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Lorenz Hartmann and David Kelsey

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Location Invariance and Games with Ambiguity

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Samantha Stapleford-Allen
Centre for Decision Research and Experimental Economics
School of Economics
University of Nottingham
University Park
Nottingham
NG7 2RD
Tel: +44 (0)115 74 86214
Samantha.Stapleford-Allen@nottingham.ac.uk

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Location Invariance and Games with Ambiguity*

Lorenz Hartmann† David Kelsey‡

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Abstract

This paper proposes that the ambiguity reflected by a set of priors remains unchanged when the set is translated within the probability simplex, i.e. ambiguity is location invariant. This unifies and generalises numerous influential definitions of ambiguity in the literature.

Location invariance is applied to normal form games where players perceive strategic ambiguity. The set of translations of a given set of priors is shown to be isomorphic to the probability simplex. Thus considering mixtures of translations has a convexifying effect similar to considering mixed strategies in the absence of ambiguity. This leads to the proof of equilibrium existence in complete generality using a fixed point theorem. We illustrate the modelling capabilities of our solution concept and demonstrate how our model can intuitively describe strategic interaction under ambiguity.

Keywords: Ambiguity, Multiple Priors, Translations, Games, Equilibrium Existence.

JEL Codes: C72, D81,

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†University of Basel, Faculty of Business and Economics, Peter Merian-Weg 6, 4002 Basel, Switzerland, Email: lorenz.hartmann@unibas.ch.

‡Department of Industrial Economics, Nottingham University Business School, Wollaton Road, Nottingham, NG8 1BB, UK. Email David.Kelsey@nottingham.ac.uk.

1 Introduction

Contemporary social science issues, such as environmental risks, the implications of new technology (e.g. artificial intelligence), and threats from terrorism or rogue states, frequently involve uncertainty. Assigning probabilities to potential outcomes is often challenging due to the intrinsic difficulty of predicting the behaviour of other people. Uncertainty regarding probabilities is commonly referred to as ambiguity.

This paper presents a theory on how ambiguity impacts interactions among a group of agents. The strategic interactions between individuals are represented as normal form games. Ambiguity is modelled using α-maxmin expected utility preferences (Marinacci, 2002).[1]

The paper has two main parts. In the first part we present a new definition of ambiguity based on the intuition that the ambiguity reflected by a set of priors does not depend on where it is located in the space of probabilities, i.e. ambiguity is *location invariant*. Put differently, the ambiguity reflected by a set of priors is not affected when translated within the probability simplex, see the below example for a demonstration of this concept. We justify our approach behaviourally by comparing location invariance to existing definitions of ambiguity from the literature. In particular, we analyse location invariance in contexts such as bid-ask spreads of assets (Dow and Werlang, 1992) and compare it to ambiguity indexes (Baillon et al., 2021). This links our theory of ambiguity to behavioural properties.

In the second part we apply our approach to strategic ambiguity in games. Location invariance enables us to model a situation where a player perceives a fixed level of ambiguity about the opponent's action regardless of which strategy the opponent plays. We define equilibrium and provide a general existence result. Our proof is novel and does not require preferences to be convex. This is achieved because we discover a novel isomorphism between the set of translations of a set of priors and the probability simplex. Thus translations can convexify a game analogous to the way mixed strategies convexify games with finite strategy sets. This enables us to demonstrate existence of equilibrium using a fixed point theorem.

When beliefs of players are unambiguous, our solution concept coincides with Nash equilibrium. Our model is thus a generalization of Nash equilibrium to strategic ambiguity. We are able to accommodate a variety of ambiguity-attitudes including both ambiguity-seeking

[1] There is an earlier literature on games which models strategic ambiguity using Choquet Expected Utility preferences, see Dow and Werlang (1994), Eichberger and Kelsey (2000) and Marinacci (2000).

and ambiguity-averse behaviour, allowing rich modelling capabilities.

Motivating example

This example aims to intuitively explain the concept of location invariance of ambiguity. An urn containing 100 balls, which are either white or black, is considered in two scenarios. In the first scenario, the decision maker (DM) is informed that at least 20 balls are white and at least 40 balls are black. In the second scenario, the DM is informed that at least 10 balls are white and at least 50 balls are black. In both scenarios, the DM faces ambiguity regarding the colour of the ball. We argue that the level of ambiguity is the same despite the different information provided. To illustrate, imagine the balls are numbered from 1 to 100. In the first scenario, the balls numbered 1 to 20 are white, the balls numbered 21 to 60 are black, and the balls numbered 61 to 100 are of unknown colour. In the second scenario, the balls numbered 1 to 10 are white, the balls numbered 11 to 60 are black, and balls numbered 61 to 100 are of unknown colour. In both cases, there are 40 balls with an unknown colour. Recalling that ambiguity means uncertainty about probabilities leads us to the conclusion that the DM faces the same amount of ambiguity in both scenarios.

An alternative way to reason is to observe that the probability of selecting a white ball is known to be between $\frac{1}{5}$ and $\frac{3}{5}$ in the first scenario and between $\frac{1}{10}$ and $\frac{1}{2}$ in the second scenario. The length of these probability ranges is the same across scenarios ($= \frac{2}{5}$). As this range is an intuitive measure of the "degree" of ambiguity that the DM faces, the scenarios reflect the same level of ambiguity. The ranges are "translations" of each other. Our concept of location invariance of ambiguity is based on this idea.

Organisation of the paper

In the next section we describe our framework and definitions. In Section 3 we introduce location invariance of ambiguity. We highlight the intuitive appeal and show in Section 4 that our approach unifies many suggested measures from the literature. For instance, we study the bid-ask spread and show that it is location invariant. In Section 5 we derive the isomorphism between the set of probability distributions and the set of translations of a prior set. Section 6 applies this to normal form games and discussed in detail a concrete example to illustrates our model. Section 7 discussed some applications and Section 8 concludes. The appendix contains the proofs of all results which are not proved in the text.

2 Preliminaries

This section introduces α-MEU preferences, following which we define translations of sets of priors.

2.1 α-MEU Preferences

We consider a finite state space S. Let $\Delta(S)$ denote the set of all probability distributions over S and 2^S the set of all events, i.e. subsets of S. An *act* is a function $a : S \to \mathbb{R}$. We shall interpret the pay-offs of acts to be monetary values. The extension to a more general consequence space is straightforward. Let $A(S)$ denote the set of all acts. Preferences $\succcurlyeq$ are modelled as a binary relation over $A(S)$. Let $\mathcal{C} \subseteq \Delta(S)$ be a set of priors, i.e. a non-empty, closed and convex subset of $\Delta(S)$.

Definition 2.1 (α-MEU) *The binary relation $\succcurlyeq$ is a multiple priors (or α-MEU) preference relation if there exists a set of priors $\mathcal{C}$, a utility function $u : \mathbb{R} \to \mathbb{R}$ and $\alpha \in [0, 1]$ such that for all $a, b \in A(S)$:*

$$a \succcurlyeq b \Leftrightarrow \alpha \min_{p \in \mathcal{C}} \mathbf{E}_p u(a) + (1 - \alpha) \max_{p \in \mathcal{C}} \mathbf{E}_p u(a) \geqslant \alpha \min_{p \in \mathcal{C}} \mathbf{E}_p u(b) + (1 - \alpha) \max_{p \in \mathcal{C}} \mathbf{E}_p u(b),$$

where $\mathbf{E}_p u(a)$ denotes the expected utility of a with respect to the probability distribution p.

If $\alpha = 1$, preferences coincide with maxmin expected utility (MEU) preferences, Gilboa and Schmeidler (1989), and reflect ambiguity aversion. If $\alpha = 0$ preferences are maxmax expected utility and reflect ambiguity seeking. For intermediate values of α preferences are neither uniformly ambiguity-averse nor ambiguity seeking. This is compatible with experimental evidence, see Kilka and Weber (2001). For axiomatisations of α-MEU, see Ghirardato et al. (2004), Gul and Pesendorfer (2015), Hartmann (2023), and Klibanoff et al. (2022).[2]

[2]However note that the axioms of Ghirardato et al. (2004) are inconsistent when the state space is finite, see Eichberger et al. (2011).

We interpret $\mathcal{C}$ as the DM's beliefs. However these are ambiguous beliefs. The DM perceives all elements of $\mathcal{C}$ as a candidates for the true probability distribution. His/her reaction to this ambiguity is in part pessimistic in the sense that weight is given to the least favourable probability distribution in $\mathcal{C}$. It is also partially optimistic since weight is also given to the most favourable probability in $\mathcal{C}$. The parameter α measures the DM's ambiguity-attitude. Higher values of α correspond to more ambiguity-averse preferences.

2.2 Translations

Our concept of ambiguity builds on *translations of prior sets*, which are defined as follows.

Definition 2.2 (Translation of prior sets) *Two prior sets $\mathcal{C}, \mathcal{C}' \subseteq \Delta(S)$ are translations of each other if there exists a $t \in \mathbb{R}^S$ such that:*

- $\sum_{s \in S} t(s) = 0$.
- $\mathcal{C}' = \{p + t | p \in \mathcal{C}\}$, *where* $(p + t)(s) := p(s) + t(s)$ *for all* $s \in S$.

In such a case we write $\mathcal{C}' = \mathcal{C} + t$. Prior sets that are translations are identical accept for their location, see Figure 1 for an illustration. The following definition defines a notation for the set of translations of a prior set. These sets become relevant later as the ambiguity classes of prior sets.

Definition 2.3 (Set of translations) *For a prior set $\mathcal{C} \subseteq \Delta(S)$, we define*

$$[\mathcal{C}] := \{\mathcal{C} + t \subseteq \Delta(S) | \sum_{s \in S} t(s) = 0\}.$$

The set $[\mathcal{C}]$ contains all translations of $\mathcal{C}$ within the simplex $\Delta(S)$. It is an equivalence class. We illustrate the concept of translations and the equivalence classes just introduced in the following examples. All three examples are graphically illustrated in Figure 1.

Example 2.1 (Singleton prior sets) *For $p, p' \in \Delta(S)$, consider the two prior sets $\mathcal{C} = \{p\}$ and $\mathcal{C}' = \{p'\}$. Define $t(s) = p(s) - p'(s)$ for all $s \in S$. Then $\mathcal{C}' + t = \mathcal{C}$. Thus singleton prior sets are translations of each other. We have $[\{p\}] = [\{p'\}] = \Delta(S)$.*

Example 2.2 (Closed neighborhoods) *For some probability* $q \in \Delta(S)$ *and* $\delta \in [0,1]$ *consider the set* $\mathcal{D}_q = \{p \in \Delta(S) | p(E) \geq (1-\delta)q(E), \forall E \in 2^S\}$. *For a fixed* δ *and any* $q, q', \in \Delta(S)$, *the sets* $\mathcal{D}_q$ *and* $\mathcal{D}_{q'}$ *are translations of one another. To see this, define* $t(s) = q(s) - q'(s)$ *for all* $s \in S$. *Then* $\mathcal{D}_{q'} + t = \mathcal{D}_q$. *Thus each* δ *determines a class of translations.*[3] *We have* $[\mathcal{D}_q] = [\mathcal{D}_{q'}] = \{\mathcal{D}_r | r \in \Delta(S)\}$.

Example 2.3 (Two-color urn) *Consider our motivating urn example from the introduction. In scenario 1, the information of the DM is reflected by the prior set* $\mathcal{C} = \{p(W) \in [\frac{1}{5}, \frac{3}{5}]\}$, *where* $p(W)$ *denotes the probability of the drawn ball being white. That is,* $\mathcal{C}$ *conforms to the set of probability distributions that the DM knows are possible. In scenario 2, the information of the DM is reflected by the prior set* $\mathcal{C}' = \{p(W) \in [\frac{1}{10}, \frac{1}{2}]\}$. *Define* $t : \{W, B\} \to \mathbb{R}$ *by* $t(W) = -\frac{1}{10}$ *and* $t(B) = \frac{1}{10}$. *Then* $\mathcal{C}' = \mathcal{C} + t$. *Thus* $\mathcal{C}$ *and* $\mathcal{C}'$ *are translations. The set of translations of* $\mathcal{C}$ *coincides with* $[\mathcal{C}] = \{[q_1, q_2] \subset \Delta(S) | q_2(W) - q_2(W) = \frac{2}{5}\}$.

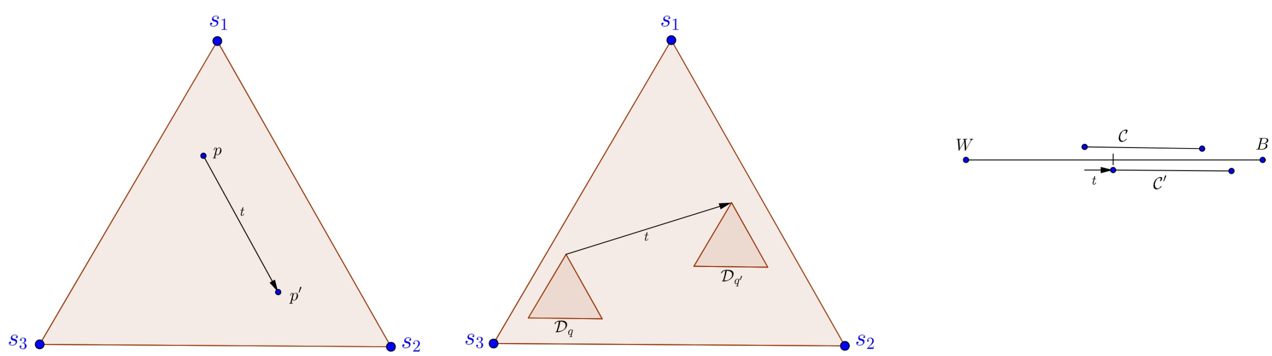

Figure 1: Illustration of the three examples. In Example 2.1 and 2.2 we consider $S = \{s_1, s_2, s_3\}$.

3 Location Invariance of Ambiguity

What does it mean for two prior sets to reflect the same ambiguity? This question is interesting in its own right. Moreover we believe that it is important for the study of ambiguity in games. We may wish to consider a situation where a player perceives his/her opponents' behaviour to be ambiguous. However we do not wish to specify which strategies the opponents will play before we have determined equilibrium. Location invariance enables us to

[3] These sets of priors arise from neo-additive preferences, see Section 4.2.1.

specify that a player perceives ambiguity without restricting this ambiguity to a particular strategy combination.

We propose that ambiguity is *location invariant*, meaning that the exact "location" of the prior set within the set of probabilities over a state space is irrelevant for the level of ambiguity that it reflects. Put differently, two prior sets reflect the same ambiguity whenever they are translations of each other. We thus have the following definition.

Definition 3.1 (Location Invariance of Ambiguity) *Two prior sets reflect the same ambiguity if and only if they are translations.*

Definition 3.1 implies that the DM faces the same ambiguity in both scenarios of our motivating urn example, as we suggest is logical. More generally, Definition 3.1 implies that the set of translations of $\mathcal{C}$, $[\mathcal{C}]$, coincides with the set of prior sets that reflect the same ambiguity as $\mathcal{C}$. We therefore, from now onwards, refer to $[\mathcal{C}]$ as the *ambiguity class* of $\mathcal{C}$.

We believe Definition 3.1 to be intuitively appealing in general and further explain our motivation for this approach in the following subsection. In Section 4 we compare our approach to established definitions of ambiguity from the literature and show that it in line with and unifies them, hereby strengthening the case for our approach.

3.1 Why Ambiguity is Location Invariant

Two prior sets that are translations, i.e. that differ only in location, can reflect very different information or beliefs regarding the likelihood of events. We believe that this alone is no reason to suggest that they reflect different ambiguities. Indeed we suggest, through Definition 3.1, that the ambiguity that they reflect is the same.

To illustrate our reasoning, first consider two singleton prior sets as in Example 2.1. Clearly, they reflect the same ambiguity, namely none at all! The likelihood of events are (subjectively) known, thus there is no ambiguity in both cases. This holds independently of these prior sets possibly reflecting completely different beliefs regarding the likelihood of events. As illustrated in Example 2.1, two singleton prior sets differ only in "location". Thus ambiguity is location invariant for singleton prior sets.

We suggest that this logic generalizes to non-singleton prior sets. Indeed, translating a prior set to a different location within the probability simplex does not manipulate any other indicator of the set (e.g. structure, size, kinks, curvatures) that may be relevant for the ambiguity that it reflects. To illustrate this point further, consider the following function.

Definition 3.2 (The function $\delta_{\mathcal{C}}$) *Let $\mathcal{C} \subseteq \Delta(S)$ be a prior set. Define $\delta_{\mathcal{C}} : A(S) \to [0,1]$ by*

$$\delta_{\mathcal{C}}(a) = \begin{cases} \frac{\max_{p\in\mathcal{C}} \mathbf{E}_p a - \min_{p\in\mathcal{C}} \mathbf{E}_p a}{\max_{p\in\Delta(S)} \mathbf{E}_p a - \min_{p\in\Delta(S)} \mathbf{E}_p a}, & a \notin X_{\mathcal{C}}; \\ 0, & a \in X_{\mathcal{C}}; \end{cases} \tag{1}$$

where $X_{\mathcal{C}} = \{a \in A(S) : \max_{p\in\mathcal{C}} \mathbf{E}_p a = \min_{p\in\mathcal{C}} \mathbf{E}_p a\}$.

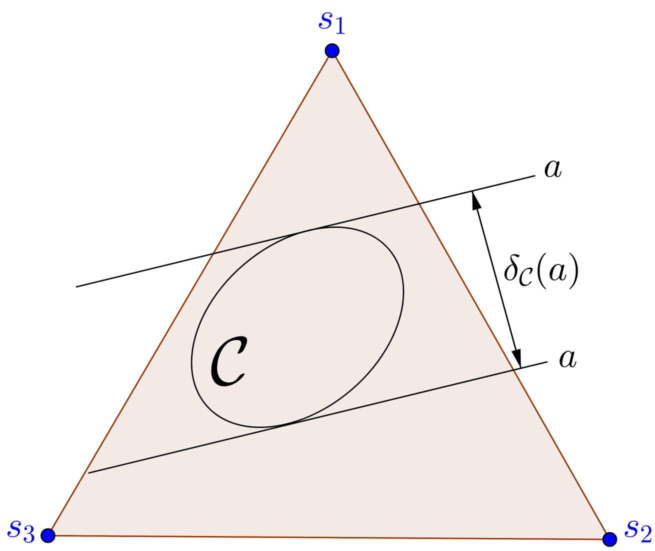

Figure 2: There are 3 states $S = \{s_1, s_2, s_3\}$. $\mathcal{C}$ is a prior set. The two parallel lines illustrate the best and worst case scenarios for the act $a \in A(S)$ given $\mathcal{C}$. The difference between these two lines is $\delta_{\mathcal{C}}(a) \in [0,1]$.

The δ-function is illustrated in Figure 2. For a prior set $\mathcal{C}$ and an act $a : S \to \mathbb{R}$, the value $\delta_{\mathcal{C}}(a)$ is the normalized difference between the highest and lowest expectation of the act a given $\mathcal{C}$. It is thus a natural indicator of ambiguity: the larger this number, the more ambiguity is reflected by $\mathcal{C}$ for the act a. If $\delta_{\mathcal{C}}(a) = 0$, there is no ambiguity relevant to the act a.[4]

Now consider two prior sets $\mathcal{C}$ and $\mathcal{C}'$ and assume that $\delta_{\mathcal{C}}(a) \neq \delta_{\mathcal{C}'}(a)$ for some $a \in A(S)$. The indicator δ suggests different levels of ambiguity regarding the act a, thus $\mathcal{C}$ and $\mathcal{C}'$ reflect different ambiguities. Thus for two prior sets to reflect the *same* ambiguity their δ-functions must coincide for all acts. As shown in Proposition 3.1, the δ-function is invariant under

[4]Note that, since prior sets are compact, there indeed exist priors in $\mathcal{C}$ that induce the (weakly) highest and lowest expectations of the act a. Thus $\delta_{\mathcal{C}}$ is well defined.

translations. Put differently, δ is the same for all elements of a given ambiguity class, thus arguing in favour of our location invariance.

Proposition 3.1 (Location invarience and $\delta_{\mathcal{C}}$) *If two prior sets $\mathcal{C}$ and $\mathcal{C}'$ are translations of one another then $\delta_{\mathcal{C}}(a) = \delta_{\mathcal{C}'}(a)$ for all $a \in A(S)$.*

3.2 Comparative Ambiguity

We now introduce, in light of Definition 3.1, the logical comparative notion for ambiguity. This is vital for comparative static exercises in ambiguity, in particular in our application to games in Section 6.

Definition 3.3 (Comparative Ambiguity for Prior Sets) *The prior set $\mathcal{C} \subseteq \Delta(S)$ reflects more ambiguity than another prior set $\mathcal{C}' \subseteq \Delta(S)$ whenever there exists a $t \in \mathbb{R}^S$ with $\sum_{s \in S} t(s) = 0$ such that*

$$\mathcal{C}' + t \subseteq \mathcal{C}.$$

This implies the following comparative notion for ambiguity classes.

Definition 3.4 (Comparative Notion for Ambiguity Classes) *Let $\mathcal{C}, \mathcal{C}' \subseteq \Delta(S)$ be prior sets and $[\mathcal{C}], [\mathcal{C}']$ their respective ambiguity classes. Then $[\mathcal{C}]$ reflects more ambiguity than $[\mathcal{C}']$ if and only if $\mathcal{C}$ reflects more ambiguity than $\mathcal{C}'$, i.e. there exists a $t \in \mathbb{R}^S$ with $\sum_{s \in S} t(s) = 0$ such that $\mathcal{C}' \subseteq \mathcal{C} + t$.*

We end this section with a sneak preview of how the concepts of location invariance, ambiguity classes and the just introduced comparative notion play a role in our application to games. Consider a normal form game Γ. The concept of an ambiguity class enables us to specify levels of ambiguity for the players in Γ without attaching it to a specific strategy combination. This allows us to demonstrate an equilibrium existence result for any such Γ and arbitrary levels of ambiguity and ambiguity attitudes for the players. The comparative notion of ambiguity allows comparative statics exercises in ambiguity.

4 Comparison with other Approaches to Ambiguity

In this section we compare our concept of location invariance of ambiguity with the previous literature. First we show that translations preserve the bid ask spread of Dow and Werlang (1992). Then we compare our measure of ambiguity with belief hedges as suggested by Baillon et al. (2021) and the ambiguity measure proposed for neo-additive preferences by Chateauneuf et al. (2007). We show that location invariance is in line with all these above approaches and thus unifies them, hereby strengthening the case for this approach.[5]

4.1 The Bid Ask Spread

Expected utility preferences are locally risk neutral at certainty (provided the utility function is continuously differentiable). This implies that for a given asset a, the individual will buy (resp. short-sell) the asset if the price is below (resp. above) its expected value. They will not trade only in the knife-edge case where the price is equal to its expected value. This prediction is not supported by actual behaviour as most people do not trade most financial assets. A possible explanation is transaction costs. However intuition suggests that even if transaction costs could be reduced to zero most people would still not trade most assets. Dow and Werlang (1992) have an alternative explanation based on ambiguity. They have shown that if individuals have MEU preferences there will be a bid-ask spread. This implies that for any given asset there will be an interval of prices in which an individual neither wishes to buy nor sell the asset. (This result has been extended by Chateauneuf and Ventura (2010).) The bid-ask spread naturally reflects the amount of ambiguity that the investor faces. It is the basis of many applications of ambiguity, see for instance, Mukerji and Tallon (2001) and Mukerji (2002).

In this subsection we show that for α-MEU preferences the bid-ask spread (or the natural counterpart which we call ask-bid spread) is location invariant for all assets. This links our theory of ambiguity to a behavioural property.

Consider an investor in a financial market. There are three options. The investor can buy a particular asset, short-sell the asset or do neither (status-quo). The investor has α-MEU

[5] Dominiak and Eichberger (2016), Eichberger and Kelsey (2014) and Marinacci (2000) also introduce comparative measures of ambiguity that are also special cases of Definition 3.1.

preferences, characterized by $\alpha, \mathcal{C}, u$, where we assume that $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing. The investor has initial wealth is ω. The asset is reflected by an act $a \in A(S)$, i.e. it results in the monetary payoff $a(s) \in \mathbb{R}$ in state $s \in S$. The price of the asset is denoted by $q \in \mathbb{R}$.

If the investor chooses the status-quo, she gets ω for sure, that is the evaluation of this option is $V(\omega) = u(\omega)$. If she decides to buy the asset, the evaluation is

$$V_q(buy) = \alpha \min_{p \in \mathcal{C}} E_p u(\omega + a - q) + (1 - \alpha) \max_{p \in \mathcal{C}} E_p u(\omega + a - q).$$

If she decides to short-sell the asset, the evaluation is

$$V_q(short - sell) = \alpha \min_{p \in \mathcal{C}} E_p u(\omega - a + q) + (1 - \alpha) \max_{p \in \mathcal{C}} E_p u(\omega - a + q).$$

Note that whereas $V_q(buy)$ is decreasing in q, $V_q(short - sell)$ is increasing in q.

Now consider the price $\overline{q}$ such that $V_q(buy) = u(\omega)$ as well as the price $\underline{q}$ such that $V_q(short - sell) = u(\omega)$. These are the reservation prices for buying and short-selling the asset.[6] If $\overline{q} \leq \underline{q}$, then for all $q \in [\overline{q}, \underline{q}]$ the investor weakly prefers the status-quo over both buying and short-selling, we call $[\overline{q}, \underline{q}]$ the *no-trade interval* of asset a. The length of this interval, $\underline{q} - \overline{q}$ is what we call the **bid-ask spread** of asset a. If $\underline{q} \leq \overline{q}$, then for all $q \in [\underline{q}, \overline{q}]$ the investor weakly prefers *both* buying and short-selling over the status-quo, we call $[\underline{q}, \overline{q}]$ the *both-trade interval* of asset a (note that the investor will not both buy and short-sell the asset). The length of this interval, $\overline{q} - \underline{q}$, is what we call the **ask-bid spread** of the asset a.

Every asset has a bid-ask or an ask-bid spread. The spreads are simultaneously non-empty if and only if they coincide, in which case they both coincide with a singleton q^*. In this case the investor buys the asset whenever $q > q^*$ and short-sells the asset whenever $q < q^*$. She is indifferent amongst all three options when $q = q^*$.

We first consider the special case of risk-neutrality, meaning that we can choose the utility function to be the identity. Under this assumption, $\alpha \geq \frac{1}{2}$ implies that the no-trade interval is non-empty for all assets and $\alpha \leq \frac{1}{2}$ implies that the both-trade interval is non-empty for all assets. This is in line with intuition since $\alpha \geq (\leq)\frac{1}{2}$ means a tendency towards ambiguity aversion (seeking).[7]

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[6]These values exist due to the preferences being continuous and monotonic. They are unique since u is strictly increasing.

[7]This result breaks down when risk-neutrality is dropped. For instance, $\alpha \geq \frac{1}{2}$ and a concave utility

Crucially, the following proposition shows that the spreads are invariant under translations. It also provides the general formula for the spreads.

Proposition 4.1 (Location invariance of spreads) *Let $\succsim_1, \succsim_2$ be risk-neutral α-MEU preferences characterized by $(\mathcal{C}_1, \alpha)$ and $(\mathcal{C}_2, \alpha)$, respectively. Then if $\mathcal{C}_1$ and $\mathcal{C}_2$ are translations, their bid-ask and ask-bid spreads coincide for all assets. The spread for asset a, given ambiguity class $[\mathcal{C}]$ and α, is*

$$|(2\alpha - 1)(\underline{p}_a - \overline{p}_a) \cdot a|,$$

where $\underline{p}_a \in \mathrm{argmin}_{p\in\mathcal{C}}\, \mathbf{E}_p a$ and $\bar{p}_a \in \mathrm{argmax}_{p\in\mathcal{C}}\, \mathbf{E}_p a$ (for an arbitrary prior set in $[\mathcal{C}]$) and $\cdot$ denotes the scalar product.

We now drop the assumption of risk-neutrality and merely assume that the utility function is continuously differentiable. Note that expected utility preferences are locally risk neutral at certainty. From the above proposition we can thus derive a local (in a neighborhood of certainty) corollary for the general case.

Corollary 4.1 *Let $\succsim_1, \succsim_2$ be α-MEU preferences characterized by $(\mathcal{C}_1, \alpha, u)$ and $(\mathcal{C}_2, \alpha, u)$, respectively. Assume that u is continuously differentiable. Then if $\mathcal{C}_1$ and $\mathcal{C}_2$ are translations, their local bid-ask and ask-bid spreads coincide for all assets.*

4.2 Belief Hedges

Baillon et al. (2021) introduce two ambiguity indexes: An ambiguity-aversion index b and an (ambiguity generated-) insensitivity index a. They highlight that their approach generalizes most ambiguity indices suggested in the literature, including the ones from Baillon et al. (2018). To introduce the indices some definitions from Baillon et al. (2021) are needed.

Definition 4.1 (Measurement design) *A measurement design $\mathcal{D}$ is a finite collection of events. $\{E_1, \ldots, E_n\}$ denotes the smallest nonempty intersections of events in $\mathcal{D}$, called design atoms. Let v denote the normalized event size, i.e. $v(E) = \frac{|E|}{n}$, where $|E|$ is the number of atoms in E.*

function may result in an empty no-trade interval.

Definition 4.2 (Level hedged) *A measurement design $\mathcal{D}$ is called l(evel)-hedged if each state $s \in S$ appears in exactly half of the events in $\mathcal{D}$. It is called v(ariation)-hedged if $\sum_{s\in E} v(E)$ is the same for all states $s \in S$. If $\mathcal{D}$ is both l-hedged and v-hedges, it is called a belief hedge.*

For a belief hedge, the design atoms form a partition of S. For a function $a : \mathcal{D} \to \mathbb{R}$, $\overline{a} = \frac{\sum_{E\in\mathcal{D}} a(E)}{|\mathcal{D}|}$ denotes the average of a. The indexes of Baillon et al. (2021) rely on the concept of probability matching. For each event $E \subseteq S$ and consequences x, y with $x \succ y$ there exists (by monotonicity) a unique number $m(E) \in [0, 1]$ such that $x_E y \sim x_{m(E)} y$.[8] We can now introduce the indices of Baillon et al. (2021).

Definition 4.3 (Ambiguity indexes of Baillon et al. (2021)) *If l-hedging holds then the index of ambiguity aversion is $b = 1 - 2\overline{m}$. If in addition v-hedging holds (i.e. $\mathcal{D}$ is a belief hedge) then the index of a(mbiguity generated) insensitivity is*

$$a = 1 - \frac{\mathrm{Cov}(m, v)}{\mathrm{Var}(v)}.$$

We refer to Baillon et al. (2021) for the motivation and discussion of these indices. The following result shows that these indices are unaffected by translations.

Proposition 4.2 (Location invariance and belief hedges) *Consider $\alpha \in [0, 1]$ and prior sets $\mathcal{C}_1, \mathcal{C}_2 \subseteq \Delta(S)$. Let $\succcurlyeq_i$ be the α-MEU preference relations represented by α and $\mathcal{C}_i$, for $i \in \{1, 2\}$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are translations, then $\succcurlyeq_1$ and $\succcurlyeq_2$ have the same ambiguity aversion and a-insensitivity index for any belief hedge $\mathcal{D}$.*

The reverse implication in Proposition 4.2 is not true, i.e. location invariance is finer than the indices from Baillon et al. (2021). There exist α-MEU preferences with the same indices but where none of their representations are translations. This is not surprising. The indices are one-dimensional and only depend on preferences over binary acts whereas there is more variability in multiple prior models. They depend on preferences over non-binary acts as well. The following example illustrates such a case.

[8] As usual, $x_E y$ denotes the act which gives consequence x on E y on E^c, and $x_p y$ denotes the lottery which gives x with probability p and y with probability $1 - p$.

Example 4.1 *Consider the state space $S = \{s_1, s_2, s_3\}$, the belief hedge $\mathcal{D} = \mathcal{P}(S)$, and let p_{unif} be the uniform distribution over S. Consider the following two prior sets:*

$$\mathcal{C}_1 = B_{\frac{1}{6}}(p_{unif}),$$
$$\mathcal{C}_2 = \{q \in \Delta(S) | q(s_i) \geq \frac{1}{6}, q(s_i, s_j) \geq \frac{1}{2}, i, j \in \{1, 2, 3\}, i \neq j\}.$$

The prior set $\mathcal{C}_1$ is the ball with radius $\frac{1}{6}$ and centre p_{unif}. The prior set $\mathcal{C}_2$ is the core of a convex capacity. Obviously they are not translations. However, it is immediate that

$$\min_{p\in\mathcal{C}_1} p(E) = \min_{p\in\mathcal{C}_2} p(E) = \begin{cases} \frac{1}{6}; |E| = 1, \\ \frac{1}{2}; |E| = 2; \end{cases}$$
$$\max_{p\in\mathcal{C}_1} p(E) = \max_{p\in\mathcal{C}_2} p(E) = \begin{cases} \frac{1}{2}; |E| = 1, \\ \frac{5}{6}; |E| = 2. \end{cases}$$

Let $\succcurlyeq_i$ be the α-MEU preference induced by $\alpha \in [0, 1]$ and $\mathcal{C}_i$, $i \in \{1, 2\}$. For all events E we have $m(E) = \alpha \min_{p\in\mathcal{C}} p(E) + (1 - \alpha) \max_{p\in\mathcal{C}} p(E)$, thus $m_1 = m_2$ and in particular $\overline{m}_1 = \overline{m}_2$. This implies that the indices coincide for $\succcurlyeq_1$ and $\succcurlyeq_2$.

4.2.1 Neo-Additive Preferences

Neo-additive preferences, (Chateauneuf et al. (2007)) (henceforth CEG) lie within the overlap of α-MEU and Choquet Expected Utility (CEU) preferences (Schmeidler (1989)). They are defined as follows.

Definition 4.4 (Neo-Additive Capacity) *Let $\alpha, \delta \in [0, 1]$. A* neo-additive-capacity *ν on S is a normalized set-function defined by $\nu(A) = \delta(1 - \alpha) + (1 - \delta)\pi(A)$, for $\emptyset \subsetneq A \subsetneq S$, where π is an additive probability distribution on S, i.e. $\pi \in \Delta(S)$. A preference relation $\succcurlyeq$ is* neo-additive *if it can be represented by a Choquet integral with respect to a neo-additive capacity.*

CEG interpret neo-additive capacities as describing a situation where the DM's "beliefs" are represented by the probability distribution π. However these are ambiguous beliefs, where ambiguity is captured by the parameter δ. The highest possible level of ambiguity corresponds to $\delta = 1$, while $\delta = 0$ corresponds to no ambiguity. The parameter α, as in our more general model, reflects the attitude towards this ambiguity.

CEG show that neo-additive preferences allow the following multiple prior representation. Let ν be a neo-additive capacity characterized by π, δ and α. Then for any act $a \in A(S)$,

$$W(a) = \alpha \min_{p \in \mathcal{B}} \mathbf{E}_p a + (1 - \alpha) \max_{p \in \mathcal{B}} \mathbf{E}_p a, \tag{2}$$

where $\mathcal{B} = \{p \in \Delta(S) | p(E) \geq (1 - \delta)\pi(E), \forall\ E \in 2^S\}$.

Neo-additive preferences are thus represented by a weighted average of the highest expected pay-off and the lowest expected pay-off of the priors in $\mathcal{B}$. The following proposition shows that our approach to ambiguity based on location invariance coincides precisely with the measure of ambiguity proposed by CEG.

Proposition 4.3 (Location invariance and neo-additive capacities) *Let ν and ν' be two neo-additive capacities characterized by π, δ, α and π', δ', α', respectively. Let $\mathcal{B}$ and $\mathcal{B}'$ be the respective prior sets from* (2). *Then $\delta = \delta'$ if and only if $\mathcal{B}$ and $\mathcal{B}'$ are translations.*

5 An Isomorphism from $\Delta(S)$ to $[\mathcal{C}]$

Building on the concept of location invariance from Section 2.2, we now introduce a result which is crucial for existence of equilibrium in Section 6. We show that any non-singleton ambiguity class $[\mathcal{C}]$ is isomorphic to the simplex $\Delta(S)$. Thus taking translations within an ambiguity class has a convexifying effect similar to taking mixed strategies in the absence of ambiguity. This enables us to prove existence of equilibrium using fixed point theorems.

5.1 Ambiguity Classes are Isomorphic to the Simplex

We first need the following support notion for prior sets.

Definition 5.1 (Support of a prior set) *Let $\mathcal{C} \subseteq \Delta(S)$ be a prior set. Define the support of $\mathcal{C}$ by* $\text{supp}(\mathcal{C}) = \bigcap_{p \in \mathcal{C}} \text{supp}(p)$.

We thus define the support of a set of priors to be those states which are in the support of all of the probability distributions in the set. This is a strong notion of support. It coincides with the set of states in which the DM "believes" in the sense that they receive positive decision-weight no matter which of the priors is the true distribution.[9]

To derive the isomorphism we first show that when a prior set $\mathcal{C}$ has a non-empty support, then for each $s \in S$ there exists a unique element $\mathcal{C}^s$ in $[\mathcal{C}]$ such that $\text{supp}(\mathcal{C}^s) = \{s\}$. Intuitively this prior set is in the "s-corner" of $\Delta(S)$.

Lemma 5.1 *If S is a finite state space and $\mathcal{C} \subseteq \Delta(S)$ is a prior set with* $\text{supp}(\mathcal{C}) \neq \emptyset$, *then for every state $s \in S$ there exists a unique prior set $\mathcal{C}^s \in [\mathcal{C}]$ such that* $\text{supp}(\mathcal{C}^s) = \{s\}$.

Now consider an arbitrary $p \in \Delta(S)$, i.e. a probability distribution over S. We construct $\mathcal{C}^p$ as the p-mix of the "corner" prior sets $\mathcal{C}^s$. That is for $s \in S$, $p(s)$ is the weight of $\mathcal{C}^s$ in the construction of $\mathcal{C}^p$, i.e. $\mathcal{C}^p$ is the p-convex combination of the sets $\mathcal{C}^s$. This is expressed formally in the following definition.

Definition 5.2 (The ψ-function) *Let $[\mathcal{C}]$ be an ambiguity class with supp$(\mathcal{C}) \neq \emptyset$. Define a function $\psi : \Delta(S) \to [\mathcal{C}]$ by*

$$\psi(p) = \mathcal{C}^p := \{q \in \Delta(S) | \exists q^s \in \mathcal{C}^s : q = \sum_{s \in S} p(s) q^s\}.$$

Thus $\mathcal{C}^p$ is the set of all pointwise p-mixtures of the elements of the corner prior sets $\mathcal{C}^s$. In Proposition B.1 in Appendix A we show in general that pointwise mixtures of sets that are translations of a given set, as in Definition 5.2, always result in a set that is again a translation of the given set. A direct consequence of that proposition is that the set $\mathcal{C}^p$ as defined above is indeed an element of $[\mathcal{C}]$. The function ψ is thus well-defined. The following proposition shows that ψ is an isomorphism.

Proposition 5.1 (Isomorphism) *Let $[\mathcal{C}]$ be an ambiguity class with supp$(\mathcal{C}) \neq \emptyset$. The function $\psi : \Delta(S) \to [\mathcal{C}]$, $p \mapsto \mathcal{C}^p$ is an isomorphism.*

Remark 5.1 *The requirement supp$(\mathcal{C}) \neq \emptyset$ cannot be dropped. Indeed supp$(\mathcal{C}) = \emptyset$ holds if and only if $[\mathcal{C}]$ only contains $\mathcal{C}$ itself as shown in Lemma B.1.*

[9]This support notion is the logical one for prior sets and is introduced in Ryan (2002), who refers to it as the *inner support*.

Note that when the prior set $\mathcal{C}$ is a singleton we have $[\mathcal{C}] = \Delta(S)$ and the isomorphism is the identity. Figure 3 illustrates the case $S = \{s_1, s_2, s_3\}$. The isomorphism maps the state s_i to the set $\mathcal{C}^{s_i}$. It maps the probability distribution $\frac{1}{2}s_1 + \frac{1}{2}s_3$ to the set $\mathcal{C}^{\frac{1}{2}s_1+\frac{1}{2}s_3}$.[10]

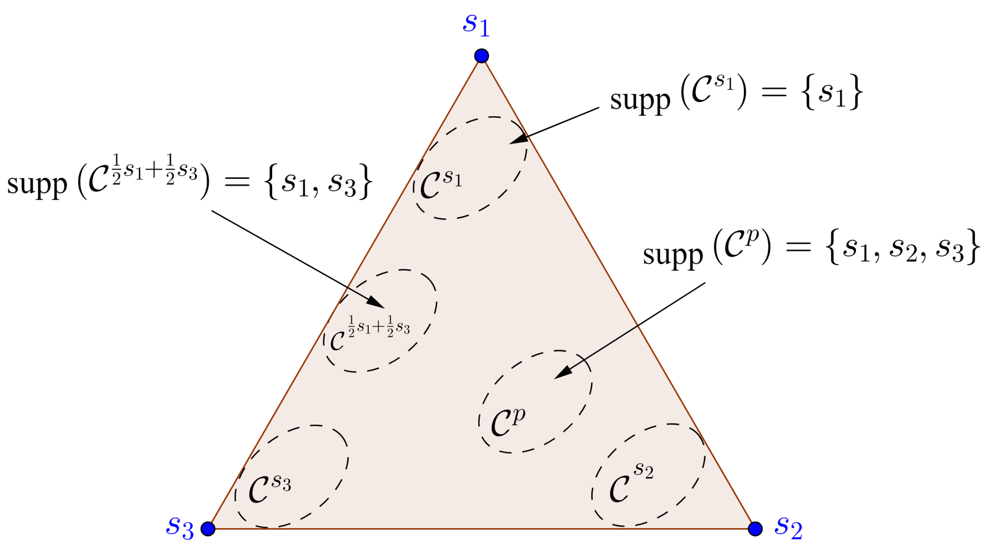

Figure 3: The sets in $[\mathcal{C}]$ and an illustration of the isomorphism.

5.2 Multiple Sources

In many applications there are multiple sources of uncertainty, in which case the state space is a Cartesian product, $S = S_1 \times ... \times S_M$. The previous results imply the existence of an isomorphism between an ambiguity class over S and the set of probability distributions over S. However it is often useful to consider beliefs which are independent over the components. In particular, if S is the set of strategies of one's opponents in a game it is usual to assume that they are independent. Our previous analysis needs to be adapted since not all translations preserve independence.[11] This can be achieved as follows.

Let $\Lambda(S)$ denote the set of all *independent* probability distributions over S, i.e.

$$\Lambda(S) = \{p \in \Delta(S) | \exists p_i \in \Delta(S_i), 1 \leq i \leq M; p(s_1, \ldots, s_M) = \prod_{i=1}^{M} p_i(s_i), \forall s = (s_1, \ldots, s_M) \in S\}.$$

Definition 5.3 (Set of independent priors) *A* set of independent priors *is a set of in-*

[10]Here, $\frac{1}{2}s_1 + \frac{1}{2}s_3$ denotes the probability distribution which results in s_1 and s_3 with probability $\frac{1}{2}$ each.

[11]Consider for instance the state space $S = S_1 \times S_2$ with $S_i = \{s_{i1}, s_{i2}\}$, $i = \{1, 2\}$, as well as the probability distribution characterized by $p(s_{11}, s_{21}) = 1$. Consider the translation characterized by $t(s_{11}, s_{21}) = -\frac{1}{2} = -t(s_{12}, s_{22})$ and 0 otherwise. Then for the prior set $\mathcal{C} = \{p\}$, the unique element of $\mathcal{C} + t = \{p + t\}$ cannot be written as an independent product of probabilities on S_1 and S_2.

dependent probability distributions induced by a prior set in each subspace, e.g. the prior sets $\mathcal{C}_i \in \Delta(S_i)$, $i \in \{1, \ldots, M\}$, induce the set of independent priors

$$\mathcal{C} = \{p \in \Lambda(S) | \exists p_i \in \mathcal{C}_i, 1 \leq i \leq M; p(s_1, \ldots, s_M) = \prod_{i=1}^{M} p_i(s_i), \ \forall s = (s_1, \ldots, s_M) \in S\}. \tag{3}$$

We denote this set by $\mathcal{C} = \times_{i=1}^{M} \mathcal{C}_i$.

Remark 5.2 *As defined $\mathcal{C}$ is not convex. However we could take the convex hull of $\mathcal{C}$. This would not change decisions since they only depend on the extremal points of $\mathcal{C}$.*

Next we shall define independent ambiguity classes. These contain sets of independent priors which represent the same ambiguity.

Definition 5.4 (Independent ambiguity class) *Let $\mathcal{C}_i$ be a set of priors over S_i, and let $[\mathcal{C}_i]$ denote its ambiguity class for $1 \leqslant i \leqslant M$. Denote by $\mathcal{C} = \times_{i=1}^{M} \mathcal{C}_i$ the set of independent priors as in equation* (3). *The resulting* independent ambiguity class $[\mathcal{C}]_I$ *is the family of sets of priors induced by the ambiguity classes $[\mathcal{C}_i]$, i.e.*

$$[\mathcal{C}]_I = \{\mathcal{C}' \subseteq \Lambda(S) | \exists \mathcal{C}_i' \in [\mathcal{C}_i] : \mathcal{C}' = \times_{i=1}^{M} \mathcal{C}_i'\}.$$

We write $[\mathcal{C}] = \times_{i=1}^{M} [\mathcal{C}_i]$.

Every element of $[\mathcal{C}]_I$ can be written as $\mathcal{C} = (\mathcal{C} + t_1) \times \ldots \times (\mathcal{C}_M + t_M)$, where t_i is a translation for $1 \leqslant i \leqslant M$. We can show that the independent ambiguity class $[\mathcal{C}]$ is isomorphic to the set of independent probability distributions over S.

Proposition 5.2 *Let $\mathcal{C} = \times_{i=1}^{M} \mathcal{C}_i \subseteq \Lambda(S)$ be an independent prior set and $[\mathcal{C}] = \times_{i=1}^{M} [\mathcal{C}_i]$ its independent ambiguity class. The function $\xi : \Lambda(S) \to [\mathcal{C}]_I$ defined by $\xi(p_1, \ldots, p_M) = \times_{i=1}^{M} \psi^i(p_i)$ is an isomorphism, where $\psi^i : \Delta(S_i) \to [\mathcal{C}_i]$ denotes the isomorphism from Definition 5.2.*

Proof. The fact that ψ^i is 1-1 for $1 \leqslant i \leqslant M$ implies that ξ is 1-1. Moreover ξ is onto since if $(p_1, \ldots, p_M) \in \Lambda$, then $(p_1, \ldots, p_M) = \xi\left(\left(\psi^1\right)^{-1}(p_1), \ldots, \left(\psi^M\right)^{-1}(p_M)\right)$. ■

In the next section we apply the concept of translation invariance to ambiguity in games. The isomorphism which we have discovered will enable us to prove existence of equilibrium using fixed point theorems.

6 Games with Multiple Priors

In this section we shall apply the concept of location invariance of ambiguity to behaviour in strategic interactions. We consider games where players perceive *strategic ambiguity* about others' strategy choice. For comparability with the existing literature we shall assume that beliefs are consistent in the sense that any two players have the same beliefs over the behaviour of a third party. Furthermore we assume that players believe that the other players act independently. These restrictions may be relaxed in applications if the context suggests it is desirable, see Section 8. Our use of ambiguity classes enable us to specify that a player perceives ambiguity about his/her opponents' behaviour without linking the ambiguity to a specific strategy combination. We formally define our equilibrium concept and prove existence. The isomorphism between ambiguity classes and the probability simplex enables us to prove existence using fixed point theorems.

6.1 Framework

We consider normal-form games $\Gamma = \langle \mathcal{H}, S_i, u_i : 1 \leq i \leq N \rangle$, where $\mathcal{H}$ is a set of N players, S_i and u_i denote respectively the strategy set and pay-off function of player i. The pure strategy sets S_i for each of the N players are finite, Σ_i denotes the set of probability distributions over S_i.[12] The sets of pure strategy combinations and distributions over them are denoted by $S = \times_{i=1}^{N} S_i$ and $\Sigma = \times_{i=1}^{N} \Sigma_i$. As usual, $S_{-i} = \times_{j \neq i} S_j$, denotes the set of pure strategies of i's opponents and $\Sigma_{-i} = \times_{j \neq i} \Sigma_j$ denotes the set of distributions over S_{-i}.[13] For player i, the function $u_i : \Sigma \to \mathbb{R}$ is the expected pay-off function.

When there are more than two players, strategic ambiguity is multi-dimensional. Thus we use the results from Section 5.2. Player i perceives ambiguity about the behaviour of player j, each reflected by a prior set $\mathcal{C}_i^j, j \neq i$. Let $\mathcal{C}_{-i}$ denote player i's overall belief about the behaviour of his/her opponents. In game theory it is usual to assume that a given player believes his/her opponents act independently. The following assumptions defines independence for games with ambiguity.

Assumption 6.1 (Independent Beliefs) *We say that individual i's beliefs about oppo-*

[12] See Remark 6.2 for why we do not call Σ_i the set of *mixed* strategies.

[13] Note that when there are more than two players, $\Sigma_i = \Delta(S_i)$ for all i but $\Sigma \subsetneq \Delta(S)$, because $\Delta(S)$ also contains correlated mixed strategies.

nents' behaviour are independent *if his/her overall belief is represented by the Cartesian product of his/her beliefs about individual opponent's behaviour,*[14] *i.e.* $\mathcal{C}_{-i} = \times_{j \neq i} \mathcal{C}_i^j \subseteq \Sigma_{-i}$.

Independence will be a maintained assumption throughout the analysis of games with ambiguity. The ambiguity-attitude of player i is represented by the parameter $\alpha_i \in [0, 1]$. Player i evaluates a strategy $s_i \in S_i$ by

$$V_i(s_i) = \alpha_i \min_{\sigma_{-i} \in \mathcal{C}_{-i}} u_i(s_i, \sigma_{-i}) + (1 - \alpha_i) \max_{\sigma_{-i} \in \mathcal{C}_{-i}} u_i(s_i, \sigma_{-i}). \tag{4}$$

The ambiguity-attitude parameter tells us how pessimistic or optimistic (s)he is towards strategic ambiguity. If α_i is small (large), the player is optimistic (pessimistic). Thus (s)he considers the worst and the best scenario over the beliefs and weights them according to his/her ambiguity-attitude.

The following definition says that any two players have the same beliefs about the behaviour of a third player. This is standard in game theory.

Assumption 6.2 (Consistent Beliefs) *We say that players have* consistent beliefs *if* $\mathcal{C}_i^k = \mathcal{C}_j^k$ *for all pairwise distinct* $i, j, k \in \{1, \ldots, N\}$.

Assumption 6.2 will be a maintained hypothesis throughout this section. We can thus drop the lower index and write $\mathcal{C}^k$ for the belief about the behaviour of player k. This allows the following definition.

Definition 6.1 (Overall Belief) *The overall belief of player* i *is* $\mathcal{C}_{-i} = \times_{j \neq i} \mathcal{C}^j \subseteq \Sigma_{-i}$.

Recall that Definition 6.1 implies both independence and consistency of beliefs. Both these assumptions can be dropped. However the resulting model then is not a generalization of Nash equilibrium to ambiguous beliefs (which is our aim). To see this, assume that prior sets are singletons, i.e. there is no ambiguity in beliefs. Then assuming both independence and consistency reduces the model precisely to Nash equilibrium, as highlighted in Remark 6.3. However dropping independence and/or consistency leads to non-Nash behaviour.[15]

Next we apply the concept of ambiguity classes from Section 3 to games.

[14]Indeed when prior sets are singletons, the notion conforms precisely to independence. We thus believe this to be the logical extension of independence to an ambiguity context.

[15]Without independence players may for instance collude, without consistency the beliefs of two players about a third player may differ. Both of these phenomena are ruled out in Nash equilibrium. See also Remark 6.3 below.

Definition 6.2 (Ambiguity Classes Games) *Let $\mathcal{C}_{-i} = \times_{j\neq i}\mathcal{C}^j$ be the belief of player i. Then $\hat{\mathcal{C}}_{-i} = \times_{j\neq i}\hat{\mathcal{C}}^j$ is a translation of $\mathcal{C}_{-i}$ if $\hat{\mathcal{C}}^j$ is a translation of $\mathcal{C}^j$ for all $j \neq i$. The ambiguity class of $\mathcal{C}_{-i}$, denoted by $[\mathcal{C}_{-i}]$, is the set of all such translations.*

Remark 6.1 *By Proposition 5.2 there exists an isomorphism ξ_i from Σ_{-i} to $[\mathcal{C}_{-i}]$. This implies that any given ambiguity class is isomorphic to the set Σ_{-i}.*

6.2 Definition of Equilibrium

In this section we present our definition of equilibrium for games with ambiguity. Our solution concept demands that players believe that others only play strategies that are best responses given their beliefs.

Definition 6.3 (Equilibrium under Ambiguity (EUA)) *Let $\Gamma = \langle \mathcal{H}; S_i, u_i : 1 \leq i \leq N\rangle$ be a normal-form game. The tuple $(\mathcal{C}_{-i}, \alpha_i)_{i=1}^N$ is an Equilibrium Under Ambiguity if for all $1 \leq i \leq N$ there exist $\mathcal{C}^i \subseteq \Sigma_i$ such that $\mathcal{C}_{-i} = \times_{j\neq i}\mathcal{C}^j$ and*

$$\emptyset \neq \operatorname{supp}(\mathcal{C}^i) \subseteq \operatorname{argmax}_{s_i\in S_i} V_i(s_i). \tag{5}$$

An EUA is said to be a singleton EUA if $\operatorname{supp}(\mathcal{C}^i)$ consists of a single strategy profile for all $1 \leq i \leq N$.

Equation (5) is our consistency notion and mimicks the idea from Nash equilibrium that beliefs are "correct" in the sense that players believe that other players play best responses given their belief.

Remark 6.2 *We interpret EUA to be a situation where players choose pure strategies (and not possibly mixed). Their beliefs, represented by prior sets, reflect the fact that they perceive ambiguity regarding the pure strategies that the other players choose.*

Remark 6.3 *When prior sets only contain a single element, i.e. $\mathcal{C}_{-i} = \{\sigma_{-i}\}$ for $1 \leq i \leq N$, then if $(\mathcal{C}_{-i}, \alpha_i)_{i=1}^N$ is an EUA, the strategy profile $\sigma = \langle\sigma_1, \ldots, \sigma_N\rangle$ is a Nash equilibrium of Γ for any values of $\alpha_1, \ldots, \alpha_N$ (as ambiguity attitude plays no role when there is no ambiguity).*[16]

[16] Recall that due to the consistency assumption we have $\sigma_{-i} = \times_{j\neq i}\sigma_j$.

In light of Remark 6.3, EUA can be viewed as the natural extension of Nash Equilibrium (possibly in mixed strategies) to strategic ambiguity. [17]

6.3 Equilibrium Existence

In this section we prove existence of equilibrium, for arbitrary perceived ambiguity and ambiguity-attitude for every player. This enables us to analyse the impact of ambiguity by conducting comparative static exercises. An *ambiguous normal-form game* is characterized by the players, their strategies and pay-offs as usual and in addition by an ambiguity class and ambiguity-attitude for each player. We rely in the concept of location invariance introduced in Section 3 and on the ambiguity classes $[\cdot]$ from Definition 6.2.

Definition 6.4 (Ambiguous Normal-form Game) *Consider a normal form game* $\Gamma = \langle \mathcal{H}, S_i, u_i : 1 \leq i \leq N \rangle$ *as well as ambiguity classes* $[\mathcal{C}^1], \ldots, [\mathcal{C}^N]$ *for the beliefs about player* i, $1 \leq i \leq N$, *and ambiguity-attitudes* $\alpha_1, \ldots, \alpha_N$ *for all players. Then we call this an ambiguous game and denote it by*

$$\Gamma = \langle \mathcal{H}, S_i, u_i, [\mathcal{C}^i], \alpha_i : 1 \leq i \leq N \rangle. \tag{6}$$

The following theorem shows that an EUA exists for every ambiguous game. The theorem thus guarantees existence for any ambiguity and ambiguity-attitude of the players. This enables comparative static analysis by comparing equilibria with different perceived ambiguities and ambiguity-attitudes.

Theorem 6.1 (Equilibrium Existence) *Let* $\Gamma = \langle \mathcal{H}, S_i, u_i, [\mathcal{C}^i], \alpha_i : 1 \leq i \leq N \rangle$ *be an ambiguous normal-form game such that* $\text{supp}(\mathcal{C}^i) \neq \emptyset$, *for* $1 \leq i \leq N$. *Then there exist* $\bar{\mathcal{C}}_{-i} \in [\mathcal{C}_{-i}] := \times_{j \neq i} [\mathcal{C}^j]$ *such that* $(\bar{\mathcal{C}}_{-i}, \alpha_i)_{i=1}^N$ *is an Equilibrium under Ambiguity.*

The existence of the sets $\bar{\mathcal{C}}_{-i} \in [\mathcal{C}_{-i}]$ means that no matter what shape and size we fix for the ambiguity about player i's behaviour, reflected by $[\mathcal{C}^i]$, there always exist elements of $[\mathcal{C}^i]$, i.e. translations of $\mathcal{C}^i$, compatible with equilibrium as in Definition 6.3. Such an equilibrium exists for any normal-form game and any possible ambiguity and ambiguity-attitude. This

[17] See also the discussion after Definition 6.1.

result does not need any restrictive assumptions such as convexity of preferences or strategic complementarity. Note that in the special case of singleton prior sets, i.e. no ambiguity, the result is a restatement of Nash's theorem.

6.4 Modelling capabilities

In this subsection we demonstrate what our model is capable of. We start by analysing the game in Figure 4 in detail. As a proof of concept, we outline some of the phenomena that can be modelled with our approach by considering different combinations of perceived ambiguities for the players and conducting comparative statics in the ambiguity attitude parameter α. Throughout our analysis we restrict attention to singleton equilibria. After that we provide a more general analysis of the modelling capabilities of our approach and discuss the possibility of refinements of our model to eliminate undesirable equilibria.

6.4.1 A concrete example

The game in Figure 4 is a symmetric two player game with three strategies for each player. Strategy s_3 is the "safe strategy", guaranteeing a payout of 0, regardless of the behaviour of the opponent. For the other two strategies the payout depends on the behaviour of the other player, s_1 being more "risky" than s_2.[18]

We demonstrate that asymmetric equilibria are possible when we allow for heterogenous ambiguity attitudes. Indeed all pure-strategy combinations of the game can be modelled as an equilibrium, demonstrating the flexibility of our model. Since this is not necessarily an advantage we discuss at the end of this subsection how "refinements" can eliminate undesirable equilibria.

For $\delta \in [0,1)$ and $S = \{s_1, s_2, s_3\}$, consider the prior set $\mathcal{C} = \{P \in \Delta(S) | P(s_3) = 0, P(s_1) \in [0,\delta]\}$. Consider the ambiguity class $[\mathcal{C}]$. This class consists of all prior sets whose elements assign a constant probability to s_3 and a probability within an interval of length δ to s_1. The sets $\mathcal{C}^{s_i}$, i.e. the extreme elements of $[\mathcal{C}]$ in the corners of the simplex

[18] The game for instance models a scenario in which two people want to meet in one of three places (e.g. s_1 = cinema, s_2 = restaurant, s_3 = stay at home), but are unable to communicate where to meet beforehand. If they manage to coordinate then there is a clear ranking amongst the three places ($s_1 > s_2 > s_3$), but it is worse being at s_1 alone than being at s_2 alone. s_3 is the outside option for which it is irrelevant what the other person does.

		Player 2		
		s_1	s_2	s_3
Player 1	s_1	$(2, 2)$	$(-4, -1)$	$(-4, 0)$
	s_2	$(-1, -4)$	$(1, 1)$	$(-1, 0)$
	s_3	$(0, -4)$	$(0, -1)$	$(0, 0)$

Figure 4: A game with two players and three strategies

(see Lemma 5.1), are illustrated in Figure 5. In this picture, the length of the prior sets corresponds to δ.

The interpretation of the ambiguity class $[\mathcal{C}]$ is that the player has a concrete belief about the likelihood that the opponent plays s_3, but perceives ambiguity about the likelihoods of the opponent playing the strategies s_1 and s_2. The parameter δ reflects this ambiguity.

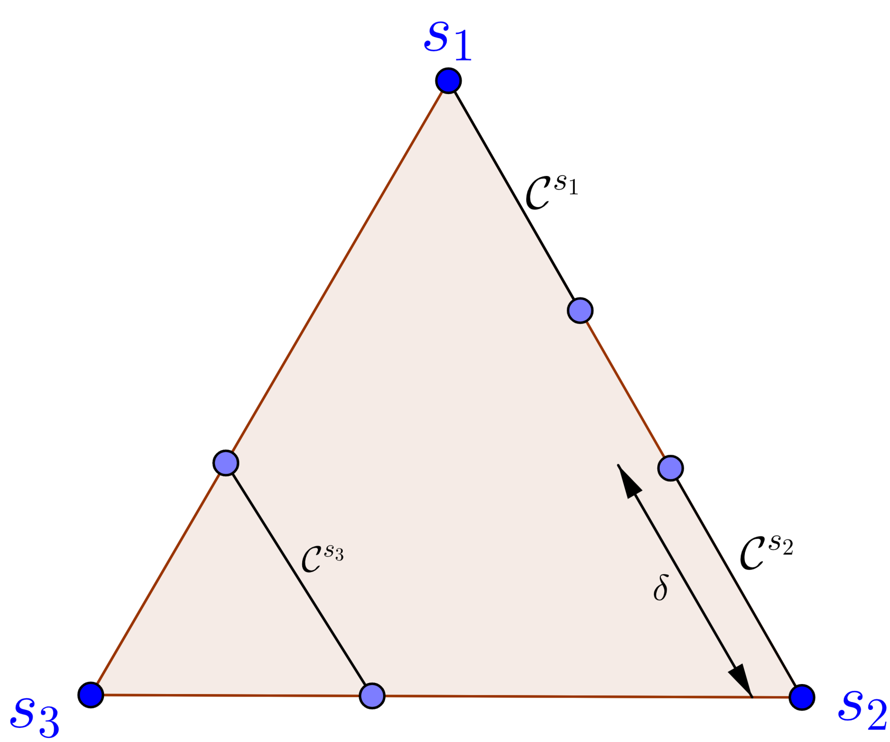

Figure 5: The corner-sets in $[\mathcal{C}]$.

We now demonstrate that all pure-strategy combinations of the game can be modelled with this $[\mathcal{C}]$, where for the asymmetric strategy combinations we need to, as you would expect, allow for asymmetry in ambiguity attitude.[19]

[19]The whole analysis is meant to provide a proof of concept and illustration of what our model is capable of. We do not claim that the assumptions made through $[\mathcal{C}]$ are realistic.

Let $[\mathcal{C}]$, characterized by $\delta \in [0,1)$, be the ambiguity class of both players.[20] Then in general, (s_i, s_j) is an EUA if

$$V_1(s_i)_{\mathcal{C}^{s_j}} \geq \max\{V_1(s_{i'})_{\mathcal{C}^{s_j}} | i' \neq i\} \qquad \text{and} \qquad V_2(s_j)_{\mathcal{C}^{s_i}} \geq \max\{V_2(s_{j'})_{\mathcal{C}^{s_i}} | j' \neq j\}, \tag{7}$$

where $V(s)_{\mathcal{C}^{s_i}}$ denotes the α-maxmin expectation of s, given the belief $\mathcal{C}^{s_i}$. The inequalities reflect that both players correctly "believe" that the other player plays what is optimal for him/her and that the players have no incentive to deviate to another strategy, given these consistent beliefs.

To make the following analysis transparent, we derive the concrete formulas that we need to determine equilibria. For instance, when the belief is $\mathcal{C}^{s_1}$, then the evaluation of the strategy s_1 is

$$V(s_1)_{\mathcal{C}^{s_1}} = \alpha \underbrace{(\delta(-4) + (1-\delta)2)}_{\text{Expectation of the worst case}} + (1-\alpha) \underbrace{2}_{\text{Expectation of the best case}} = 2 - 6\alpha\delta.$$

The other relevant equations are the following.

$V(s_1)_{\mathcal{C}^{s_2}} = -4 + 6\delta(1-\alpha)$	$V(s_1)_{\mathcal{C}^{s_3}} = -4 + 6\delta(1-\alpha)$	$V(s_2)_{\mathcal{C}^{s_1}} = -1 + 2\delta(1-\alpha)$
$V(s_2)_{\mathcal{C}^{s_2}} = 1 - 2\delta\alpha$	$V(s_2)_{\mathcal{C}^{s_3}} = -1 + 2\delta(1-\alpha).$	

Of course $V(s_3) = 0$ for all possible beliefs and ambiguity attitudes. These equations are all we need to analyze the equilibria of the game for these ambiguity classes. For each strategy combination we can determine for which α/δ combinations the considered strategy combination is an EUA. For instance, plugging in above formulas into (7) gives that (s_1, s_1) is an EUA if and only if for both players $V(s_1)_{\mathcal{C}^{s_1}} \geq \max\{V(s_2)_{\mathcal{C}^{s_1}}, V(s_3)_{\mathcal{C}^{s_1}}\}$, i.e.

$$2 - 6\alpha\delta \geq \max\{-1 + 2\delta(1-\alpha), 0\}.$$

We now illustrate some comparative statics in ambiguity δ and ambiguity attitude α. First we consider the symmetric case $\alpha_1 = \alpha_2 = \alpha$.

In the case $\delta = 0$, the equilibria coincide precisely with the Nash equilibria of the game

[20] This is a simplifying assumption. Of course we could allow for heterogeneity in the ambiguity class. However this is not necessary for our analysis. Furthermore, since the game is symmetric, it is not unreasonable to assume that the players perceive the same ambiguity.

since there is no ambiguity involved. That is (s_1, s_1), (s_2, s_2) and (s_3, s_3) are the equilibria. As δ increases, this breaks down and it depends on α whether a given strategy combination is an equilibrium or not. For instance, when $\delta = \frac{1}{2}$ and $\alpha > \frac{2}{3}$, (s_1, s_1) is no longer an EUA, but the other two symmetric strategy combination are. When δ is further increased to $\delta = \frac{3}{4}$, then $\alpha > \frac{2}{3}$ implies that also (s_2, s_2) is not an EUA anymore, indeed (s_3, s_3) remains as the unique EUA. This is in line with intuition. When there is sufficient ambiguity, then sufficient pessimism means that the "risky" Nash equilibia drop out. Even when the player believes that the other player will not play s_3, he will nonetheless play s_3 himself due to the high level of ambiguity and the pessimistic attitude towards it. Thus (s_1, s_1) and (s_2, s_2) are not equilibria in such a case.

As intuition suggests, the contrary phenomenon occurs when players are sufficiently optimistic. Indeed, when $\delta = \frac{2}{3}$, $\alpha < \frac{1}{4}$ eliminates (s_3, s_3) as an EUA but retains the other two symmetric strategy combinations. When δ further increases to $\delta = \frac{8}{9}$, then $\alpha < \frac{1}{4}$ eliminates also (s_2, s_2), thus (s_1, s_1) is the unique EUA. Again we suggest that this is in line with intuition. When there is sufficient ambiguity, then sufficient optimism means that the players are more willing to play the "risky" strategies. Even when the player believes that the other player will play s_3, he will nonetheless play s_1 or s_2 himself due to the high level of ambiguity and the optimistic attitude towards it. Thus (s_3, s_3) is not an equilibrium in such a case.

We now allow for heterogeneity in ambiguity attitude. We demonstrate that this allows us to intuitively model non-symmetric equilibria. Again with the formulas in (7) we can determine that (s_2, s_3) is an EUA if $\delta = \frac{2}{3}$, $\alpha_1 \leq \frac{1}{4}$ and $\alpha_2 \geq \frac{3}{4}$. That is, when there is sufficient ambiguity, then one player being optimistic and the other being pessimistic can model that the pessimistic player plays the safe s_3, whereas the optimistic player plays the risky s_2. Further increasing the ambiguity to $\delta = \frac{5}{6}$, $\alpha_1 \leq \frac{1}{10}$ and $\alpha_2 \geq \frac{2}{5}$ makes the even more extreme (s_1, s_3) an EUA. The strategy combination (s_1, s_2) is an EUA for $\delta = \frac{9}{10}$, $\alpha_1 \leq \frac{1}{9}$ and $\alpha_2 \in [\frac{1}{3}, \frac{4}{9}]$. Here, the first player is sufficiently optimistic, whereas the second player is rather optimistic but not too optimistic. The latter thus chooses the middle strategy s_2.

We have thus demonstrated that all pure-strategy combinations of the game in Figure 4 can be modelled.

6.4.2 General analysis of the modelling capabilities

In the following we demonstrate more generally the modelling capabilities of our approach.

Proposition 6.1 *Let $\Gamma = \langle \mathcal{H}, S_i, u_i : 1 \leq i \leq N \rangle$ be a normal-form game. Assume that for $s = (s_1, \ldots, s_n) \in S$ there exist $s', s'' \in S$ and $\alpha_1, \ldots, \alpha_n \in [0,1]$ such that for all $i \in \{1, \ldots, n\}$*

$$\begin{aligned} &\alpha_i \min\{u_i(s_i, s'_{-i}), u_i(s_i, s''_{-i})\} + (1-\alpha_i)\max\{u_i(s_i, s'_{-i}), u_i(s_i, s''_{-i})\} \\ &> \alpha_i \min\{u_i(s'_i, s'_{-i}), u_i(s'_i, s''_{-i})\} + (1-\alpha_i)\max\{u_i(s'_i, s'_{-i}), u_i(s'_i, s''_{-i}) \end{aligned} \tag{8}$$

for all $s'_i \in S_i$. Then s can be modelled as a singleton EUA.

Numerous strategy combinations are covered by Proposition 6.1. Indeed, all strategy combinations in the game in Figure 4 fall into this category. To get some intuition consider the case of pure pessimism (optimism), i.e. $\alpha_1 = \cdots = \alpha_n = 1$ $(= 0)$. Then the existence of strategy combinations s' and s'' as in Proposition 6.1 means that s_i has a better worst (best) case than any other strategy s'_i, when attention is restricted to the opponents playing either s'_{-i} or s''_{-i}. This explains in particular that minimax (maximax) strategy combinations can be modelled with our approach, see also Proposition 6.2.

Above we have already mentioned that our model corresponds precisely to Nash equilibrium when there is no ambiguity, i.e. prior sets are singletons. We now ask the reverse question: How do the equilibria look like when ambiguity is large? First we need to define what large ambiguity means.

Definition 6.5 *Let S be a state space and $\mathcal{C} \subseteq \Delta(S)$ a prior set. Define*

$$\delta(\mathcal{C}) = \min_{s \in S} \max_{\sigma \in \mathcal{C}} \sigma(s).$$

For an ambiguity class $[\mathcal{C}]$ define

$$\delta([\mathcal{C}]) = \min_{\mathcal{C}' \in [\mathcal{C}]} \delta(\mathcal{C}').$$

When $\delta(\mathcal{C})$ is close to 1, ambiguity is so large that $\mathcal{C}$ contains almost all priors in $\Delta(S)$.

In such a case preferences are close to Hurwicz-preferences (Hurwicz (1951)). The following proposition is thus not surprising and answers the question of what happens in equilibrium when ambiguity is large.

Proposition 6.2 *When perceived ambiguity is sufficiently large for all players, i.e.* $\delta([\mathcal{C}_{-i}])$ *close enough to 1 for all* $i \in \{1, \ldots, n\}$*, then in equilibrium players maximize*

$$\alpha_i \min_{s_{-i} \in S_{-i}} u(s_i, s_{-i}) + (1 - \alpha_i) \max_{s_{-i} \in S_{-i}} u(s_i, s_{-i}). \tag{9}$$

Proof. Assume that s_i' does not maximize $\alpha_i \min_{s_{-i} \in S_{-i}} u(s_i, s_{-i}) + (1 - \alpha_i) \max_{s_{-i} \in S_{-i}} u(s_i, s_{-i})$. Then if $\delta([\mathcal{C}_{-i}])$ is sufficiently close to 1, s_i' cannot be optimal for any element of $[\mathcal{C}_{-i}]$. Since the support of the prior set must be non-empty, it must contain some strategy that maximizes $\alpha_i \min_{s_{-i} \in S_{-i}} u(s_i, s_{-i}) + (1 - \alpha_i) \max_{s_{-i} \in S_{-i}} u(s_i, s_{-i})$. ■

In particular this means that under sufficient ambiguity, in equilibrium pessimists play pure minimax strategies and optimists play pure maximax strategies. Assuming that the maximisers of equation 6.2 are unique, this result implies that equilibrium under ambiguity is unique when ambiguity is sufficiently large.

6.4.3 Refinements

The example demonstrates how flexible and general our model is. Furthermore (we believe that) the equilibria that we get in our analysis of the game in Figure 4 are intuitive. The example is meant to demonstrate this flexibility and is a "proof of concept": It demonstrates what our model is capable of and that our theory has practical potential.

But the flexibility demonstrated in both the previous subsections is also a disadvantage, since a theory that can model almost everything is not very convincing. However "refinements" are possible. For instance, if we restrict attention to homogenous ambiguity attitudes (i.e. $\alpha_1 = \alpha_2$ in the example), then the asymmetric equilibria disappear. One could further restrict attention to MEU preferences (i.e. $\alpha = 1$), resulting in a model in which the players are purely pessimistic regarding the strategic ambiguity that they face, or to preferences that have a neo-additive representation (see Section 4.2.1).

Our equilibrium existence result works for *any* ambiguity and ambiguity attitude, which is why we present our theory in its fully general form. Desirable assumptions about preferences will lead to refinements of our general model.[21]

7 Applications and Examples

In the following we apply our model to some concrete and typical game-theoretic examples. The intuitive appeal of the resulting equilibria is highlighted.

7.1 Example 1: A public goods game

There are N players with two strategies each: contribute, c or not contribute, nc. If all players contribute then they receive a payout of 1. If not all players contribute then the ones who do contribute receive -1 and the ones who do not contribute receive 0. There are two pure-strategy Nash equilibria. In the first all contribute, in the second, no player contributes.[22]

Assume now that the players perceive ambiguity about the strategy choice of the other players. The players belief about what P_j does is reflected by $\mathcal{C}^j \subseteq \Sigma_j = \Delta(\{c, nc\})$. A prior set is thus an interval with length δ^j. The ambiguity class $[\mathcal{C}^j]$ consists of the prior sets in Σ_j with the same interval length δ^j. To fix perceived ambiguity thus means to fix such an interval-length.

The overall belief of player i is $\mathcal{C}_{-i} = \times_{j \neq i} C^j$. The following analysis shows that whether cooperation can be sustained in equilibrium depends on both the interval lengths δ^j and ambiguity attitudes α_i.

The strategy combination $(c, \ldots, c)$ is a singleton EUA, i.e. the prior sets in equilibrium are $\mathcal{C}^j = \{\sigma_j \in \Sigma_j; \sigma_j(nc) \leq \delta^j\}$, whenever we have that for all $i \in \{1, \ldots, N\}$:[23]

$$1 - \alpha_i \left(2 - \prod_{j \neq i} (1 - \delta_j) \right) \geq 0.$$

[21] This is analogous to certain refinements of Nash Equilibrium eliminating certain undesirable Nash equilibria.

[22] There is also a mixed equilibrium which depends on N.

[23] This can be seen by calculating that $V_i(nc) = 0$ and $V_i(c) = \alpha_i(1 - \Pi_{j \neq i}(1 - \delta^j))(-1) + (1 - \alpha_i)1$.

That is cooperation is an equilibrium whenever the players are sufficiently optimistic and ambiguity is not too large. Even with high levels of ambiguity, cooperation can be sustained when players are sufficiently optimistic. However sufficient ambiguity about a single other player can lead to a breakdown of cooperation when players are pessimistic. For instance complete ambiguity about a single player, i.e. $\delta^{j'} = 1$ for some j means that cooperation can only be sustained when all other players are more optimistic than pessimistic, i.e. $\alpha_i \leq \frac{1}{2}$ for all $i \neq j'$. It is sufficient to have one player for whom the above inequality fails in order for cooperation to break down. In such a case the only EUA is when no player cooperates. We believe that these results are quite intuitive.

7.2 Example 2: A public goods game with a threshold

Consider two players that play a public goods game with a threshold. Both players simultaneously choose a contribution $x_i \in \{0, \dots, m\}$. If contributions exceed a certain threshold $T \in \mathbb{R}$, then both players receive a prize with value $V \in \mathbb{R}$. If contributions are less than T, then the players get nothing. The payout functions are thus

$$u_i(x_i, x_{-i}) = V\mathbf{1}_{\{x_1+x_2 \geq T\}} - x_i.$$

There are two kinds of Nash equilibria: The contributions add up to reach T or both players contribute nothing. If $V > m \geq T$, then the latter Nash equilibria drop out as each player can reach the threshold alone and it is worthwhile to do so.

Now assume that the players perceive ambiguity about the other players strategy choice. In the following we perform some comparative static exercises to illustrate the effect of a change in ambiguity attitude when there is a lot of ambiguity. Furthermore we assume that $V > m$ and $T > m - 1$ ensuring that we do not have any strictly dominated strategies.

Assume that $m > T$, i.e. each player can reach the threshold independently of the other player's contribution. It is the safe strategy to play m. Strategy 0 is the risky choice hoping for m by the other player. When there is sufficient ambiguity, then according to Proposition 6.2 the players will choose the strategy $s_i \in S_i$ which maximizes $\alpha_i \min_{s_{-i} \in S_{-i}} u(s_i, s_{-i}) + (1 - \alpha_i) \max_{s_{-i} \in S_{-i}} u(s_i, s_{-i})$. Only $s_i = 0$ and $s_i = m$ are candidates for this. It depends on α_i what the equilibrium looks like. Simple calculations show that under sufficient ambiguity,

(m, m) constitutes the unique EUA whenever $\alpha_i > \frac{m}{V}$ and $(0, 0)$ constitutes the unique EUA whenever $\alpha_i < \frac{m}{V}$. Furthermore when $\alpha_1(\alpha_2) > \frac{m}{V}$ and $\alpha_2(\alpha_1) < \frac{m}{V}$ then $(m, 0)$ $((0, m))$ is the unique EUA.

Assume now that $m < V$, i.e. the players cannot reach the threshold alone. The safe strategy is thus to play 0, the risky strategy is to play $\lceil T - m \rceil$ hoping for m by the other player.[24] Again when there is sufficient ambiguity the players maximize

$$\alpha_i \min_{s_{-i} \in S_{-i}} u(s_i, s_{-i}) + (1 - \alpha_i) \max_{s_{-i} \in S_{-i}} u(s_i, s_{-i}).$$

Only $s_i = 0$ and $s_i = \lceil T - m \rceil$ are candidates for this. Simple calculations show that under sufficient ambiguity, $(\lceil T - m \rceil, \lceil T - m \rceil)$ constitutes the unique EUA whenever $\alpha_i < \frac{V - \lceil T - m \rceil}{V}$ and $(0, 0)$ constitute the unique EUA whenever $\alpha_i > \frac{V - \lceil T - m \rceil}{V}$. Furthermore when $\alpha_1(\alpha_2) < \frac{V - \lceil T - m \rceil}{V}$ and $\alpha_2(\alpha_1) > \frac{V - \lceil T - m \rceil}{V}$, then $(\lceil T - m \rceil, 0)$ $((0, \lceil T - m \rceil))$ constitute the unique EUA.

Thus we can see that when ambiguity is high it crucially depends on the parameters m and V whether more optimism increases contributions and thus the chance for cooperation. Furthermore asymmetries in ambiguity-attitude can lead to asymmetric behaviour.

8 Conclusion

This paper has two main contributions. Firstly we suggest that ambiguity is location invariant. We show that location invariance is in line with and unifies numerous influential definitions of ambiguity in the literature. Furthermore we show that the set of translations of a given set of priors is isomorphic to the simplex. Secondly we propose a solution concept for games where players have ambiguous beliefs as represented by the α-MEU model. We illustrate the modelling capabilities of our approach in general as well as concretely in several examples.

A number of extensions are possible in future research. When studying ambiguity in games, we assume that any two players have the same beliefs about the behaviour of a third party and that players believe that their opponents act independently. These assumptions are

[24] $\lceil x \rceil$ is the smallest integer bigger or equal than x.

standard in the previous literature. It is useful to retain them, since they enable us to isolate the effect of ambiguity in games. However we note these assumptions act as constraints. Our proof of existence of equilibrium with these constraints easily implies existence in their absence. In a strategic interaction, a given player may perceive ambiguity concerning whether or not his/her opponents are colluding. This may be relevant for competition policy. If the independence assumption is relaxed our framework would be suitable for analysing such questions. The model can thus be made more general by relaxing/dropping the consistency Assumption 6.2 as well as the assumption of independent beliefs. For singleton prior sets (no ambiguity) the model would be more general than Nash equilibrium.

Our results can be extended to the case where the ambiguity attitude depends on the strategy chosen, as in Hartmann (2019). It would be relatively straightforward to include games of incomplete information by adding a type space.

Our theory can be extended to extensive form games. In this context players will get new information during the course of the game. It is necessary to consider how they update their beliefs and whether the updated preferences will satisfy dynamic consistency. Dynamic consistency as well as consequentialism can be ensured by requiring players to have preferences of the recursive multiple priors form, Sarin and Wakker (1998). Thus multiple prior preferences can be applied to problems involving many time periods without violating dynamic consistency.[25] This will allow an extension of our solution concept to extensive form games, which enables us to study the impact of ambiguity in dynamic models. If dynamic consistency and consequentialism are satisfied it is relatively easy to solve extensive form games by backward induction. One application would be to study how herding and bubbles in financial markets are affected by ambiguity. Secondly it would be possible to study how ambiguity affects the possibility of cooperation in repeated games.

Appendix

Throughout this appendices, we denote by ϕ_t a translation, that is if $\mathcal{C}' = \mathcal{C} + t$, then we write $\phi_t(\mathcal{C}) = \mathcal{C}'$.

[25] Sarin and Wakker (1998) propose recursive multiple prior preferences which are dynamically consistent while also allowing for non-neutral attitudes to ambiguity. Epstein and Schneider (2003) develop the recursive multiple priors model and apply it to financial markets.

A Proofs for results in Section 3 and Section 4

Proposition 3.1 If two prior sets $\mathcal{C}$ and $\mathcal{C}'$ are translations of one another then $\delta_{\mathcal{C}}(a) = \delta_{\mathcal{C}'}(a)$ for all $a \in A(S)$.

Proof. Let $a \in A(S)$ be an act, ϕ_t a translation, and $p \in \Delta(S)$ a prior. Notice that $\mathbf{E}_p(a) = p \centerdot a$. Then,

$$\mathbf{E}_{p+t}(a) = (p+t) \centerdot a = p \centerdot a + t \centerdot a = \mathbf{E}_p a + t \centerdot a.$$

Now assume that the prior sets $\mathcal{C}$ and $\mathcal{C}'$ are translations, i.e. $\mathcal{C}' = \phi_t(\mathcal{C})$ for some translation ϕ_t. The above implies that $q \in \text{argmax}_{p \in \mathcal{C}'} \mathbf{E}_p(a)$ if and only if $q + t \in \text{argmax}_{p \in \mathcal{C}} \mathbf{E}_p(a)$. In particular, $\min_{p \in \mathcal{C}'} \mathbf{E}_p(a) = \min_{p \in \mathcal{C}} \mathbf{E}_p(a) + t \centerdot a$. The same holds when "min" is replaced by "max". Thus

$$\begin{aligned}\max_{p \in \mathcal{C}'} \mathbf{E}_p(a) - \min_{p \in \mathcal{C}'} \mathbf{E}_p(a) &= \max_{p \in \mathcal{C}} \mathbf{E}_p(a) + t \centerdot a - (\min_{p \in \mathcal{C}}(\mathbf{E}_p(a) + t \centerdot a) \\ &= \max_{p \in \mathcal{C}} \mathbf{E}_p(a) - \min_{p \in \mathcal{C}} \mathbf{E}_p(a).\end{aligned}$$

This implies $\delta_{\mathcal{C}}(a) = \delta_{\mathcal{C}'}(a)$. ■

Proposition 4.1 Let $\succcurlyeq_1, \succcurlyeq_2$ be risk-neutral α-MEU preferences characterized by $(\mathcal{C}_1, \alpha)$ and $(\mathcal{C}_2, \alpha)$, respectively. Then if $\mathcal{C}_1$ and $\mathcal{C}_2$ are translations, their bid-ask and ask-bid spreads coincide for all assets. The spread for asset a, given ambiguity class $[\mathcal{C}]$ and α, is

$$|(2\alpha - 1)(\underline{p}_a - \overline{p}_a) \cdot a|,$$

where $\underline{p}_a \in \text{argmin}_{p \in \mathcal{C}} \mathbf{E}_p a$ and $\bar{p}_a \in \text{argmax}_{p \in \mathcal{C}} \mathbf{E}_p a$ (for an arbitrary prior set in $[\mathcal{C}]$) and $\cdot$ denotes the scalar product.

Proof. Assume that $\mathcal{C}_2 = \mathcal{C}_1 + t$. Let $\overline{q}_a^i, \underline{q}_a^i$ be the buy and short-sell reservation prices for $i \in \{1, 2\}$. Let $\underline{p}_a^i, \overline{p}_a^i$ be minimizing and maximizing values of a given $\mathcal{C}_i$, $i \in \{1, 2\}$. We have that

$$\omega = (\alpha \underline{p}^i_a + (1-\alpha)\overline{p}^i_a) \cdot (\omega + a - \overline{q}^i_a)$$

$$\omega = (\alpha \underline{p}^i_{-a} + (1-\alpha)\overline{p}^i_{-a}) \cdot (\omega - a + \underline{q}^i_a).$$

Rearranging leads to

$$\overline{q}^i_a = (\alpha \underline{p}^i_a + (1-\alpha)\overline{p}^i_a) \cdot a$$

$$\underline{q}^i_a = (\alpha \underline{p}^i_{-a} + (1-\alpha)\overline{p}^i_{-a}) \cdot a.$$

Now note that the minimizing and maximizing values can be chosen such that $\overline{p}^i_a = \underline{p}^i_{-a}$, that is the best case for a coincides with the worst case for $-a$. Thus the above can be rearranged to get the formula for the spread:

$$\overline{q}^i_a - \underline{q}^i_a = (2\alpha - 1)(\underline{p}^i_a - \overline{p}^i_a) \cdot a.$$

The minimizing and maximizing values can be chosen such that $\underline{p}^2_a = \underline{p}^1_a + t$ and $\overline{p}^2_a = \overline{p}^1_a + t$. Thus we have

$$\begin{aligned}\overline{q}^2_a - \underline{q}^2_a &= (2\alpha - 1)(\underline{p}^2_a - \overline{p}^2_a) \cdot a \\ &= (2\alpha - 1)(\underline{p}^1_a + t - \overline{p}^1_a - t) \cdot a \\ &= \overline{q}^1_a - \underline{q}^1_a.\end{aligned}$$

■

Proposition 4.2 Consider $\alpha \in [0,1]$ and prior sets $\mathcal{C}_1, \mathcal{C}_2$. Let $\succcurlyeq_i$ be the α-MEU preference relations induced by α and $\mathcal{C}_i$, for $i \in \{1,2\}$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are translations, then $\succcurlyeq_1$ and $\succcurlyeq_2$ have the same ambiguity aversion and a-insensitivity index for any belief hedge $\mathcal{D}$.

Proof. We first show that when $\mathcal{C}_1$ and $\mathcal{C}_2$ are translations, i.e. $\mathcal{C}_2 = \phi_t(\mathcal{C}_1)$, then $m_2 = m_1 + t$.

First note that in the α-MEU model, $m(E) = \alpha \min_{p\in\mathcal{C}} p(E) + (1-\alpha)\max_{p\in\mathcal{C}} p(E)$. We have

$$\begin{aligned}
m_2(E) &= \alpha \min_{p\in\mathcal{C}_2} p(E) + (1-\alpha)\max_{p\in\mathcal{C}_2} p(E) = \alpha \min_{p\in\mathcal{C}_1+t} p(E) + (1-\alpha)\max_{p\in\mathcal{C}_1+t} p(E) \\
&= \alpha[\min_{p\in\mathcal{C}_1} p(E) + t(E)] + (1-\alpha)[\max_{p\in\mathcal{C}_1} p(E) + t(E)] \\
&= \alpha \min_{p\in\mathcal{C}_1} p(E) + (1-\alpha)\max_{p\in\mathcal{C}_1} p(E) + t(E) = m_1(E) + t(E).
\end{aligned}$$

Next we show that the average of the matching probability function is unaffected by translations, i.e. that $\overline{m}_1 = \overline{m}_2$. First note that $\sum_{E\in\mathcal{D}} t(E) = \sum_{s\in S} t(s)\frac{|\mathcal{D}|}{2} = \frac{|\mathcal{D}|}{2}\sum_{s\in S} t(s) = 0$, where the first equality follows from $\mathcal{D}$ being l-hedged as well as the additivity of t and the last equality follows from t being a translation. Thus we have

$$\begin{aligned}
\overline{m}_2 = \overline{m_1 + t} &= \frac{\sum_{E\in\mathcal{D}}[m_1(E) + t(E)]}{|\mathcal{D}|} \\
&= \frac{\sum_{E\in\mathcal{D}} m_1(E)}{|\mathcal{D}|} + \frac{\sum_{E\in\mathcal{D}} t(E)}{|\mathcal{D}|} = \overline{m_1}.
\end{aligned}$$

This directly implies that the ambiguity-aversion index is the same for both preferences.

We now turn to a-insensitivity, defined by $a = 1 - \frac{\text{Cov}(m,\nu)}{\text{Var}(\nu)}$. It suffices to show that $\text{Cov}(m_2, v) = \text{Cov}(m_1, v)$. We have

$$\begin{aligned}
\text{Cov}(m_2, v) &= \overline{(m_2 - \overline{m_2})(v - \overline{v})} = \overline{m_2 v} - \overline{m_2}\ \overline{v} \\
&= \overline{(m_1 + t)v} - \overline{m_1 + t}\ \overline{v} = \overline{m_1 v} + \overline{tv} - \overline{m_1}\ \overline{v} - \overline{t}\ \overline{v} \\
&= \text{Cov}(m_1, v) + \overline{tv} - \overline{v}\ \overline{t}.
\end{aligned}$$

Since $\overline{t} = 0$ by the definition of translation it suffices to show that $\overline{tv} = 0$:

$$\begin{aligned}
\overline{tv} &= \sum_{E\in\mathcal{D}} t(E)v(E) = \sum_{E\in\mathcal{D}}\sum_{E_i\in E} t(E_i)\underbrace{v(E_i)}_{=\frac{1}{|\mathcal{D}|}} \\
&= \frac{1}{|\mathcal{D}|}\sum_{E\in\mathcal{D}}\sum_{E_i\in E}\sum_{s\in E_i} t(s) \overset{(*)}{=} \frac{1}{|\mathcal{D}|}\frac{|\mathcal{D}|}{2}\underbrace{\sum_{s\in S} t(s)}_{=0} = 0,
\end{aligned}$$

where $(*)$ holds due to l-hedging. We have thus shown that $\text{Cov}(m_2, \nu) = \text{Cov}(m_1, \nu)$ which implies that the a-insensitivity is unaffected by translations. ■

Proposition 4.3 *Let ν and ν' be two neo-additive capacities characterized by π, δ, α and π', δ', α', respectively. Let $\mathcal{D}$ and $\mathcal{D}'$ be the respective prior sets from (2). Then $\delta = \delta'$ if and only if $\mathcal{B}$ and $\mathcal{B}'$ are translations.*

Proof. Assume that $\delta = \delta'$. Consider the translation $t = (1-\delta)(\pi' - \pi)$. We show that $\mathcal{B}' = \mathcal{B} + t$. Consider some $\hat{p} \in \Delta(S)$. Then

$$\hat{p} \in \mathcal{B} + t \iff \hat{p}(E) \geq (1-\delta)\pi(E) + (1-\delta)(\pi'(E) - \pi(E)), \forall E \subseteq S;$$
$$\iff \hat{p}(E) \geq (1-\delta)\pi'(E), \forall E \subseteq S \iff \hat{p} \in \mathcal{B}'.$$

For the converse assume that $\delta > \delta'$. Consider an arbitrary state $s' \in S$ and the following translations t and t':

$$t(s) = \begin{cases} 1 - \pi(s), & s = s'; \\ -\pi(s), & s \neq s'; \end{cases} \qquad t'(s) = \begin{cases} 1 - \pi'(s), & s = s'; \\ -\pi'(s), & s \neq s'. \end{cases}$$

The prior sets $\mathcal{B} + t$ and $\mathcal{B}' + t'$ are the sets $\mathcal{B}$ and $\mathcal{B}'$ translated into the s'-corner of the simplex, i.e.

$$\mathcal{B} + t = \{\hat{p} \in \Delta(S) | \hat{\pi}(E) \geq (1-\delta)\mathbb{I}_{s'}, \forall E \subseteq S\},$$
$$\mathcal{B}' + t' = \{\hat{p} \in \Delta(S) | \hat{\pi}(E) \geq (1-\delta')\mathbb{I}_{s'}, \forall E \subseteq S\},$$

where $\mathbb{I}_{s'}$ is the degenerate distribution in $\Delta(S)$ which puts all weight on s'. Due to $\delta > \delta'$ we have that $\mathcal{B}' + t' \subsetneqq \mathcal{B} + t$. Therefore $\mathcal{B}$ and $\mathcal{B}'$ are not translations. ■

B Existence of Equilibrium in Games

In this appendix we show equilibrium existence. We first need to show that any given ambiguity class is isomorphic to the simplex as claimed in Proposition 5.1. We need some lemmas which lead up to this result.

Throughout we use the notation as introduced in Section 2. We adapt the notation accordingly once we focus on games. The next result shows that if one member of an ambiguity class has non-empty support then every other member also has non-empty support.

Lemma B.1 *Consider a finite state space S and prior sets $\mathcal{C}, \mathcal{C}' \subseteq \Delta(S)$ with $\mathcal{C}' \in [\mathcal{C}]$. Then* $\text{supp}(\mathcal{C}) \neq \emptyset$ *if and only if* $\text{supp}(\mathcal{C}') \neq \emptyset$*. Furthermore, if both supports are empty, then the sets are identical, i.e. $\mathcal{C} = \mathcal{C}'$.*

Proof. Since $\mathcal{C}' \in [\mathcal{C}]$, there exists some translation ϕ_t such that $\mathcal{C}' = \phi_t(\mathcal{C})$. First assume that $\text{supp}(\mathcal{C}) \neq \emptyset$, i.e. there exists some $\hat{s} \in S$ such that $\min_{p \in \mathcal{C}} p(\hat{s}) > 0$. If $t(\hat{s}) > -\min_{p \in \mathcal{C}} p(\hat{s})$, then $\hat{s} \in \text{supp}(\mathcal{C}')$. Otherwise $t(\hat{s}) = -\min_{p \in \mathcal{C}} p(\hat{s}) < 0$. Hence there exists some $s' \in S$ such that $t(s') > 0$. It holds that $\min_{q \in \mathcal{C}'} q(s') = \min_{p \in \mathcal{C}} p(s') + t(s') \geq t(s') > 0$. Thus $s' \in \text{supp}(\mathcal{C}') \neq \emptyset$.

Now assume that $\text{supp}(\mathcal{C}) = \emptyset$, which implies $\min_{p \in \mathcal{C}} p(s) = 0$ for all $s \in S$. Assume for contradiction that $t(s) < 0$ for some $s \in S$. Then $\min_{q \in \mathcal{C}'} q(s) = \min_{p \in \mathcal{C}} p(s) + t(s) = t(s) < 0$. But this cannot happen since $q(s) \geq 0$. Thus $t \equiv 0$ must hold, so $\mathcal{C}' = \mathcal{C}$. ■

For the second part, the intuition is that if the support is empty then $\mathcal{C}$ touches all sides of the simplex. In this case no translation will leave $\mathcal{C}$ entirely within the simplex.

Lemma 5.1 *If S is a finite state space and $\mathcal{C} \subseteq \Delta(S)$ is a prior set with* $\text{supp}(\mathcal{C}) \neq \emptyset$*, then for every state $s^* \in S$ there exists a unique prior set $\mathcal{C}^{s^*} \in [\mathcal{C}]$ such that* $\text{supp}(\mathcal{C}^{s^*}) = \{s^*\}$.

Proof. Define a prior set $\mathcal{C}^{s^*} = \phi_t(\mathcal{C})$ where t is defined by

$$t(s) = -\min_{p \in \mathcal{C}} p(s), s \neq s^*; \qquad t(s^*) = \sum_{s \in S \setminus \{s^*\}} \min_{p \in \mathcal{C}} p(s).$$

By construction, $\min_{p \in \mathcal{C}^{s^*}} p(s) = \min_{p \in \mathcal{C}} p(s) + t(s) = 0$, for all $s \in S \setminus \{s^*\}$, implying that $\text{supp}(\mathcal{C}^{s^*}) \subseteq \{s^*\}$. By Lemma B.1, $\text{supp}(\mathcal{C}^{s^*}) \neq \emptyset$ so we can conclude that $\text{supp}(\mathcal{C}^{s^*}) = \{s^*\}$.

To show uniqueness, let $\mathcal{C}' = \phi_{\hat{t}}$ be a translation of $\mathcal{C}$ such that $\text{supp}(\mathcal{C}') = \{s^*\}$. There does not exist $\hat{s} \neq s^*$ such that $\hat{t}(\hat{s}) > -\min_{p \in \mathcal{C}} p(\hat{s})$. This would imply $\min_{p \in \mathcal{C}'} p(\hat{s}) > 0$, and hence $\hat{s} \in \text{supp}(\mathcal{C}')$. For $\hat{s} \neq s^*$, we cannot have $\hat{t}(\hat{s}) < -\min_{p \in \mathcal{C}} p(\hat{s})$, since this would imply the existence of $\tilde{p} \in \mathcal{C}'$ such that $\tilde{p}(\hat{s}) < 0$. Hence $\hat{t}(s) = -\min_{p \in \mathcal{C}} p(s), s \neq s^*$. Since $\sum_{s \in S} t(s) = \sum_{s \in S} \hat{t}(s) = 0$ we have $\hat{t}(s^*) = t(s^*)$, which implies $\hat{t} = t$. This establishes uniqueness of $\mathcal{C}^{s^*}$. ■

In what follows, for $s \in S$, we denote the prior set derived in Lemma 5.1 by $\mathcal{C}^s$. The following proposition is the crucial insight towards deriving the isomorphism.

Proposition B.1 (Pointwise mixtures of translations) *Let $\mathcal{C}_1, \ldots, \mathcal{C}_m \subset \mathbb{R}^n$ be non-empty, convex and compact sets. If they are translations of each other then for any $\alpha_1, \ldots, \alpha_m \in$*

$[0,1]$ *such that* $\sum_{i=1}^m \alpha_i = 1$ *the set of pointwise* α*-mixtures*

$$\mathcal{C}_{m+1} = \left\{ \sum_{i=1}^{m} \alpha_i p_i \in \mathbb{R}^n | p_i \in \mathcal{C}_i \ \forall \ i \in \{1, \ldots, m\} \right\}$$

is also a translation of these sets.

Proof. Since $\mathcal{C}_1, \ldots, \mathcal{C}_m$ are translations there exist, for all $i, j \in \{1, \ldots, m\}$, translations $t_i^j \in \mathbb{R}^n$ such that $\mathcal{C}_j = \mathcal{C}_i + t_i^j$. Of course, $t_i^i \equiv 0$ and $t_i^j = -t_j^i$ for all $i, j \in \{1, \ldots, m\}$.

Define $t = \sum\limits_{i=1}^{m} \alpha_i t_1^i$. We now show that $\mathcal{C}_{m+1} = \mathcal{C}_1 + t$.

Consider $p_1, \ldots, p_m$ with $p_i \in \mathcal{C}_i$. We need to show that $p := \sum\limits_{i=1}^{m} \alpha_i p_i \in \mathcal{C}_1 + t$. Note that there exist $p_1^1, \ldots, p_1^m \in \mathcal{C}_1$ such that $p_i = p_1^i + t_1^i$ for all $i \in \{1, \ldots, m\}$. We have

$$p = \sum_{i=1}^{m} \alpha_i p_i = \sum_{i=1}^{m} \alpha_i (p_1^i + t_1^i) = \underbrace{\sum_{i=1}^{m} \alpha_i p_1^i}_{\in \mathcal{C}_1} + \underbrace{\sum_{i=1}^{m} \alpha_i t_1^i}_{=t},$$

where $\sum\limits_{i=1}^{m} \alpha_i p_1^i \in \mathcal{C}_1$ is true since $\mathcal{C}_1$ is convex. Thus we have shown that $p \in \mathcal{C}_1 + t$ and therefore $\mathcal{C}_{m+1} \subseteq \mathcal{C}_1 + t$.

For the other direction consider $p \in \mathcal{C}_1 + t$. For every $i \in \{1, \ldots, m\}$ define $p_i = p - t + t_1^i$. Clearly, $p_i \in \mathcal{C}_i$ for all $i \in \{1, \ldots, m\}$. We have

$$\sum_{i=1}^{m} \alpha_i p_i = \sum_{i=1}^{m} \alpha_i (p - t + t_1^i) = p - t + \underbrace{\sum_{i=1}^{m} \alpha_i t_1^i}_{=t} = p.$$

Thus we have found elements $p_1, \ldots, p_m$ of $\mathcal{C}_1, \ldots, \mathcal{C}_m$ such that their α-mix is an element of $\mathcal{C}_1 + t$. This implies $\mathcal{C}_1 + t \subseteq \mathcal{C}_{m+1}$. Overall we have shown the required $\mathcal{C}_{m+1} = \mathcal{C}_1 + t$.

■

We now prove a lemma that we need for the proof of Proposition 5.1.

Lemma B.2 *Let* $[\mathcal{C}]$ *be a non-singleton ambiguity class. The number* $\tau(s)$ *defined by* $\tau(s) = \min\limits_{p \in \mathcal{C}^s} p(s) \in (0,1]$ *does not depend on* s*.*

Proof. Let $\hat{s}, \tilde{s} \in S, \hat{s} \neq \tilde{s}$. Note that $\min\limits_{p \in \mathcal{C}^{\hat{s}}} p(\hat{s}) = \tau(\hat{s})$ and $\min\limits_{p \in \mathcal{C}^{\hat{s}}} p(s) = 0, s \neq \hat{s}$. By construction $\mathcal{C}^{\tilde{s}} = \mathcal{C}^{\hat{s}} + t'$, where $t'(\hat{s}) = -\tau(\hat{s}), t'(\tilde{s}) = \tau(\hat{s})$ and $t(s) = 0$ otherwise. We now have $\tau(\hat{s}) = t'(\tilde{s}) = \min_{p\in\mathcal{C}^{\tilde{s}}} p(\tilde{s}) - \underbrace{\min_{p\in\mathcal{C}^{\hat{s}}} p(\tilde{s})}_{=0} = \tau(\tilde{s})$. ■

Remark B.1 *This result enables us to define $\tau = \tau(s)$. The number τ can be interpreted as a measure for the space that the prior sets in $[\mathcal{C}]$ have to move around in $\Delta(S)$. It is 0 if and only if* $\operatorname{supp}(\mathcal{C}) = \emptyset$ *and 1 if and only if* $|\mathcal{C}| = 1$.

Lemma 5.1 defines a 1-1 map from S to the elements of $[\mathcal{C}]$ with singleton support. We can now prove that the extension of this as introduced in Definition 5.2 is an isomorphism between $\Delta(S)$ and $[\mathcal{C}]$.

Proposition 5.1 *The function $\psi : \Delta(S) \to [\mathcal{C}]$, is an isomorphism.*

Proof. First note that all elements of $\mathcal{C}^q$ are indeed probability distributions and thus $\psi(q) \subseteq \Delta(S)$. Furthermore Proposition B.1 guarantees that ψ indeed maps every element of $\Delta(S)$ to an element of $[\mathcal{C}]$. So ψ is well defined. Moreover ψ is 1-1, since if $\hat{q} \neq \tilde{q}$ then $t^{\hat{q}}(s) \neq t^{\tilde{q}}(s)$ for some $s \in S$, hence $\mathcal{C}^{\hat{q}} \neq \mathcal{C}^{\tilde{q}}$.

We now show that ψ is onto. If $\mathcal{C}' \in [\mathcal{C}]$ then $\mathcal{C}' = \mathcal{C}^{s^*} + t'$ for some t', such that $\sum_{s\in S} t'(s) = 1$. Notice that if $\tilde{s} \neq s^*, \exists \tilde{p} \in \mathcal{C}^{s^*}$ such that $\tilde{p}(\tilde{s}) = 0$. Thus we must have $t'(\tilde{s}) \geq 0$ to ensure that $\tilde{p} + t'$ is a well defined probability distribution.

Define a probability distribution $q' \in S$ by:

$$q'(s) = \begin{cases} \frac{t'(s)}{\tau}, & s \neq s^*; \\ \frac{t'(s)}{\tau} + 1, & s = s^*. \end{cases}$$

We can show that q' so defined is a probability distribution since $q'(s) \geq 0$ holds since $t'(s) \geq 0$ for $s \neq s^*$ and

$$\sum_{s\in S} q'(s) = \sum_{s\in S} \frac{t'(s)}{\tau} + 1 = 1.$$

By construction $\psi(q') = \mathcal{C}'$, which establishes ψ to be onto. The function ψ is thus a bijection. To show that it is an isomorphism it remains to be shown that ψ is a homomorphism.[26] Note that ψ maps one mixture space to another while preserving the mixture operation, as shown in Proposition B.1. It is thus a homomorphism. The result follows.

[26]Recall that an isomorphism is defined as a bijective homomorphism.

■

Remark B.2 *The isomorphism* ψ *implies in particular that for every prior set* $\mathcal{C}$ *with non-empty support we have*

$$[\mathcal{C}] = \{\mathcal{C}^q | q \in \Delta(S)\}.$$

The remaining result in this appendix are needed for the application to games but are interesting in their own right.

Corollary B.1 *If* $[\mathcal{C}]$ *is non-singleton, then for all* $q \in \Delta(S)$ *we have* $\text{supp}(\mathcal{C}^q) = \text{supp}(q)$.

Proof. By definition and Proposition B.1, $\mathcal{C}^p = \{q = \sum_{s \in S} p(s) p^s | p^s \in \mathcal{C}^s \; \forall s \in S\}$. It follows directly that for any state $s \in S$ we have $s \in supp(\mathcal{C}^p)$ if and only if $p(s) > 0$. ■

The next result shows that if beliefs are represented by $\mathcal{C}^q$, then preferences are linear in q. This implies that if s corresponds to the strategies of an opponent in a normal form game, then mixed strategies will convexify pay-offs in the usual way.

Lemma B.3 *The evaluation functional* V *is linear in* $q \in \Delta(S)$, *i.e. for* $a \in A(S)$ *and* $\alpha \in [0, 1]$

$$V(a|\mathcal{C}^q, \alpha) = \sum_{s \in S} q(s) V(a|\mathcal{C}^s, \alpha).$$

Proof. Assume for now $\alpha = 1$. Consider an act $a \in A(S)$ and some $q \in \Delta(S)$. Denote by $\underline{p}_q(a)$ an element of $\text{argmin}_{p \in \mathcal{C}^q} E_p a$. This exists since prior sets are compact. Now consider an arbitrary state $s^* \in S$. For each state $s \in S$ there exist translations $t^s_{s^*}$ such that $\mathcal{C}^s = \mathcal{C}^{s^*} + t^s_{s^*}$. From Proposition B.1 we know that $\mathcal{C}^q = \mathcal{C}^{s^*} + t$, where $t = \sum_{s \in S} q(s) t^s_{s^*}$.

For each $s \in S$ consider $\underline{p}_s(a) := \underline{p}_q(a) - t + t^s_{s^*}$, which is an element of $\text{argmin}_{p \in \mathcal{C}^s} E_p(a)$. We have

$$\begin{aligned}
\sum_{s \in S} q(s) \underline{p}_s(a) &= \sum_{s \in S} q(s) (\underline{p}_q(a) - t + t^s_{s^*}) \\
&= \sum_{s \in S} q(s) \underline{p}_q(a) - t + \underbrace{\sum_{s \in S} q(s) t^s_{s^*})}_{=t} \\
&= \sum_{s \in S} q(s) \underline{p}_q(a) \\
&= \underline{p}_q(a)
\end{aligned}$$

This implies

$$\begin{aligned}\min_{p\in\mathcal{C}^q}\mathbf{E}_p a &= \mathbf{E}_{\underline{p}_q}(a)\\ &= \sum_{s\in S} q(s)\mathbf{E}_{\underline{p}_s(a)}(a)\\ &= \sum_{s\in S} q(s)\min_{p\in\mathcal{C}^s}\mathbf{E}_p a.\end{aligned}$$

Analogously we can show that $\max_{p\in\mathcal{C}q}\mathbf{E}_p a = \sum_{s\in S} q(s)\max_{p\in\mathcal{C}^s}\mathbf{E}_p a$. This implies

$$\begin{aligned}V(a|\mathcal{C}^q,\alpha) &= \alpha\min_{p\in\mathcal{C}^q}\mathbf{E}_p a + (1-\alpha)\max_{p\in\mathcal{C}^q}\mathbf{E}_p a\\ &= \alpha(a)\sum_{s\in S} q(s)\min_{p\in\mathcal{C}^s}\mathbf{E}_p a + (1-\alpha(a))\sum_{s\in S} q(s)\max_{p\in\mathcal{C}^s}\mathbf{E}_p a\\ &= \sum_{s\in S} q(s)\left[\alpha(a)\min_{p\in\mathcal{C}^s}\mathbf{E}_p a + (1-\alpha(a))\max_{p\in\mathcal{C}^s}\mathbf{E}_p a\right]\\ &= \sum_{s\in S} q(s)V(a|\mathcal{C}^s,\alpha).\end{aligned}$$

This shows the required linearity.

■

Theorem 6.1 Let $\Gamma = \langle \mathcal{H}, S_i, u_i, [\mathcal{C}^i], \alpha_i : 1 \le i \le N\rangle$ be an ambiguous normal-form game such that $\text{supp}(\mathcal{C}^i) \neq \emptyset$, for $1 \le i \le N$. Then there exist $\bar{\mathcal{C}}_{-i} \in [\mathcal{C}_{-i}] := \times_{j\neq i}[\mathcal{C}^j]$ such that $(\bar{\mathcal{C}}_{-i}, \alpha_i)_{i=1}^N$ is an Equilibrium under Ambiguity.

Proof. Define a correspondence $\rho : \Sigma \to \times_{i=1}^N [2^{\Delta(S_i)}]$, by $\rho(\sigma) = \times_{i=1}^N \rho_i(\sigma)$, where $\rho_i(\sigma) = \text{argmax}_{\phi_i\in\Delta(S_i)} V_i(\phi_i|\mathcal{C}^{\sigma_{-i}}, \alpha_i)$. That is ρ maps every independent probability distribution over S to the cartesian product of the best responses for each player. By Berge's Theorem ρ is upper hemi-continuous, compact and non-empty valued. Hence by Kakutani's Theorem ρ has a fixed point $\bar{\sigma} = \bar{\sigma}_1 \times \cdots \times \bar{\sigma}_N$.

Now consider the element of $[\mathcal{C}]$ induced by $\bar{\sigma}$ through the isomorphism ψ, i.e. $\mathcal{C}^{\bar{\sigma}} = \mathcal{C}^{\bar{\sigma}_1} \times \cdots \times \mathcal{C}^{\bar{\sigma}_n}$. We claim that $\mathcal{C}^{\bar{\sigma}}$ induces an EUA of Γ, i.e. $(\mathcal{C}^{\bar{\sigma}_{-i}}, \alpha_i)_{i=1}^N$ is an EUA of Γ.

Suppose that $\bar{s}_i \in \text{supp}\left(\mathcal{C}^{\bar{\sigma}_i}\right)$. Then, by Lemma A.1, $\bar{s}_i \in \text{supp}\left(\bar{\sigma}_i\right)$. Since $\bar{\sigma}$ is a fixed point of ρ we have $\bar{\sigma}_i \in \text{argmax}_{\phi_i\in\Delta(S_i)} V_i(\phi_i|\mathcal{C}^{\bar{\sigma}_{-i}}, \alpha_i)$, and thus also

$\bar{s}_i \in \text{argmax}_{\phi_i \in \Delta(S_i)} V_i(\phi_i | \mathcal{C}^{\bar{\sigma}_{-i}}, \alpha_i)$.

This means that $\bar{s}_i$ is optimal given $\mathcal{C}^{\bar{\sigma}_{-i}}$ and α_i which finishes the proof. ■

Proposition 6.1 Assume that for $s = (s_1, \ldots, s_n) \in S$ there exist $s', s'' \in S$ and $\alpha_1, \ldots, \alpha_n \in [0, 1]$ such that for all $i \in \{1, \ldots, n\}$

$$\alpha_i \min\{u_i(s_i, s'_{-i}), u_i(s_i, s''_{-i})\} + (1 - \alpha_i) \max\{u_i(s_i, s'_{-i}), u_i(s_i, s''_{-i})\}$$
$$> \alpha_i \min\{u_i(s'_i, s'_{-i}), u_i(s'_i, s''_{-i})\} + (1 - \alpha_i) \max\{u_i(s'_i, s'_{-i}), u_i(s'_i, s''_{-i})$$

for all $s'_i \in S_i$. Then s can be modelled as a singleton EUA.

Proof. Let $s, s', s'' \in S$ and $\alpha_1, \ldots, \alpha_n$ be as in Proposition 6.1. The idea is to construct prior sets $\mathcal{C}^j$ such that s is in the support of the resulting prior sets $\mathcal{C}_{-i}$ and nonetheless sufficiently much decision weights go to s'_{-i} and s''_{-i}.

Consider $\epsilon > 0$ small. Consider the set $\mathcal{C}^j = \text{Conv}(\sigma^j_1, \sigma^j_2)$, where $\sigma^j_1, \sigma^j_2 \in \Delta(S_j)$ are probability distributions such that $\sigma^j_1(s_{-i}) = \epsilon$, $\sigma^j_1(s'_{-i}) = 1 - \epsilon$ and $\sigma^j_2(s_{-i}) = \epsilon$, $\sigma^j_2(s''_{-i}) = 1 - \epsilon$. The resulting overall belief of P_i is thus $\mathcal{C}_{-i} = \times_{j \neq i} \mathcal{C}^j$. First observe that $\text{supp}(\mathcal{C}_{-i}) = \{s_{-i}\}$ for all $\epsilon > 0$. For $s^-_i \in S_i$ denote by $x_{s^-_i} = \min_{s_{-i} \in S_{-i}} u_i(s^-_i, s_{-i})$ the worst consequence when P_i plays s^-_i. Denote by $y_{s^-_i} = \max_{s_{-i} \in S_{-i}} u_i(s^-_i, s_{-i})$ the best consequence when P_i plays s^-_i.

For ϵ sufficiently small we now have

$$\begin{aligned}
V_i(s_i) &= \alpha_i \min_{\sigma_{-i} \in \mathcal{C}_{-i}} u_i(s_i, \sigma_{-i}) + (1 - \alpha_i) \max_{\sigma_{-i} \in \mathcal{C}_{-i}} u_i(s_i, \sigma_{-i}) \\
&\geq (1 - \epsilon)^{n-1} \left[\alpha_i \min\{u_i(s_i, s'_{-i}), u_i(s_i, s''_{-i})\} + (1 - \alpha_i) \max\{u_i(s_i, s'_{-i}), u_i(s_i, s''_{-i})\}\right] \\
&\qquad + (1 - (1 - \epsilon)^{n-1}) x_{s_i} \\
&> (1 - \epsilon)^{n-1} \left[\alpha_i \min\{u_i(s'_i, s'_{-i}), u_i(s'_i, s''_{-i})\} + (1 - \alpha_i) \min\{u_i(s'_i, s'_{-i}), u_i(s'_i, s''_{-i})\}\right] \\
&\qquad + (1 - (1 - \epsilon)^{n-1}) y_{s'_i} \\
&\geq \alpha_i \min_{\sigma_{-i} \in \mathcal{C}_{-i}} u_i(s^\sigma_i, \sigma_{-i}) + (1 - \alpha_i) \max_{\sigma_{-i} \in \mathcal{C}_{-i}} u_i(s'_i, \sigma_{-i}) \\
&= V_i(s'_i)
\end{aligned}$$

for all $s'_i \in S_i \backslash \{s_i\}$, where the two inequalities follow from the fact that by construction the weight $\alpha_i(1 - \epsilon)^{n-1}$ is assigned to the worse out of s'_{-i} and s''_{-i} being played and weight

$(1-\alpha_i)(1-\epsilon)^{n-1}$ to the better out of s'_{-i} and s''_{-i} being played. ■

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