexible and allow us to change the weights of the sub-portfolio positions from one period to the next, which is not easily done with a univariate approach that implicitly takes the portfolio composition as given.

[25] Frees and Valdez (1998) give a readable introduction to the use of copulas in insurance risk measurement. For more on copulas and their methodology, see also, e.g., Cherubini *et alia* (2004).

attraction that they avoid the danger of misspecifying the distribution, which could lead to major errors in our estimated risk measures. They are based on the assumption that the near future will be sufficiently like the recent past that we can use the recent historical data (as reflected in the empirical distribution) to forecast the future. Their usefulness, in practice, therefore depends on whether this assumption holds in any situation. Fortunately, it often does hold, and non-parametric methods have a good track record. On the other hand, they can be inaccurate where this assumption does not hold. Their estimates can also be imprecise, especially in the tail regions where data are especially sparse. As a result, non-parametric methods often have difficulty handling extremes.

The most common non-parametric approach is historical simulation (HS), in which we read off quantiles from a histogram of historical losses, but can be refined in many ways. We can replace histograms with kernels; these are more sophisticated non-parametric density estimators, and, in effect, seek to smooth the jagged edges of histogram columns without imposing strong assumptions on the data. We can also refine HS using weighted HS (e.g., we can weight data by age, volatility or correlation to take account of changing market circumstances), neural networks (which also allow us to adjust for changing market circumstances), bootstrap methods (which help to gauge accuracy), and principal components and factor analysis methods (which are very useful for dimensionality reduction, which can be a concern when there is a large number of random variables to be considered).

We can also extend non-parametric methods to include non-historical scenarios. For example, we might construct some hypothetical scenarios, give these scenarios some probabilities, and then apply non-parametric methods to mixtures of historical and hypothetical scenarios. Adding hypothetical events to our data set helps remedy the main weaknesses of historically-based non-parametric approaches – namely, their complete dependence on the historical data set and their difficulties handling extremes – and also allows an integrated and consistent treatment of historical and hypothetical scenarios.[26]

3.3. Stochastic simulation methods

The third class of approach is stochastic simulation (or Monte Carlo simulation) methods. These methods simulate the loss distribution using a random number simulation engine,[27] and they are much more powerful and flexible than the earlier methods. This is because the loss distribution is derived from an underlying calculation engine that can take account of virtually any level of complexity. The basic method is to specify the model 'behind' the loss distribution: we specify all the factors (e.g., we specify the exogenous and random variables, the distributions, the relationships between different variables, parameter calibration,[28] etc.) that together determine the loss. Having set this model up, we then carry out a large number of simulation trials, each of which produces a simulated loss based on a set of simulated realizations of the 'driving' risk factors, which are obtained by taking random drawings from their specified distributions. If we carry out a large number of such trials, then the distribution of simulated losses obtained in this way will provide a good approximation to the true but unknown loss distribution that we are seeking. We

[26] Ways of implement this type of approach are discussed further by Berkowitz (2000) and Aragonés *et alia* (2001).

[27] These 'random' numbers are of course never truly random. They are either 'pseudo' random numbers generated by (so-called) random number generators, which produce sequences of numbers that try to mimic many of the properties of 'true' random numbers, or they can be 'quasi' random numbers, sometimes known as low-discrepancy numbers, which do not seek to produce 'random-looking' numbers, but which often produce superior results in higher-dimension problems. For more on all these issues, see, e.g., Broadie and Glasserman (1998) or Jäckel (2002).

[28] Parameters might be calibrated using statistical methods and/or 'judgement'. It is important to appreciate that the parameters are forward-looking and even the best statistical methods are inevitably backward looking. Thus, we need to 'adjust' any historically based estimates with our best judgement on how the future might differ from the past. This can be important in such areas as modelling mortality or financial returns, where the foreseeable future [sic] might plausibly be rather different from the recent past.

then obtain estimates of our risk measures by applying non-parametric methods to this simulated loss distribution.

Stochastic methods are ideally suited to a great range of risk measurement problems, and will often provide the best way of dealing with the problems we are likely to encounter: they are particularly good at dealing with complicating factors – such as those considered in the next section – that other approaches often cannot handle. We can also refine stochastic approaches (e.g., using stratified sampling and importance sampling) to focus on any particular features of the loss function that we might be especially interested in (e.g., such as its extreme tail). Stochastic methods are therefore the methods of choice for the vast majority of 'complex' risk problems, and we shall have more to say on them in the next two sections.[29]

4. COMPLICATING FACTORS IN INSURANCE RISK MEASUREMENT PROBLEMS

Insurance risk problems often involve many complicating features, which credible risk measurement can often not ignore, and which will often necessitate that we use stochastic simulation methods. These complicating factors include:

- *Valuation problems*: Most insurance positions are valued using accounting standards. These have the advantages that they are formula-driven and their limitations are well understood. However, book values are often very misleading and frequently lead to excessive smoothing: this means that they can have great difficulty valuing derivatives positions such as

[29] The downsides of stochastic methods are that they can be less easy to use than some alternatives, they require a lot of calculations (which can be time-intensive, and sometimes be beyond our computing capability), and they often have difficulty with early-exercise features in options. However, the downsides are fairly minor: ease of use is not much of a problem for actuaries (and there is also a lot of good user-friendly software available as well), calculation times are falling rapidly thanks to improvements in computing power, and methods have recently become available to handle early exercise (such as those of Andersen (2000), Longstaff and Schwartz (2001), etc.).

options. The problems of book valuation have prompted accounting authorities to move toward market value (or marked-to-market) accounting. This works well for assets traded in liquid secondary markets, but becomes more subjective when markets are thin or non-existent, as is the case for many insurance positions Mark-to-model valuation is sometimes used, but this is subjective and unreliable, and open to abuse as well as error. In practice, many insurance companies therefore have little choice but to use an uneasy (and generally inconsistent) combination of book, marked-to-market and marked-to-model valuation.

- *'Badly behaved' and heterogeneous risk factors*: Risk factors often exhibit mean-reversion, asymmetry, heavy tails, jumps and nonstationarity, and such features often necessitate some form of stochastic simulation.[30] Insurance problems also typically involve multiple risk factors that are often very heterogeneous. In fact, the risk factors driving underwriting risks, market risks, credit risks and operational risks are likely to have very different distributions: underwriting risks will often require models of underlying 'real' processes (e.g., for weather, temperature, earthquakes, mortality, etc.), and often require models of both the frequency and the severity of loss events;[31] market risks will require models of key stock and bond indices, exchange rates, commodity prices etc. (and some of these might 'badly behaved' too); credit risks will require models of default processes, default correlations, ratings migrations, and so on;[32] operational risks involve many further complicating factors;[33] and there are also

[30] Some examples of how these processes might be modelled as given in Dowd (2005a, chs. 8 and 9).

[31] There are many examples of such models in the references cited in this paper. Some more general discussions of these models are to be found in Klugman *et alia* (1998) and Denuit *et alia* (2005).

[32] The complexities of credit risk issues are discussed further, e.g., in Crouhy *et alia* (2000; 2001a, chs. 7-13), and in Alexander and Sheedy (eds) (2004).

[33] For more on operational risks, see Alexander (ed.) (2003) the articles in the Fall (2002) issue of the *Journal of Risk Finance*, or Tripp *et alia* (2004).

liquidity risk issues to be considered as well.[34] The heterogeneity of these various risk factors also makes aggregation difficult, and will often require us to model multivariate distributions using copulas.

- *Non-linearity*: Position values are often non-linear functions of risk factors. For example, property losses might be complicated functions of risk factors such as hurricanes, earthquakes or temperatures, and many derivatives positions will have varying sensitivities to underlying risk factors (e.g., the sensitivities of options positions depend on the extent to which they are in or out of the money). There can also be complicated conditionality relationships between position values and risk factors. Conditionality can arise from many sources, but in insurance contexts often arises from the presence of loading factors and thresholds. These thresholds might be deductibles, reference losses, stop-loss limits, retention levels, etc., and they might be nominal, proportional, per occurrence, per risk, deterministic or stochastic. We can also get multiple thresholds (e.g., layers in reinsurance treaties) and these can sometimes be stochastically related.

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- *Optionality*: Insurance problems are often riddled with financial options (e.g., minimum interest guarantees, surrender options, guaranteed annuity rates, etc.; see also Jørgensen (2001)). Often, these options are quite exotic (e.g., have ‘non-standard’ underlying processes) and/or allow for early exercise, and such features often require that they be valued using stochastic methods. Insurance problems also often exhibit many real options too (e.g., franchise options, options to expand/contract, etc.). Real options have their own difficulties: valuation is often complicated by market incompleteness, and many real options are intrinsically complicated compound options (i.e., options on options) that are not easy to handle.

[34] These issues are discussed further in Kelliher *et alia* (2004) and Dowd (2005a, ch. 14).

- *Parameter and model risk*: Insurance risk problems are often subject to considerable parameter and model risk, and this can lead to major errors in estimates of risk measures. These problems can be especially difficult in situations where we have relatively little data to estimate parameters or select models reliably (e.g., where we have small sample sizes or are dealing with extremes).[35] These sorts of problems also manifest themselves for many insurance-specific reasons as well, e.g., where losses that are reported but not adjusted, incurred but not reported, etc., and where actuaries are working with mortality tables that are subject to aggregate mortality risk, where there are issues about smoothing methods, etc.[36]
- *Long horizons*: Insurance problems can have long horizons (e.g., problems involving life insurance, annuities and runoffs can have horizons of decades). Longer horizons increase the pressure to face up to the problems of specifying a dynamic portfolio management strategy, and taking account of policyholder behavior; they increase the importance of inbuilt options (e.g., such as options to cash out, convert, etc., barrier options, and real options); they aggravate vulnerability to parameter and model risk, because we are effectively extrapolating from the estimated model over a longer period; and they make model validation more difficult, because it becomes harder to accumulate a good track record that can be used for validation purposes.

[35] For example, evidence from a number of empirical studies suggests that the VaR models used by financial institutions can be quite inaccurate (see, e.g., Beder (1995), Marshall and Siegel (1997) and Berkowitz and O'Brien (2002)). In the insurance area, there is also significant evidence that published mortality tables are subject to significant parameter risk (see, e.g., Olivieri (2001) and Cairns *et alia* (2005)).

[36] The problems of parameter and model are discussed in more detail in Derman (1997), Geske (1999), Cairns (2000), Dowd (2000; 2005a ch. 16), Kato and Yoshiba (2000), Siu *et alia* (2001b) Hirsa *et alia* (2003), or Cairns *et alia* (2005).

- *Other complicating factors*: There can also be many further complicating factors, such as: coinsurance features; captive arrangements (e.g., commingling arising from multiparent captives, etc.); unit-linking; with-profits and endowment features; securitisation and ART[37]; integrated risk management; committed capital features; liquidity risks; credit risks (e.g., collateral requirements, credit enhancement features, etc.); the impact of risk management strategies; and tax issues.

5. EXAMPLES OF INSURANCE RISK MEASUREMENT PROBLEMS

We now consider some illustrative insurance problems that involve the estimation of QBRMs. These are chosen to illustrate how some of the issues and methods we have discussed can be applied to insurance problems, and to highlight the thinking behind the application.[38]

5.1. Estimating a univariate loss distribution[39]

[37] For more on securitisation and ART issues, see, e.g., Canabarro *et alia* (2000), Culp (2002) and Swiss Re (2003).

[38] We would also like to make some points about good practice that apply in every case: (1) It is always important to carry out some preliminary data analysis to 'get to know' our data, identify their characteristics (e.g., their sample moments) and possible outliers. This analysis should also involve some graphical analysis (e.g., QQ plots) to form a tentative view of what distributions might fit the data. (2) We should always be aware of potential parameter and model risk, and it is often good practice to look for ways of trying to estimate our exposure to them (e.g., by estimating confidence intervals for our risk measures).We can estimate parameter risk if we treat the parameters as random variables in their own right, specify the distributions from which the parameters are drawn, and then embed this parameter model is a broader stochastic simulation framework. We can quantify model risk by resorting to mixtures of distributions. These methods are explained further in Dowd (2000; 2005a, ch. 16). Alternatively, we can quantify these risks using Bayesian methods (e.g., as in Siu *et alia* (2001b)).

[39] Naturally, this problem of modelling a loss distribution also implies a corresponding probability-of-ruin problem: if we obtain a risk measure from the loss distribution and interpret this as the initial reserve or capital, then the probability of ruin is the probability of a loss exceeding this reserve. We can therefore look at this same problem in probability-of-ruin terms. From this perspective, we can see that the risk measure estimation methods discussed here

One of the simplest applications is the estimation of a univariate loss distribution. We would begin by identifying the risk factor(s) involved, which might be the loss itself, or the random variables driving it (e.g., seismic activity, outbreaks of disease, mortality experience, etc.). Where the loss is to be modelled conditional on risk factors, we also need to specify the nature of the relationship between the loss variable and the risk factor(s). We then need to model the risk factor(s), and these models can be quite sophisticated. We may also have to deal with further complications such as embedded options, unobserved variables, and so forth. We also have to estimate (and/or calibrate) any parameters. Depending on the context, we might be more interested in the central mass of the loss distribution (e.g., if we are concerned with expected claims, etc.) or the tails (e.g., if we are working with exceedances, extremes, etc.), and this will also influence the distribution(s) chosen. The resulting model might then be parametric, nonparametric, or some combination (i.e., semiparametric), but the actual calculations are more likely than not to involve some form of stochastic simulation.[40]

5.2. Modelling multiple loss functions

A natural extension is where we wish to obtain the distribution of some aggregate loss from the loss distributions of constituent positions. We can sometimes handle this type of problem using a multivariate version of the univariate approach just mentioned, going through much the same steps but

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allow actuaries a much more flexible approach to probability of ruin than is possible using classical actuarial theory, because it no longer requires them to make the old assumptions (e.g., of compound Poisson processes, independence, etc.) that were made in the past to achieve tractable closed-from solutions. Instead, we can make any assumptions we like about loss processes and if necessary rely on simulation methods to give us numerical answers.

[40] Some illustrative examples are: Klugman and Parsa (1999) and Bolancé *et alia* (2003), who model insurance losses; O'Connor *et alia* (1999), who model defaults in corporate lending portfolios; Cebrián *et alia* (2003), who model large insurance claims using a generalized Pareto model; and Canabarro *et alia* (2000), Clark (2002), Woo (2002) and Daneshvaran and Morden (2004), who model natural catastrophes.

with multivariate rather than univariate models. However, such an approach is often restrictive (e.g., because it forces marginal distributions to be the same, and because it only allows a limited range of dependence between the different loss variables). It is therefore generally better to model multivariate losses using copulas, and this would involve the following two-stage modelling strategy. In the first stage, we model the various individual loss distributions in exactly the same way as before (i.e., we obtain their marginal distributions). The second stage then involves the copula analysis: we select and fit a copula function to represent the dependence structure between the different loss variables. This function provides a way of representing the multivariate distribution which takes as its arguments the marginal distribution functions of our individual loss variables. We then input the fitted marginals to the fitted copula to obtain the aggregate loss distribution that we are seeking,[41] and we estimate our desired risk measures from the estimated aggregate loss distribution.

5.3. Scenario analyses

Scenario analyses are very commonly used for insurance/actuarial problems. To construct a scenario, we first specify the portfolio we are concerned about (e.g., property, casualty, life, country, type of product, etc.). We then identify the risk factors involved, and, in doing so, it often helps to distinguish between general economic factors and underwriting factors. The former refer to macroeconomic conditions, fluctuating interest rates, equity indices, inflation, FX rates, changes in insurance market conditions, etc. The latter refer to factors that might trigger large claims, and are often (but not always) specific to product lines: these might be earthquakes, windstorms, cyclones, hurricanes, fires, environmental pollution, asbestos, tobacco, motorway pile ups, natural catastrophes, plane crashes, infectious diseases, war, terrorism, pollution, changes in mortality or morbidity trends, changes in legal opinions, regulatory changes, etc.

[41] Some nice examples of copula modelling in insurance are given in Frees and Valdez (1998) and Klugman and Parsa (1999): both analyze cases where the marginal distributions for losses and allocated loss adjustment expenses (ALAE) are modelled using Pareto distributions, and their joint dependence is modelled using Archimedean copulas.

Underwriting risk factors might be specific to particular portfolios or product lines, but there are connections across portfolios to be considered as well, and identifying common risk factors can sometimes be difficult.

Once risk factors are identified, we then have to model them. Often, modelling boils down to specifying some kind of loss distribution that would be calibrated using past data and/or expert judgement. It is also sometimes possible to assume that loss events are independent, which greatly simplifies the modelling process. However, we also have to bear in mind that some types of event (e.g., changes in liability) have long-term effects, so assumptions of independence are often not appropriate: this leads us into methods of handling temporal dependence (e.g., comonotonic approaches, etc.). We might also be concerned about scenarios affecting investments, credit risk exposures (e.g., in relation to reinsurers), and so on. The latter are becoming increasingly important as ART practices spread, and raise difficult and complex issues relating to default probabilities (including default probability transition matrices), recovery rates, the modelling of creditworthiness, the impact of credit enhancement features, and so forth. Once we have modelled our risk factors, we specify our exposures to them, and then aggregate to form some sense of the overall impact of the scenarios considered. It goes without saying that we should try to consider all relevant risk factors, take account of interdependencies across portfolios, and also take account of the impact of risk management strategies.

Various guidelines have been suggested to assist the process of scenario generation. It is important to ensure that scenarios are plausible, consistent, and do not violate no-arbitrage conditions. Scenarios should also embody plausible assumptions about the sensitivities of positions to underlying risk factors. Modellers need to take account of plausible interactions between market, credit and liquidity risks, and also be on their for potential pitfalls (such as the failing to consider losses that occur when risk factors only move a little, or not at all, e.g., as would be the case on straddle positions). It is also important for scenarios to respect the principle of parsimony and avoid unnecessary complexity. Key assumptions should also be highlighted and subjected to critical scrutiny, and modellers should have some idea of the sensitivity of their results to such assumptions. Finally, it is very important that modellers are able to articulate the basic 'stories' embodied in their scenarios and successfully communicate these to non-specialist audiences (e.g., senior management).[42]

Scenario analyses can come in a variety of forms and involve any number of scenarios. In its traditional form, an institution would construct a limited number of deterministic scenarios that typically involved asking 'what if?' questions and then trying to work through their likely consequences. Scenarios like these can be on historical experience or on alternate history (i.e., asking what might have been?), but can also be based on more 'mechanical' possibilities (e.g., what happens if the stock market falls by x amount, interest rates rise by y amount, etc.).[43] However, most modern scenario analyses are stochastic, and involve large numbers of simulation trials.[44] Risk measures are then very easy to determine: each trial produces a loss outcome that can be

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[42] For more on some of the guiding principles behind scenario generation – and a sense of the diversity of practice and intricacies involved – see, e.g., Breuer and Krenn (1999), Smith and Riley (1999), Reynolds (2001), Altschull and Robbins (2003), Longley-Cook and Kehrberg (2003), and Dowd (2005a, ch. 13).

[43] We can also get more elaborate forms of mechanical scenarios analysis, sometimes known as factor push or maximum loss optimization: we identify the risk factors, shock them all (e.g., by a couple of standard deviations), and then have an algorithm mechanically search through these scenarios to identify the worst one. These methods are often used for estimating the risks of derivatives positions, and some well-known examples are the quadratic programming method suggested by Wilson (1994), and the delta-gamma methods suggested by Rouvinez (1997). These 'mechanical' methods are reviewed in Dowd (2005b, chapter 13.3).

[44] Stochastic simulation exercises can sometimes be extremely time-intensive, and in such cases we would also have to think carefully about ways of economizing on calculation time. For example, we might resort to variance reduction methods (such as antithetics, control variates, importance sampling, stratified sampling, etc. We can also resort to methods such as principal components analysis. A good example of such a method is the 'scenario simulation' approach of Jamshidian and Zhu (1997): this is a computationally highly efficient approach that involves simulating the principal components for a set of risk factors.

regarded as having the same probability as any other loss outcome; obtaining an estimate of one's preferred risk measure from the simulated loss outcomes is then trivial. Stochastic scenario analyses have been applied to a great variety of insurance problems and can sometimes be very sophisticated.

The following applications and associated references give some indication of what stochastic models involve and their diverse range of possible applications:

- Asset-return modelling (Wilkie (1984), Exley *et alia* (2000), Blake *et alia* (2001), Hibbert *et alia* (2001), Bayliffe and Pauling (2003), and Longley-Cook and Kehrberg (2003)).
- Catastrophe modelling (Clark (2002), Woo (2002), Canabarro *et alia* (2000)), Daneshvaran and Morden (2004)).
- Mortality modelling (Olivieri (2001), Czernicki *et alia* (2003), Cairns *et alia* (2005), Dowd *et alia* (2005)).
- Reserving and capital (Li and Tsai (2000), Hürlimann (2001), Turnbull (2002), Czernicki *et alia* (2003), Hibbert and Turnbull (2003), Watson Wyatt (2003, 2004), Moody's (2004)).
- Various miscellaneous insurance applications, such as modelling asset-liability management (Jagger and Mehta (1997), Swiss Re (2000), Exley *et alia* (2000), Babbel (2001)), embedded options (Jørgensen (2001)), with-profit guarantees (Hibbert and Turnbull (2003)), the impact of policyholder behavior on insurance company positions (Altschull and Robbins (2003)) and insurance company defaults (Moody's (2004), Ekström (2005)).

We emphasise that this list is merely illustrative, but even so it does give some indication of the state of the art.

6. FURTHER ISSUES

Our discussion has focused on three key themes: the type of risk measure, the estimation method, and the type of insurance application. Inevitably, this focus means that there are other (often important) issues that we have barely touched on. These include:

- *Capital allocation and risk budgeting*: Typically, these involve breaking down the aggregate risk measure into its component risks (i.e., determining the extent to which each business unit contributes to the overall risk measure) so that the firm can then allocate capital to 'cover' these component risks. Allocating capital 'correctly' is an important issue not just because we want to avoid risk 'black holes' and ensure that firms are adequately capitalised at the firmwide level, but also because it is necessary if they are to price products properly and avoid inadvertent cross-subsidisation. However, capital allocation is a very difficult subject, and closed-form solutions for component risks are known only for special cases, the best known of which is where risk factors are elliptically distributed (see Garman (1997), Wang (2002), and Valdez and Chernih (2003)). Capital allocation is also further complicated by overhead allocation issues[45] and by the fact that it is sometimes necessary to distinguish between the capital allocation and the risk measure on which the allocation is based (Goovaerts *et alia* (2003b)).[46]
- *Risk-expected return analysis and performance evaluation*: In practice, we might want to identify the risk-expected return trade-off faced by the firm at

[45] For example, one of the best known contributions to this subject in insurance is the capital allocation model of Myers and Read (2001): this model satisfies varies 'nice' properties and has had a major influence on the literature. However, Gründl and Schmeiser (2005) point out that this model is open to a number of problems – of which the central one is the difficulty of allocating equity capital to different business lines, which is a form of overhead allocation problem – and typically leads to incorrect decision-making. Thus, capital allocation is much more difficult than it looks at first sight.

[46] In practice, firms tend to resort to ad-hoc rules of thumb along the lines discussed at length in, e.g., Matten (2000). For more on capital allocation and risk budgeting, see also Panjer (2002) Fischer (2003), Tsanakas (2004) or Dowd (2005a, ch. 13).

an aggregate level, and we want systems of performance evaluation that correctly allow for the risks taken as well as profits earned by individual managers. Achieving these objectives is very difficult, not just because they raise issues of risk aggregation/disaggregation, but also because they raise organizational issues (e.g., moral hazard) and the classic risk-expected return decision rules (e.g., Sharpe ratios) are not reliable outside the restrictive world of elliptical distributions. It is therefore necessary to think in terms of risk-expected return analysis outside the familiar but limiting confines of ellipticality (and some ways of doing so are suggested by Acerbi (2004, pp. 186-199) and Dowd (2005b)). However, there has also been considerable progress made with these types of problem in an insurance setting, where a number of recent studies have looked at risk measures in optimal insurance, resinsurance and hedging contexts (see, e.g., Young (1999), Gajek and Zagrodny (2004), Kaluszka (2005) and Korn (2005)).

- *Long-run and strategic issues*: The estimation and uses of measures in long-run and strategic contexts, where we might be concerned with the impacts of alternative asset allocation strategies (e.g., Blake *et alia* (2001)), the effects of long-term trends (e.g., Dowd *et alia* (2004), Gilles *et alia* (2003) or the value of real options (e.g., such as franchise options; see Panning (1999)).
- *Model evaluation*: In the financial risk area the subject of model evaluation (or backtesting) has received a considerable amount of attention. Most of these backtests focus on the frequency of losses exceeding a VaR (see, e.g., Kupiec (1995), Christoffersen (1998)), but more sophisticated backtests compare sets of complete density forecasts against subsequently realized outcomes (e.g., Berkowitz (2001). These methods all require matching sets of forecasts and realized outcomes, and are therefore only practically feasible for models with short forecast horizons where such track records can be accumulated. However, the subject has received little attention in the actuarial field, and this is unfortunate as evaluating insurance risk models is likely to be even more difficult than evaluating the risk models commonly used in capital market institutions. The best advice we can offer insurance

practitioners is to use some of the backtests developed in the financial risk area, where possible. Beyond that, they need to think innovatively and above all take the problem of evaluation seriously: they might identify key assumptions in their models and test or critique these as best they can; they might bootstrap from their models and see if bootstrapped outcomes (e.g., such as long-run returns or mortality rates) are consistent with target outcomes; they might use simulation models to estimate shortfall probabilities and check if the estimated shortfall probabilities are acceptable; and so forth.

- *Enterprise-wide risk management (ERM)*: Risk measurement systems also have an important role to play in ERM systems, which seek to counteract 'silo mentalities' in risk management and manage risks across the firm on a holistic, consistent and proactive basis. [47] ERM systems require reliable risk measures, and also highlight the issue of how to aggregate risks across heterogeneous positions in a consistent and intellectually defensible manner. There have also been major ERM initiatives in recent years, one by the CAS in the United States and another being the launch in 2005 of the ERM Institute International led by Shaun Wang, both of which seek to promote ERM education in the insurance industry.
- *Regulatory risk measures*: There is a large literature on the regulatory uses of financial risk measures. The best known instance of these is the use of VaR measures to determine regulatory capital requirements in the Basel capital adequacy regime (see e.g., Basel Committee (2003)): this has been heavily criticized on a number of grounds (see, e.g., Danielsson *et alia* (2001), Danielsson (2002)). There have also been many other initiatives by financial regulators involving the regulatory uses of financial risk measures: these include the EU's Solvency II initiative, the UK Financial Services

[47] For more on ERM and its applications in insurance, see Tillinghast Towers Perrin (2000, 2001) and Wang (2002). Some recent surveys on the practice of ERM in the insurance industry are given by CAS (2001), Tillinghast-Towers Perrin (2002) and PWC (2004). Mikes (2005) gives a good account of some of the practical problems that can arise with the implementation of ERM systems.

Authority's capital adequacy regime for insurers (e.g., FSA (2002, 2005)), the use of risk-based capital requirements by the NAIC and by other regulators (e.g., NAIC (1995)), and the regulatory use of early warning systems to identify institutions that are likely to get into difficulties (see, e.g. Gunther and Moore (2002), Pottier and Sommer (2002)).[48]

7. CONCLUSION

The subject of financial risk measurement has come a long way since the appearance of VaR in the early 1990s. In retrospect, it is clear that VaR was much overrated, and is now discredited as a 'respectable' risk measure – despite the ostrich-like reluctance of many of its adherents to face up to this fact. Risk measurement has moved on, and we now have many 'respectable' risk measures to choose from: these include coherent risk measures, spectral risk measures, distortion risk measures, and many others. Indeed, in some ways, we now have *too many* risk measures available to us, and there are (usually) no easy ways to determine which might be best: the most appropriate risk measure depends on the assumptions we make (e.g., whether we are prepared to 'buy into' risk aversion theory, whether we prefer to work with distortion functions, etc.), and would appear also to be sometimes context-dependent. Any search for a single 'best' risk measure – one that is best in all conceivable circumstances – would therefore appear to be futile, and practitioners should be pragmatic. *De gustibus non disputandum est.*

Estimating any of these risk measures is straightforward: if we can estimate quantiles then we can easily estimate any QBRMs. This implies that upgrading from a VaR to a more 'respectable' risk measure is easy: we just add a few more lines of code to our program. And, as for estimation method, our

[48] We also ignore the related issues raised by non-regulatory rating systems. For more on these, see, e.g., Crouhy *et alia* (2001b) or Krahnen and Weber (2001).

preferred approach will almost always be some form of stochastic simulation: most insurance risk problems are complex, and these complexities can (in general) only be handled using stochastic methods.

In short, our main advice to insurance practitioners is that they should usualy avoid the VaR as a risk measure and choose risk measures that appear to be 'respectable' and appropriate to the problem at hand; they should also face up to the complexity of insurance risk problems and think in terms of stochastic methods as their preferred estimation vehicles.

Forecasting the future is of course a very uncertain business, but it seems to us that there are certain trends in risk measurement that are very likely to continue over the near future:

- We already have more than enough risk measures to choose from, and the business of producing new risk measures would appear to generating rapidly diminishing returns. We would therefore anticipate that significant new developments in this area will become harder to achieve and less frequent.
- Estimation methods will continue to improve: new estimation methods are being developed in other fields, and there is a process of intellectual arbitrage whereby methods developed fields such as engineering, physics, and statistics gradually make their way over to financial risk management or actuarial science, and so lead to improvements in the estimation and management of financial risks. We have seen this with extreme-value theory and copulas, and we would anticipate further developments of a similar nature. However, our sense is that this process is also slowing down.
- We perceive that practitioners are gradually becoming more aware of the importance of parameter and model risk, and we expect this trend to continue. From a practical point of view, it is much better to have a rough and ready estimate of a risk measure and be aware of its limitations, than have a 'fancy' estimate and be unaware of its weaknesses. Indeed, we would argue that any estimated risk measure reported on its own is close

to meaningless without some indicator of how precise the estimate might be. Fortunately, practitioners are becoming more aware of these issues, and we would anticipate that a time will eventually come when precision metrics will be reported as a matter of course.

- There is also the troublesome subject of model evaluation. This is difficult enough in the simpler contexts of, say, the 1-day VaR models used by securities firms, and is generally much more difficult for the risk models appropriate for insurance companies (because of their greater complexity, longer horizons, etc.). Recent disasters such as Equitable Life (where the firm failed to value the options it had written; see Blake (2001)) only underline this point: the failure to evaluate models properly is one of the key weaknesses of modern actuarial practice. We therefore anticipate that this subject will receive increasing attention from actuaries, and it is especially important that work be done on the evaluation of longer-horizon risk forecasting models.

Finally, it is also helpful to see developments in risk measurement in their broader cross-disciplinary context. For a very long time, actuaries have been accustomed to think of themselves of 'the' risk experts. Generation after generation of actuaries took this as given, and for many years there was no-one to challenge it. Then in the last decade or two the upstart discipline of 'financial risk management' emerged out of nowhere, as it were, and the new breed of financial risk managers started laying claim to much of the territory that actuaries had traditionally considered as their own (e.g., over the modelling of long-term asset returns). The stage was set for a classic turf war, and the FRM profession had the advantage that it had a flagship, the VaR, that took center stage: VaR was the flavor, not just of the month, but of the entire decade, and everyone wanted a 'VaR model'. The actuarial profession was then criticized because insurance companies were generally well behind capital markets institutions in their risk management practices, and because spectacular disasters (such as Equitable Life in the UK) highlighted the limitations of the assumptions on which many actuarial projections had been based. And yet the

‘VaR revolution’ itself became unstuck, and some of the standard risk measures used by actuaries for many years (such as ES-type measures and the outcomes of stress tests) turned out to be very respectable when viewed from the perspective of recent risk measure theory. Those actuaries (and others) who were skeptical of the siren calls of the VaR had been right all along. Furthermore, the state of risk management understanding reflected in actuarial journals is in many respects well ahead of that reflected in FRM journals. Indeed, the only area where we can see that actuaries are visibly behind their FRM counterparts is in terms of risk model evaluation. On the other hand, for their part, most FRM practitioners are quite ignorant of the actuarial literature – so much so, in fact, that we would argue that the failure of FRM practitioners to acknowledge developments in actuarial science is little less than a professional disgrace.

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