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Claudio Zoli is Lecturer, School of Economics, University of Nottingham

July 2000

Contents

Inverse sequential stochastic dominance: rank-dependent welfare, deprivation and poverty measurement.

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Abstract

We provide characterizations of sequential stochastic dominance conditions which are dual to those introduced in Atkinson and Bourguignon (1987). Instead of evaluating social welfare according to the utilitarian approach, we apply the dual approach to the measurement of welfare and inequality suggested in Weymark (1981) and Yaari (1987, 1988). Different interpretations of the results, in terms of either welfare comparisons of populations decomposed into needs-based subgroups, or intertemporal income comparisons are suggested.

The dual SWF is shown to be consistent with a class of satisfaction and deprivation indices. The sequential dominance criteria based on this class of indices is introduced, they require comparisons involving generalized satisfaction curves and deprivation curves of the reference groups in the population. The connections with dominance criteria associated to rank-dependent poverty indices and the sequential dominance results introduced is investigated.

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1 Introduction

The well known criterion of sequential stochastic dominance introduced in Atkinson and Bourguignon (1987) allows to compare, in terms of utilitarian social welfare functions, income distributions of populations decomposed into subgroups of individuals homogeneous in needs.[1] This evaluation is made under the hypothesis that the needier is an individual the higher is his/her marginal evaluation of income at any income level, and that this difference decreases as income increases. Karcher, Moyes and Trannoy (1995) apply similar tools in evaluating intertemporal income structures, basing welfare judgements upon the discounted sum of expected utilities. Moreover, related results are available in the context of poverty measurement. Atkinson (1992) considers dominance criteria which allow to compare distributions of incomes belonging to individuals with different needs, when poverty is evaluated according to the class of additively decomposable poverty indices. Even in this case the poverty reduction effects of marginal changes in incomes is supposed to be positively related to the needs. Jenkins and Lambert (1993) and Chambaz and Maurin (1998) extend some of the previous results, concerning respectively welfare and poverty rankings, to comparisons between populations showing different marginal distributions of needs.

All these criteria are strongly related to the utilitarian representation of social preferences, but no criterion is available for the dual social welfare representation based on linear rank-dependent evaluation functions.

In this paper instead of applying the utilitarian approach we evaluate need based social welfare functions or intertemporal distributions according to the dual approach introduced in Donaldson and Weymark (1980), Weymark (1981) and Yaari (1987, 1988)[2]. Preference orderings over income distributions are represented by a welfare function which is a weighted average of ordered incomes, where each income is weighted according to its position in the ranking.

As stated in Atkinson and Bourguignon (1989): "*It is possible that a dual approach might also be informative, although decomposition by population subgroups appears less tractable in the dual* (p.14)". The reason is that, since every income is weighted according to its position in the ranking, if we

[1] For a different approach to social welfare evaluations when needs differ see Ebert (1997) and particularly Ebert (2000a) which considers rank-dependent evaluation functions.

[2] See also Ebert (1988), Chew and Epstein (1989), Quiggin (1993), Ben Porath and Gilboa (1994), Weymark (1995) and Safra and Segal (1998).

confine our attention within subgroups, the ranking of every income changes in absolute terms with respect to that of other individuals' income evaluated over the whole population. Then also the weight associated with this income changes.

The family of rank-dependent evaluation functions fails therefore to satisfy the most common separability assumptions. Usually we require the welfare evaluation over an aggregate population to be independent from the way in which the population is partitioned into subgroups. If this is the case, the only relevant information to take into account in evaluating individuals' contributions to overall welfare is the level of their characteristics (income, needs, location, resources,..). Rank-dependent models, even in their linear representation, require the use of extra information concerning the relative position of individuals. This information should not in principle be affected by the decomposition procedure if we still want to use similar models in evaluating populations partitioned into subgroups. This peculiarity makes these representation models particularly suitable to be applied within the context of relative deprivation measurement. In this context, in order to obtain a *Dual SWF* consistent with the decomposition of population into homogeneous subgroups we need to suppose that every reference group within which each individual confines his/her aspirations, and compares his/her position, is *needs based.*

Social welfare could be seen as an aggregate of the welfare of each subgroup weighted by the share of individuals belonging to it, or in general by a function representing the policy maker opinion regarding the contribution of each group to the overall final welfare evaluation. It follows that every individual welfare contributes to the social welfare to extent to which it contributes to his/her community (or group) welfare[3].

Population is considered to be either continuous or discrete and denoted with S. It is partitioned into non overlapping and exhaustive population subgroups S^i, and $Y^i(s) \geq 0$ is the income level of agent $s \in S^i$, therefore Y^i is called the *income profile of group* i, while the whole distribution income profile is denoted with Y. If $S^i = \{s_1, s_2, ...s_l, ..s_{m_i}\}$, with each agent s_l showing equal weight $1/m_i$, we have the m_i^{th}*-dimensional discrete case,* where $Y^i(s_l) = y_l^i$, $l = 1, 2, ..m_i$ and the income profile of group i is represented by the m_i^{th}-dimensional vector $\mathbf{y}^i = (y_1^i, y_2^i, .y_l^i, .y_{m_i}^i)$. The set of all m^{th} dimensional vectors such that $Y^i(s) \geq 0$, and at least for one individual

[3]See Ok and Lambert (1999) for a similar approach within the utilitarian framework.

$Y^i(s) > 0$ is denoted $\mathbf{Y}^m$, while $\mathbf{Y}$ denotes all such vectors for all $m > 1$.

Denoting with $F_i(y)$ the cumulative income distribution (continuous or discrete) of subgroup i of the population F for $i = 1, 2, ..n$. The share of individuals belonging to group i is q_i^F, it follows $F(y) = \sum_{i=1}^n q_i^F F_i(y)$ where $F(y)$ is the cumulative income distribution function of an income profile with support $(0, +\infty)$ and finite mean $\mu(F) = \int_0^{+\infty} ydF(y)$. Let $\mathcal{F}$ the set of all such cumulative distributions.

Moreover, $F_i^{-1}(p) = \inf\{y : F_i(y) \geq p\}$ with $p \in [0, 1]$ is the left continuous inverse of $F_i(y)$ showing the income y of an individual at the $100p^{th}$ percentile of the distribution of group i.

The decomposable *Dual SWF* can be represented as:

$$W(F) = \sum_{i=1}^{n} q_i^F \int_0^1 v_i(p) F_i^{-1}(p) dp. \tag{1}$$

where $v_i(p) \geq 0$ is the weight attached to the income of an individual ranked at the $100p^{th}$ percentile in group i, which we will consider being continuous and twice differentiable[4]. At this preliminary stage we will take this welfare formulation as the primitive concept, and we will discuss the partial ranking criteria associated with subsequent restrictions on the set of groups income weights. However, it is possible to provide justifications for the welfare representation in (1), and also different interpretations of the partial orderings obtained.

Notice that since $W(F)$ is defined over distribution functions it is *within group anonymous* and *population replication invariant*. That is, welfare evaluations within each subgroup depend only on the income distribution and are invariant w.r.t. replications of the income profile. In the discrete case these properties corresponds to the situation in which evaluation is invariant w.r.t. permutations of the income profiles $Y^i = (y_1^i, y_2^i, .y_l^i, .y_{m_i}^i)$, and replication of the whole profile Y, where to each individual is associated a finite number of "clones".[5]

[4]Notice that for the empirical case $W(F)$ reduces to

$$W(\mathbf{y}) = \sum\nolimits_{i=1}^{n} \frac{m_i}{\sum_{i=1}^n m_i} \sum\nolimits_{l=1}^{m_i} y_{(l)}^i \left(V^i[l/m_i] - V[(l-1)/m_i] \right)$$

where $y_{(l)}^i \leq y_{(l+1)}^i$ and $V^i(t) = \int_0^t v(p)dp$.

[5]See Dalton (1920), and Dasgupta, Sen and Starrett (1973).

The next section discusses some plausible restrictions on the set of income weights, introduces the basic results, and provides some comments.

In Section 3 we will introduce a class of individual deprivation (and satisfaction) indices, and the related aggregate indices of total deprivation (and satisfaction) within the population. It will be shown that the class of aggregate satisfaction indices obtained provides an interpretation for (1). The indices suggested differ from the standard one because they aggregate the average income gaps, measuring the extent of the feeling of deprivation of each individual, weighting them with an "envy parameter" which depends on the income ranking of the individual considered. Different assumptions regarding the correlation between the envy coefficients and the needs of the individuals will allow to specify sequential deprivation and satisfaction criteria. Moreover, some considerations will be made regarding a generalization of variable population evaluations within the context of additive welfare representations. In particular in characterizing the aggregate deprivation (and satisfaction) index it will be made use of a general version of population replication property which will allow to obtain a parametric characterization of the variable population evaluations.

The final section introduces a general version of the Sen (1976), and Kakwani (1980) rank-dependent poverty indices. It provides some partial ranking criteria which are associated with the unanimous ranking of distributions in terms of those indices, and highlights the similarities between the poverty rankings and the results obtained for the *Dual SWF.* Moreover, some plausible extensions to the set of rank-dependent poverty indices, comprising the revision of the Sen index suggested by Shorrocks (1995), and of the needs based poverty ranking criteria suggested in Atkinson (1992), Jenkins and Lambert (1993) and Chambaz and Maurin (1998) are suggested. These are consistent with the basic results discussed in next section, and provide a rationale for using transformations of the Jenkins and Lambert (1997) TIPs curves or Shorrocks (1998) poverty gap profiles[6].

2 Properties and fundamental results

The welfare function in (1) could be interpreted as a needs based SWF. In what follows we will consider restrictions on the set of weights based on normative grounds. The approach is in spirit similar to the one suggested

[6]See also Spencer and Fisher (1992).

by Atkinson and Bourguignon (1987), Bourguignon (1989) and Jenkins and Lambert (1993) although it is applied to the rank-dependent welfare representation. We will also make use of arguments regarding inter and intra groups comparisons of effectiveness of transfers occurring both between and within groups, which allow for an immediate characterization of the obtained sequential dominance criteria in terms of transfer principles. The characterization suggested could therefore be considered dual to the one obtained in Ebert (2000) for Atkinson-Bourguignon sequential generalized Lorenz dominance criteria.

As a starting point it seems plausible to impose that welfare in (1) is a non-decreasing function for all individuals' incomes. This *welfare monotonicity* property corresponds to a specification of non negative weights $v_i(p)$. Moreover, higher concern in the evaluation function is given to the income of individuals belonging to needier groups. Given the linearity of the function in terms of incomes, this condition requires to give higher weight to the income of needier individuals, at a given position in the ranking. In comparing two individuals in the same position but with differing needs, the social welfare function in (1) should exhibit *concern for needs*, that is attach higher weight to the income of the needier individual. Therefore, *if needs are ranked in decreasing order*, that is individuals in group i are needier than those in group $i+1$, then $v_i(p) \geq v_k(p)$ for any $i < k$ and for any p. In other words, both *welfare monotonicity* and *concern for needs* applied over (1) could be formalized as follows:

$$v_i(p) = \sum_{j=i}^{n} w_j(p), \quad w_j(p) \geq 0 \quad \forall i = 1, 2, ..n; \quad \forall p \in [0, 1]. \text{ (\textbf{Property A})}$$

Where $w_i(p)$ is a non-negative, continuous and twice differentiable function which measures the gap between the weights attached to the income at the p^{th} position in the income ranking in group i and income at the same position in group $i+1$. Notice that the use of a linear rank-dependent SWF, in the absence of any other relevant information, forces to make interpersonal comparisons only between individuals situated at the same ranking in the different subgroup incomes distributions. Such comparisons are therefore independent from the absolute income levels. This restriction comes from the specific separability or independence properties underlying the dual welfare representation[7]. As a result the relative position of individuals within their homogeneous group could play a normative role.

If the social decision maker is inequality averse in evaluating the welfare of every subgroup then $v_i'(p) \leq 0 \ \forall i = 1, 2...n$, and $p \in [0, 1]$, (see Mehran, 1976, and Yaari 1987, 1988).

Property A as well as inequality aversion could also be obtained as a result of a combination of welfare monotonicity and the requirement of satisfaction of a needs based version of the Principle of Transfers. The *Needs based (Positional) Principle of Transfers* states that

Axiom 1 (Needs Based Positional PT) *A small transfer of a given amount of income $\delta > 0$ from a tiny fraction dp of the population at the $100(p + \rho)^{th}$ percentile of the distribution of group i to a fraction dp of individuals at the $100p^{th}$ percentile of the distribution of group j, where $\rho \geq 0$ and $j \leq i$ does not lead to a welfare decrease.*

Such transfer is called *progressive transfer,* transfers of opposite sign are called regressive[8]. The welfare effect of a progressive transfer, according to (1), is formally

$$\delta v_i(p) - \delta v_j(p + \rho) \geq 0 \quad \forall p, \rho, i > j$$

If ρ tends to 0, given welfare monotonicity, we get Property A $v_i(p) \geq v_j(p + \rho) \geq 0$. When $j = i$ we get the standard Principle of Transfers applied over homogeneous populations which requires within our context $v_i'(p) \leq 0$.

Given property A, inequality aversion implies therefore $\sum_{j=i}^{n} w_j'(p) \leq 0 \ \forall i, p$. The following assumption strengthen this condition:

$$w_j'(p) \leq 0 \quad \forall j = 1, 2...n, \quad \forall p \in [0, 1]. \qquad \textbf{(Property A1)}$$

This property is equivalent to the requirement that:

Axiom 2 (Between Group Positional Transfer Sensitivity (BGPTS)) *The impact on the social welfare of a progressive transfer involving individuals within the same group is greater, ceteris paribus, the needier is the group.*

[7]See Weymark (1981), Yaari (1987, 1988), Ben Porath and Gilboa (1994).

[8]For a discussion of the role of progressive transfers in inequality and welfare measurement see Mosler and Muliere (1998).

That is, consider a small progressive transfer $\delta > 0$ from a tiny fraction dp of the population at the $100(p+\rho)^{th}$ percentile of the distribution to a fraction dp of poorer individuals at the $100p^{th}$ percentile both belonging to group i. The impact of this transfer on W is:

$$\Delta_i W(p,\rho,\delta) = -\delta v_i(p+\rho)dp + \delta v_i(p)dp \geq 0 \tag{2}$$

Compare this effect to that of a similar transfer involving individuals at the same percentiles but belonging to group $k > i$: $\Delta_k W(p,\rho,\delta) = -\delta v_k(p+\rho)dp + \delta v_k(p)dp \geq 0$. Given Property A, Property A1 is equivalent to impose that the first transfer shows a greater impact on W for any $p, \rho, \delta, k > i$. Formally, $\Delta_i W(p,\rho,\delta) \geq \Delta_k W(p,\rho,\delta) \geq 0$ that is

$$v_i(p+\rho) - v_i(p) \leq v_k(p+\rho) - v_k(p) \qquad \forall p, \rho, k > i$$

which is equivalent to $v_i'(p) \leq v_k'(p)$ $\forall p \in [0,1]$, $i, k = 1, 2...n$, such that $k > i$, which is the requirement of Property A1.

The welfare representation in (1) consistent with the suggested properties, has different justifications and allows for various economic interpretations of the results. Moreover it could be informative in other frameworks when appropriate variables are substituted to the income. For instance in section 4 an application of the results associated to (1) in the poverty measurement will be provided, in that context incomes are substituted by relative or absolute income gaps.

The social evaluation in (1) can also be seen as an intertemporal evaluation of income distributions, where $F_i(y)$ denotes the distribution of income at time $t+i-1$, and q_i^F is the size of the population at that time. The present value of the intertemporal social welfare is a weighted average of the welfare in every period evaluated according to the dual approach. Time impatience is incorporated into the function by considering lower weights $v_i(p)$ for future periods, as stated in Property A. The social welfare evaluation shows inequality aversion in every period. Moreover, Property A1 could be interpreted as an effect of time impatience in implementation of redistributive policies, that is, given the same redistributive effect of a policy, the welfare impact is considered greater the sooner the policy is implemented.[9]

[9]A special case is the one in which every welfare function in any period is evaluted applying the same set of weights, i.e. $v_i(p) = v_j(p) = v(p)$ $\forall i, j$, such that $v(p) \geq 0$, and $v'(p) \leq 0$, but the welfare of any period is discounted according to a rate $\delta^{i-1}, 0 \leq \delta \leq 1$. Then $v_i(p) = \delta^{i-1} v(p)$ satisfies Properties A and A1.

In their extension of the Atkinson-Bourguignon dominance criteria to populations with different marginal distributions of needs, Jenkins and Lambert (1993) suggested an additional condition, namely that there exists a top level of income at which differences in needs do not play any role in the evaluation[10]. In our context an analogous condition could be imposed. Suppose that the weight associated to the income of the richest individual in every subgroup is the same, that is whatever is the income or the needs of every individual, being the richest in a subgroup is evaluated in the same way:

$$v_i(1) = v_k(1) \qquad \forall i, k = 1, 2, ...n. \qquad \textbf{(Property A*)}$$

Such a condition is more restrictive than the version suggested by Jenkins and Lambert (1993): it requires that being at the top of the distribution within a group is evaluated in the same way irrespective of incomes and needs. As we will show, for all the dual sequential dominance results we obtain, for populations showing different marginal distributions of needs, this condition is in fact irrelevant.

Before introducing the first results, it should be worth specify the meaning of *inverse stochastic dominance.* This dominance criterion has been suggested by Muliere and Scarsini (1989). For $F \in \mathcal{F}$, and $K = 1, 2, ..$ we denote[11]

$$F^{-1}_{[1]}(p) = F^{-1}(p), \quad F^{-1}_{[K]}(p) = \int_0^p F^{-1}_{[K-1]}(t)dt, \qquad 0 \leq p \leq 1$$

and define the K-th degree inverse stochastic dominance $\succcurlyeq_{-K}$ as follows:

Definition 1 *Given two income distributions F and $G \in \mathcal{F}$, $F \succcurlyeq_{-K} G$ if and only if $F^{-1}_{[K]}(p) \geq G^{-1}_{[K]}(p)$ for all p, $0 \leq p \leq 1$.*

As shown by Muliere and Scarsini (1989) direct and inverse stochastic dominance are not equivalent for degrees higher than the second. In what

[10] Moyes criticizes this condition arguing that the dominance results associated to Jenkins and Lambert's (1993) procedure are not independent from the choice of the top level. Therefore two pairs of distributions differing only in the top level could be ranked in different ways. Since the top income level is evaluated in the same way for all needs its choice in general should not have a relevance for the final solution, at least within an utilitarian context. According to Moyes's remark it does.

[11] Notice that $F_i^{-1}(p)$ is the inverse income distribution of group i, while $F^{-1}_{[K]}(p)$ is the K^{th} degree inverse distribution function of population F. Therefore $F^{-1}_{i[K]}(p)$ is the K^{th} degree inverse distribution function of $F_i(y)$.

follows we will discuss sequential dominance conditions which are dual to the well know sequential stochastic dominance in Atkinson and Bourguignon (1987), in that they are associated to the inverse approach.

2.1 Some results

Denote by $\mathcal{Y}$ the set of social welfare functions $W(.)$ in (1) satisfying Property A, and by $\mathcal{Y}_1$ the set of social welfare functions $W(.)$ in $\mathcal{Y}$ satisfying also Property A1.

In what follows we will provide a set of dominance criteria which are both necessary and sufficient for welfare dominance for all Dual SWFs in $\mathcal{Y}$ and $\mathcal{Y}_1$ respectively. After a discussion of these preliminary results, a weaker dominance criterion is suggested which is consistent with welfare dominance for a subset of $\mathcal{Y}_1$.

A complete proof is provided in the main text only for the first proposition. For all other results in this section only the sufficiency parts of the proofs follow the propositions. The necessity parts, which are more tedious and less interesting, are to be found in the appendix.[12]

Proposition 1 *Given two distributions* F, $G \in \mathcal{F}$, $W(F) \geq W(G)$ *for all SWFs in* $\mathcal{Y}$ *if and only if* $\sum_{i=1}^{k} \phi_i(p) \geq 0$ *for any* $k = 1, 2, ...n$ *and every* $p \in [0,1]$, *where* $\phi_i(p) = q_i^F F_i^{-1}(p) - q_i^G G_i^{-1}(p)$.

Proof:
Sufficiency: Denoting $\Delta W = W(F) - W(G)$, we obtain:

$$\Delta W = \sum_{i=1}^{n} \int_0^1 v_i(p)[q_i^F F_i^{-1}(p) - q_i^G G_i^{-1}(p)]dp = \sum_{i=1}^{n} \int_0^1 v_i(p)\phi_i(p)dp. \qquad (3)$$

Since $v_i(p)$ satisfies Property A, substituting for $v_i(p) = \sum_{j=i}^{n} w_j(p)$, we have:

$$\Delta W = \int_0^1 \sum_{i=1}^{n} \left\{ [\sum_{j=i}^{n} w_j(p)]\phi_i(p) \right\} dp.$$

[12] The proofs of Propositions 1 and 2 follow the same arguments introduced in Atkinson and Bourguignon (1987).

Rearranging, this becomes:

$$\Delta W = \int_0^1 \sum_{i=1}^{n} \left(w_i(p) \sum_{j=1}^{i} \phi_j(p) \right) dp. \tag{4}$$

Then, since $w_i(p) \geq 0 \ \ \forall i, p$, the condition $\sum_{j=1}^{i} \phi_j(p) \geq 0$ for any $i = 1, 2, ..n$ is sufficient for ΔW being non-negative.

Necessity: We need to use the following lemmas[13]:

Lemma 1 *Let $I = [0, 1]$ be an interval, V the set of all continuous functions over I, and V^+ the set of all continuous non-negative functions over I. If $\{w_1(p), ..w_i(p), ..w_n(p)\}$ is a set of continuous functions over I, $w_i(p) \in V \ \forall i$ and $\omega_i(p) \in V^+ \ \forall i$, then $\sum_{i=1}^{n} w_i(p)\omega_i(p) \in V^+$ if and only if $w_i(p) \in V^+ \ \forall i$.*

Lemma 2 *Following the same notation as in Lemma 1, let $\psi(p)$ be a continuous function over the interval I in $[0, 1]$. Then $\int_0^1 \psi(p)z(p)dp \geq 0, \ \forall z(p) \in V^+$ if and only if $\psi(p) \in V^+$.*

Let $\Delta_k(p) = \sum_{i=1}^{k} \phi_i(p)$. Suppose there exists a k such that $\Delta_k(p) < 0$ for some $p \in [0, 1]$. Then, following Lemma 1, there exists an interval $\mathcal{J} \subset [0, 1]$ and a set of values $\hat{w}_1(p), \hat{w}_2(p), ...\hat{w}_i(p), ...\hat{w}_n(p) \in V^+$; such that $\sum_{i=1}^{n} \hat{w}_i(p)\Delta_i(p) < 0$ for each $p \in \mathcal{J}$. Following Lemma 2 we obtain:

$$\int_0^1 z(p) \left(\sum_{i=1}^{n} \hat{w}_i(p)\Delta_i(p) \right) dp < 0 \tag{5}$$

for some $z(p) \in V^+$. Define: $v_i(p) = \sum_{j=i}^{n} \hat{w}_j(p)z(p)$ where $z(p) \in V^+$. Since $\hat{w}_j(p) \in V^+$, $v_i(p)$ satisfies Property A, indeed $v_i(p) - v_{i+1}(p) = \hat{w}_i(p)z(p) = w_i(p) \geq 0$. Substituting into (4) we obtain:

$$\Delta W = \int_0^1 \sum_{i=1}^{n} (\hat{w}_i(p)z(p)\Delta_i(p))\, dp \geq 0. \tag{6}$$

[13] Lemma 1 has been introduced in Chambaz and Maurin (1998), it is a continuous function version of Lemma 2 in Atkinson and Bourguignon (1987). It provides necessary conditions for the standard sequential stochastic dominance when the shares of populations q_i^F and q_i^G do not coincide. Lemma 2 in the text, which is a straightforward extension of the previous, appears as Lemma 1 in Atkinson and Bourguignon (1987).

The assumption of welfare dominance for all SWFs in $\mathcal{Y}$ implies the relation in (6) which is inconsistent with (5). Then $\Delta_k(p)$ must be non-negative for any k and p. ■

That is, the necessary and sufficient condition for first order welfare dominance evaluated according to any SWF belonging to $\mathcal{Y}$ is a *sequential inverse stochastic dominance criterion of first order*, or in other words a *sequential rank-dominance* condition[14] in which each inverse subgroup distribution is weighted according to the share (or size, if we apply the intertemporal valuation procedure) of its population q_i. This criterion is dual to that suggested by Atkinson and Bourguignon (1987), who required first order direct sequential dominance (that is a criterion where the weighted averages of distributions functions, aggregated at every level of income, are compared). In our case, the aggregation and the comparisons take place at every percentile as a result of the application of the rank-dependent model where appropriate restrictions are imposed on the weights.

We denote the dominance conditions introduced as

$$F \succcurlyeq_{-1}^{k} G \Leftrightarrow \sum_{i=1}^{k} \phi_i(p) \geq 0 \quad \forall p \in [0,1],$$

which is first order sequential inverse stochastic dominance at stage k, while $F \succcurlyeq_{-1}^{S} G$ denotes first order sequential inverse stochastic dominance of F on G, that is

$$F \succcurlyeq_{-1}^{S} G \Leftrightarrow F \succcurlyeq_{-1}^{k} G \quad for\ all \quad k = 1, 2, ..n.$$

We now turn to dominance conditions for all $W \in \mathcal{Y}_1$.

Proposition 2 *Given two distributions F, $G \in \mathcal{F}$, $W(F) \geq W(G)$ for all SWFs in $\mathcal{Y}_1$ if and only if $\sum_{i=1}^{k} \psi_i(p) \geq 0$ for any $k = 1, 2, ...n$ and every $p \in [0,1]$, where $\psi_i(p) = \int_0^p \phi_i(t)dt$.*

Proof:

Sufficiency: Integrating by parts in (4) we obtain:

$$\Delta W = -\sum_{i=1}^{n} \int_0^1 \left([\sum_{j=i}^{n} w_j'(p)] \int_0^p \phi_i(t)dt \right) dp + \sum_{i=1}^{n} \left[[\sum_{j=i}^{n} w_j(p)] \int_0^p \phi_i(t)dt \right]_0^1 . \tag{7}$$

[14] The rank-dominance condition simply requires that at every relative position in the income parade the dominating distribution shows higher incomes, see Saposnik (1981).

Simplifying and substituting:

$$\Delta W = -\sum_{i=1}^{n}\int_{0}^{1}[\sum_{j=i}^{n} w_j'(p)]\psi_i(p)dp + \sum_{i=1}^{n}\left([\sum_{j=i}^{n} w_j(1)]\psi_i(1)\right). \qquad (8)$$

Rearranging we obtain:

$$\begin{aligned}\Delta W &= -\int_{0}^{1}\sum_{i=1}^{n}[\sum_{j=i}^{n} w_j'(p)]\psi_i(p)dp + \sum_{i=1}^{n}\left(w_i(1)\sum_{j=1}^{i}\psi_j(1)\right) \qquad (9)\\ &= -\int_{0}^{1}\sum_{i=1}^{n} w_i'(p)[\sum_{j=1}^{i}\psi_j(p)]dp + \sum_{i=1}^{n}\left(w_i(1)\sum_{j=1}^{i}\psi_j(1)\right).\end{aligned}$$

Since $w_i(1) \geq 0$ and $w_i'(p) \leq 0$ $\forall i = 1, 2, ...n$, $p \in [0, 1]$, then $\sum_{j=1}^{i}\psi_j(p) \geq 0$ for any i, and $p \in [0, 1]$, becomes a sufficient condition for ΔW being non negative.

Necessity: See appendix ■

Notice that, rewriting $\psi_i(p)$ explicitly in terms of the inverse distribution function, and denoting

$$GL_{F_i}(p) = \int_{0}^{p} F_i^{-1}(t)dt,$$

which is the generalized Lorenz curve of distribution F_i, we obtain:

$$\begin{aligned}\psi_i(p) &= \int_{0}^{p} q_i^F F_i^{-1}(t)dt - \int_{0}^{p} q_i^G G_i^{-1}(t)dt = q_i^F\int_{0}^{p} F_i^{-1}(t)dt - q_i^G\int_{0}^{p} G_i^{-1}(t)dt\\ &= q_i^F GL_{F_i}(p) - q_i^G GL_{G_i}(p). \qquad (10)\end{aligned}$$

Thus, $\sum_{i=1}^{k}\psi_i(p) \geq 0$ for any $k = 1, 2, ...n$ and every $p \in [0, 1]$ means:

$$\sum_{i=1}^{k} q_i^F GL_{F_i}(p) \geq \sum_{i=1}^{k} q_i^G GL_{G_i}(p) \qquad \forall k = 1, 2, ...n;\ \forall p \in [0, 1].$$

That is, in order to check welfare dominance for all SWFs in $\mathcal{Y}_1$, we have to compare at every percentile p, and every stage k, a weighted average of generalized Lorenz curves of all groups with index of needs not higher than k belonging to the two distributions, where the weights are given by the shares of the population within each subgroup. The link with the traditional sequential dominance criterion is clear: while Atkinson-Bourguignon require sequential weighted averages of curves associated to the second degree stochastic dominance condition for every group, the criterion we introduce requires convex combination of the curves associated to the second degree *inverse* stochastic dominance condition, that is the generalized Lorenz curves.

While direct and inverse stochastic dominance are equivalent if we consider the first and the second degree of comparison, this is not the case when we consider sequential dominance conditions. In both cases, only the first stage of the sequential procedure is equivalent between the direct and inverse approach, namely dominance for the neediest group. Notice that the final stage in the *direct* procedure corresponds to respectively first degree or generalized Lorenz dominance for all the population according to the degree of refinement of the ordering. The final stage in the *inverse* procedure corresponds to an average over all subgroups of the inverse distribution functions, in the first proposition, or an average of the generalized Lorenz curves, in the second proposition. These conditions in principle are different from those associated to the direct procedure. The reason is that individuals' incomes are aggregated at each percentile, therefore the final conditions correspond to a first or second degree stochastic dominance criterion applied over distributions obtained averaging the incomes of the individuals belonging to the same percentile within each subgroup. Moreover, notice that in the extreme case in which individuals are all distinct, that is each individual constitutes a class of needs, the results in the previous propositions require in both cases generalized Lorenz dominance evaluated over income distributions ranked according to needs instead of incomes.

The dominance conditions introduced are denoted

$$F \succsim_{-2}^{k} G \Leftrightarrow \sum_{i=1}^{k} \psi_i(p) \geq 0 \qquad \forall p \in [0,1],$$

which means second order inverse sequential stochastic dominance at stage k, while $F \succsim_{-2}^{S} G$ denotes second order sequential inverse stochastic dominance of F w.r.t. G, while

$$F \succsim_{-2}^{S} G \Leftrightarrow F \succsim_{-2}^{k} G \qquad for \;\; all \;\; k = 1, 2, ..n.$$

An interesting class of dominance conditions is obtained if we restrict attention to a subclass of $\mathcal{Y}_1$ considering the set of evaluation functions exhibiting linear weights. As it could be expected, since for this class of SWFs every subgroup welfare function becomes a linear transformation of the average income and Gini coefficient of the distribution, the dominance conditions involve sequential comparisons of the Gini-based welfare functions of the subgroups, and sequential means comparisons. Denote $\mu(F_i)$ the average income of distribution F_i, and let

$$\Xi_\Gamma(F_i) = \mu(F_i)[1 - \Gamma(F_i)]$$

the Gini based welfare index, interpreted as the *equally distributed equivalent income* (see Blackorby and Donaldson, 1978), where inequality is evaluated according to the Gini coefficient $\Gamma(.)$.

Remark 1 *$W(F) \geq W(G)$ for all SWFs in $\mathcal{Y}_1$ which are linear in $v_i(p)$ if and only if $\mathcal{W}_i(F) - \mathcal{W}_i(G) \geq 0$ and $\hat{\mu}_i(F) - \hat{\mu}_i(G) \geq 0$ $\forall i = 1, 2, .., n$ where $\mathcal{W}_i(F) = \sum_{j=1}^{i} q_j^F \Xi_\Gamma(F_j)$, $\hat{\mu}_i(F) = \sum_{j=1}^{i} q_j^F \mu(F_j)$*

Proof:

Notice that all SWF in $\mathcal{Y}_1$ for which $v_i(p)$ is linear could be characterized by using only two parameters for each subgroup, namely $v_i'(p) = -\nu_i$, $\nu_i \geq 0$ $\forall i, p$, where $w_i'(p) = -\omega_i \leq 0$ (therefore $\nu_i = \sum_{j=i}^{n} \omega_j$), and $v_i(1) = \vartheta_i \geq 0$ where $\vartheta_i = \sum_{j=i}^{n} \zeta_j$, $\zeta_j = w_j(1) \geq 0$. Substituting into (9) we obtain:

$$\Delta W = \sum_{i=1}^{n} \omega_i \int_0^1 \Psi_i(p)dp + \sum_{i=1}^{n} \left(\zeta_i \Psi_i(1)\right). \tag{11}$$

where $\Psi_i(p) = \sum_{j=1}^{i} \psi_j(p)$.

Notice that $\Psi_k(p) = \sum_{i=1}^{k} q_i^F GL_{F_i}(p) - \sum_{i=1}^{k} q_i^G GL_{G_i}(p)$, and that $\Psi_k(1) = \sum_{i=1}^{k} q_i^F \mu(F_i) - \sum_{i=1}^{k} q_i^G \mu(G_i)$. Moreover, recall that

$$\int_0^1 GL_{F_i}(p)dp = \frac{1}{2}\mu(F_i)[1 - \Gamma(F_i)] = \frac{\Xi_\Gamma(F_i)}{2}$$

where $\Gamma(F_i)$ is the Gini coefficient of distribution F_i. Thus, substituting, we have:

$$\Delta W = \sum_{i=1}^{n} \left[\frac{\omega_i}{2} \left[\mathcal{W}_i(F) - \mathcal{W}_i(G)\right] + \zeta_i \left[\hat{\mu}_i(F) - \hat{\mu}_i(G)\right]\right].$$

Since ω_i and ζ_i are independent, $\mathcal{W}_i(F) \geq \mathcal{W}_i(G)$ and $\hat{\mu}_i(F) \geq \hat{\mu}_i(G)$ are not only sufficient but also necessary conditions for ΔW being not-negative. ■

The functions $\mathcal{W}_i(F)$ and $\hat{\mu}_i(F)$ could be considered as sequential Gini based welfare indices and sequential averages of the total distribution. Therefore, to every distribution partitioned into subgroups we could associate two sets of n indicators: the set of inequality averse welfare indices $\mathcal{W}_i(F)$, and those of the inequality neutral $\hat{\mu}_i(F)$. In comparing subgroups where individual incomes are equally distributed, $\mathcal{W}_i(F)$ boils down to $\hat{\mu}_i(F)$; when incomes are unequally distributed, weighted averages of the Gini coefficients of the distributions play a role. Adopting an extension of the equally distributed equivalent income approach to welfare measurement (adapted to the situation involving the functions in $\mathcal{Y}_1$) we may interpret $\mathcal{W}_i(F)$ as

$$\mathcal{W}_i(F) = \hat{\mu}_i(F)\left(1 - \mathcal{G}_i(F)\right)$$

where $\mathcal{G}_i(F)$ could be considered the sequential Gini coefficient of the distribution F evaluated at the i^{th} stage. Rearranging $\mathcal{W}_i(F)$ and $\hat{\mu}_i(F)$ it is immediate to derive

$$\mathcal{G}_i(F) = \frac{\sum_{j=1}^{i} q_j^F \mu(F_j)\Gamma(F_j)}{\sum_{j=1}^{i} q_j^F \mu(F_j)}. \qquad (12)$$

Therefore $\mathcal{G}_i(F)$ is the Gini coefficient which, if common to all subgroups with need index lower than i, leads to the same level of sequential absolute inequality. Notice that, because of the aggregation properties satisfied by (1), in the inequality evaluation of the distribution of the first i subgroups, what is relevant is the existing inequality within each group and not the levels of income: the between-group comparisons component of the overall inequality evaluation is eliminated. This is the reason why the relative deprivation interpretation of the function in (1) seems to be the most promising: individuals compare themselves only within their reference group, therefore the overall perception of inequality is confined to the within-group components. Another interpretation of this behavior is provided in Peragine (1998) where the Dual SWF specified over income distributions is applied for the purpose of evaluating inequalities in terms of opportunities, following Roemer (1998) approach, over populations partitioned into groups where individuals are homogeneous in opportunities but not in incomes.

2.2 Extensions

When two distributions cannot be ranked unanimously according to the previous criteria it is possible to refine these partial orderings adding further restrictions on the set of weights. These restrictions could be associated to extra normative properties of the original welfare criterion applied. If we restrict our attention to within-group comparisons, the most natural additional requirement is the satisfaction of the property of transfer sensitivity. Given the structure of the welfare ordering, the appropriate criterion to apply is the *Principle of Positional Transfer Sensitivity (PPTS)*, discussed in Mehran (1976), Kakwani (1980) and Zoli (1999). This requires that a small transfer from a richer to a poorer individual, *with a given proportion of the population in between them*, is valued more if it occurs at lower income levels. In other words any *Elementary Favorable Composite Positional Transfer (EFCPT)*[15] should not lead to a welfare loss.

Definition 2 (EFCPT) *An* ***Elementary Favorable Composite Positional Transfer (EFCPT)*** *is a regressive and a progressive transfer both of the amount* $\delta > 0$*, from a fraction dp of individuals at 100*p^{th} *percentile to individuals at the 100*$(p+\rho)^{th}$*, and from individuals at 100*q^{th} *percentile to individuals at the 100*$(q-\rho)^{th}$*, where* $\rho > 0$*, and* $p > q$*.*

It is straightforward to check that in the present context PPTS is equivalent to $v_i''(p) \geq 0 \ \forall i, p$ (see Mehran, 1976, see also Zoli, 1999). Given property A, we can rewrite this condition as $\sum_{j=i}^{n} w_j''(p) \geq 0 \ \forall i, p$,

When we extend our comparisons in order to add needs based inter-group evaluations of the effects of the transfers, it seems plausible to require that:

Axiom 3 (Between Group Effectiveness of PPTS) *The welfare effect of EFCPTs of the same kind (same* δ*,* p*,* q*,* ρ*) applied to different groups could not be lower the higher is the needs level of the group.*

More precisely, consider the impact on the social welfare of a EFCPT of amount $\delta > 0$ involving individuals within the same group i, composed of a regressive transfer from a tiny fraction dp of the population at the $100p^{th}$ percentile to a fraction dp of individuals at the $100(p+\rho)^{th}$ percentile, and

[15] See Zoli (1997, 1999).

a progressive transfer of the same amount involving fractions of poorer individuals, situated respectively at the $100(q+\rho)^{th}$ and $100q^{th}$ percentile, where $p = q + \varepsilon$, $\varepsilon \geq 0$. The impact of this transfer on W is:

$$\begin{aligned}\Lambda_i(q,p,\rho,\delta) &= \Delta_i W(q,\rho,\delta) - \Delta_i W(p,\rho,\delta) = \\ &= -\delta v_i(q+\rho)dp + \delta v_i(q)dp + \delta v_i(p+\rho)dp - \delta v_i(p)dp = \\ &= \delta\left[(v_i(p+\rho) - v_i(p)) - (v_i(q+\rho) - v_i(q))\right]dp \geq 0. \quad (13)\end{aligned}$$

Comparing this effect to that of a similar EFCPT involving individuals at the same percentiles but belonging to group $k > i$, and requiring a *not higher* effectiveness of this second transfer, we get $\Lambda_i(q,p,\rho,\delta) - \Lambda_k(q,p,\rho,\delta) \geq 0$ $\forall q,p,\rho,\delta,i,k > i$. Making explicit this condition and simplifying for δ and dp it could be equivalently rewritten as

$$(v_i(p+\rho) - v_i(p)) - (v_i(q+\rho) - v_i(q)) \geq (v_k(p+\rho) - v_k(p)) - (v_k(q+\rho) - v_k(q)), \quad (14)$$

which for small ρ becomes:

$$v_i'(p) - v_i'(p-\varepsilon) \geq v_k'(p) - v_k'(p-\varepsilon). \quad (15)$$

That is for small ε, $v_i''(p) \geq v_k''(p)$ $\forall p,i,k > i$, which together with the within group PPTS property leads to the following condition

$$v_i''(p) \geq v_k''(p) \geq 0 \quad \forall p,i,k > i. \quad \textbf{(Property A2)}$$

Adding property A, it becomes:

$$w_j''(p) \geq 0 \quad \forall j = 1,2...n, \quad \forall p \in [0,1]. \quad (16)$$

We will consider the class of SWFs satisfying Properties A, A1, A2, and denote this class $\mathcal{Y}_2$. The following result suggests a normative justification for a refinement of the sequential dominance conditions previously introduced. As will be shown, $\mathcal{G}_i(.)$ the Gini coefficients of every ordered subset of the subgroups play an important role.

Proposition 3 *Given two distributions F, $G \in \mathcal{F}$, $W(F) \geq W(G)$ for all SWFs in $\mathcal{Y}_2$ if and only if $\sum_{i=1}^{k}\psi_i(1) \geq 0$ for any $k = 1,2,...n$ and $\sum_{i=1}^{k}\tau_i(p) \geq 0$ for any $k = 1,2,...n$ and every $p \in [0,1]$, where $\tau_i(p) = \int_0^p \psi_i(t)dt$.*

Proof:
Sufficiency: Integrating by parts (8) we obtain

$$\begin{aligned}\Delta W &= \sum_{i=1}^{n}\left([\sum_{j=i}^{n} w_j(1)]\psi_i(1)\right) + \sum_{i=1}^{n}\int_0^1 \left([\sum_{j=i}^{n} w_j''(p)]\int_0^p \psi_i(t)dt\right) dp \\ &\quad - \sum_{i=1}^{n}\left[[\sum_{j=i}^{n} w_j'(p)]\int_0^p \psi_i(t)dt\right]_0^1 . \end{aligned} \tag{17}$$

Simplifying and substituting:

$$\begin{aligned}\Delta W &= \sum_{i=1}^{n} w_i(1)\left(\sum_{j=1}^{i}\psi_j(1)\right) + \int_0^1 \sum_{i=1}^{n} w_i''(p)[\sum_{j=1}^{i}\tau_j(p)]dp \qquad (18) \\ &\quad - \sum_{i=1}^{n} w_i'(1)\left(\sum_{j=1}^{i}\tau_j(1)\right). \end{aligned}$$

Recalling that $\Psi_i(1) = \sum_{j=1}^{i}\psi_j(1) = \sum_{j=1}^{i}\left[q_j^F\mu(F_j) - q_j^G\mu(G_j)\right]$, and denoting $\Upsilon_i(p) = \sum_{j=1}^{i}\tau_j(p)$, rearranging we obtain:

$$\Delta W = \sum_{i=1}^{n}\left[w_i(1)\Psi_i(1) - w_i'(1)\Upsilon_i(1) + \int_0^1 w_i''(p)\Upsilon_i(p)dp\right]. \tag{19}$$

Since $w_i(p) \geq 0, w_i'(p) \leq 0, w_i''(p) \geq 0$ $\forall i, p$ then $\Psi_i(1) \geq 0$ $\forall i$ and $\Upsilon_i(p) \geq 0$ $\forall i, p$ are sufficient conditions for ΔW being non negative.

Necessity: see appendix ■

These criteria are *third degree inverse sequential stochastic dominance* conditions[16]. They require a comparison of a sequence of weighted averages of the curves associated to the third degree inverse stochastic dominance condition of every subgroup, and a sequence of weighted averages of mean incomes of every subgroup, that is $\hat{\mu}_i(F) \geq \hat{\mu}_i(G)$ $\forall i$ (as defined in remark 1). Once we restrict the analysis to a single group the usual third degree inverse stochastic dominance criterion and dominance in terms of means are

[16] Lambert and Ramos (2000) suggest a dual result related to the utilitarian framework.

required. This result is consistent with that presented in Zoli (1999)[17] concerning comparisons of homogeneous populations.

The similarities with the results in Zoli (1999) simplify the analysis concerning the normative significance of the family of indices $\mathcal{G}_i(F)$ in (12) when the second degree inverse sequential dominance conditions are not satisfied.

We will discuss only a specific case. All extensions follows from this case in the similarly as done for the comparisons involving homogeneous populations in Zoli (1999). Let

$$F \succcurlyeq_{-3}^{k} G \Leftrightarrow \sum_{i=1}^{k} \tau_i(p) \geq 0 \qquad \forall p \in [0,1],$$

represent third degree inverse sequential stochastic dominance of F w.r.t. G at stage k, while $F \succcurlyeq_{-3}^{S} G$ denotes overall third degree inverse sequential stochastic dominance of F w.r.t. G i.e.

$$F \succcurlyeq_{-3}^{S} G \Leftrightarrow F \succcurlyeq_{-3}^{k} G \quad for \ \ all \ \ k = 1, 2, ..n.$$

We consider a situation in which there exists a stage k of the sequential comparison at which the curves $\sum_{i=1}^{k} q_i^F GL_{F_i}(p)$ and $\sum_{i=1}^{k} q_i^G GL_{G_i}(p)$ intersect once, while for all remaining stages distribution F dominates sequentially G, i.e. $F \succcurlyeq_{-2}^{j} G$ for all $j \neq k$, $j = 1, 2, ..n$. In this case it is impossible to obtain unambiguous ranking if we consider all SWFs in $\mathcal{Y}_1$. If we restrict the class of evaluation functions to SWFs in $\mathcal{Y}_2$, under some extra conditions it is possible to reach an unanimous ranking.

The following definition introduces the ordering $\succ_R^k$. It extends in an appropriate way within the context of evaluation discussed in this paper the definition of the *leximin criterion* $\succ_R$, considering leximin dominance of the curves $\sum_{i=1}^{k} q_i GL_i$ at the stage k. Let $AGL_F^k(p) = \sum_{i=1}^{k} q_i^F GL_{F_i}(p)$, that is the average generalized Lorenz curve of distribution F for the stage k, evaluated at the percentile p.

Definition 3 *$F \succ_R^k G$ if and only if there exists an interval $(0, p*)$ on which $AGL_F^k(p) \not\equiv AGL_G^k(p)$ and $AGL_F^k(p) \geq AGL_G^k(p)$.*

This condition requires that the weighted averages of the incomes of the poorest individuals in the first k groups of F is higher than that of the

[17]See Proposition 2 in Zoli (1999), see also Wang and Young (1998).

poorest individuals in G. Of course if leximin is satisfied in every group then $\succ_R^k$ holds. But $\succ_R^k$ is weaker than leximin for every group. It considers the opportunity to compensate the negative differences in the incomes of the poorest individuals in less needy groups with the advantages deriving from the positive differences experienced in needier groups.

Proposition 4 *If $F \succcurlyeq_{-2}^j G$ for all $j \neq k$, $j = 1, 2, ..n$, and if $AGL_F^k(p)$ and $AGL_G^k(p)$ cross once, $\hat{\mu}_k(F) = \hat{\mu}_k(G)$, and $F \succ_R^k G$, then $W(F) \geq W(G)$ for all SWFs in $\mathcal{Y}_2$ if and only if $\mathcal{G}_k(F) \leq \mathcal{G}_k(G)$.*

Proof:
Notice that $F \succcurlyeq_{-2}^j G$ for all $j \neq k$, $j = 1, 2, ..n$, implies $F \succcurlyeq_{-3}^j G$ and $\hat{\mu}_j(F) \geq \hat{\mu}_j(G)$ for all $j \neq k$, $j = 1, 2, ..n$. In order to have dominance for all SWFs in $\mathcal{Y}_2$, according to Proposition 3, since $\hat{\mu}_k(F) = \hat{\mu}_k(G)$, we need only to check that $\int_0^p AGL_F^k(t)dt \geq \int_0^p AGL_G^k(t)dt$ for all p, that is $F \succcurlyeq_{-3}^k G$. Notice that $\int_0^1 AGL_F^k(t)dt = \sum_{i=1}^k q_i^F \int_0^1 GL_{F_i}(p) = \frac{1}{2}\hat{\mu}_k^F \left(1 - \mathcal{G}_k(F)\right)$. Therefore $\mathcal{G}_k(F) \leq \mathcal{G}_k(G)$ is a necessary condition for third degree sequential inverse stochastic dominance at stage k. But it is also sufficient when $F \succ_R^k G$.

If $F \succ_R^k G$ at least the dominance is ensured for low values of p. Since the curves $AGL_F^k(p)$ and $AGL_G^k(p)$ intersect, $\Upsilon_k(p) = \int_0^p \left[AGL_F^k(t) - AGL_G^k(t)\right] dt$ reaches a maximum level, then decreases, reaching its minimum for $p = 1$. But, given $\hat{\mu}_k(F) = \hat{\mu}_k(G)$, if $\mathcal{G}_k(F) \leq \mathcal{G}_k(G)$ then $\Upsilon_k(1) \geq 0$ which ensures the third degree sequential inverse stochastic dominance. ■

The result could be easily extended to the case in which at each stage $AGL_F^k(p)$ and $AGL_G^k(p)$ intersect once, the following remark states the conditions for welfare dominance (the proof is omitted, it follows straightforwardly from that of the previous proposition)[18]

Remark 2 *If $AGL_F^k(p)$ and $AGL_G^k(p)$ cross once, $\hat{\mu}_k(F) = \hat{\mu}_k(G)$, and $F \succ_R^k G$, for all $k = 1, 2, ..n$ then $W(F) \geq W(G)$ for all SWFs in $\mathcal{Y}_2$ if and only if $\mathcal{G}_k(F) \leq \mathcal{G}_k(G)$ for all $k = 1, 2, ..n$.*

Extension to multiple intersections of $AGL^k(p)$ curves could be derived analogously at what done for the homogeneous population comparisons in Zoli (1999). In which case the welfare dominance condition will require comparisons of $\mathcal{G}_k(.)$ indices of the subsets of population in each group.

[18]See Dardanoni and Lambert (1988) for a dual result related to the utilitarian approach applied within homogeneous populations.

3 Relative deprivation indices

As introduced in the previous section, the relative deprivation evaluation framework seems an interesting context in which to provide interpretations for the welfare representation in (1).

A generalization of the framework introduced in Hey and Lambert (1980) for applying the relative deprivation approach to the measurement of welfare and inequality is discussed. The perception of deprivation felt by each individual is represented through an "envy factor" which decreases proportionately the utility of each individual according to the gap (or some increasing transformation of it) between the income of wealthier subjects and his/her income. The envy parameter is not restricted to be constant as in Hey and Lambert (1980), in order to capture the concept of relativities of deprivation it is suggested to be a function of the ranking of the individual within the income scale in his/her reference group, depending on how many individuals within the reference group are richer, poorer or have his/her level of income.

3.1 Introduction

In his seminal paper Yitzhaki (1979) provides a quantification of the concept of relative deprivation introduced by Runciman (1966). Runciman's definition of relative deprivation states:

"We can roughly say that (a person) is relatively deprived of X when (i) he does not have X, (ii) he sees some other person or persons, which may include himself at some previous or expected time, as having X (whether or not that is or will be in fact the case), (iii) he wants X, and (iv) he sees it as feasible that he should have it." (1966, p.10).

From (ii) and (iv) the relativity part of the concept is evident, that is individuals compare themselves with other persons perceived similar to them according to some criteria, and they evaluate their position relatively to the economic conditions of these reference individuals. Points (i), and (iii) provide an idea of the concept of deprivation. An individual perceives to be deprived if someone else has access to resources which he/she considers useful while he/she does not have access to them.

The relevant aspects therefore become, the identification of the *reference group,* that is the subset of individuals within which the subject confines his comparisons, and the identification of what is considered the object of the deprivation, that is what is considered to be within X.

In other terms, following the terminology applied in the context of the theories of distributive justice, we have to specify which are the ethically relevant individual characteristics, and what are the relevant objects to consider.[19]

In the original contribution of Yitzhaki (1979) and in the comment of Hey and Lambert (1980), homogeneous individuals make their comparisons in terms of income. The deprivation felt by an individual is quantified as the average of all the positive income gaps evaluated with respect to his/her income comparing the incomes of richer individuals within the reference group. Additive aggregation of all individual deprivation indices leads to the absolute Gini coefficient.

This simple but appealing approach leaves open some questions. At first stage it is not clear why should an income gap of the same extent contribute in the same way to the deprivation perception of individuals with different incomes: different intensity of deprivation is not allowed between individuals with different incomes. Moreover, the simple additive aggregation rule is deprivation inequality neutral, that is any individual level of deprivation is treated in the same way irrespective of its extent. Notice that even if the total deprivation is insensitive with respect to the differences in the individual deprivation indices, it is inequality averse if we consider the income distribution, indeed the aggregate index obtained by Yitzhaki is an inequality index.[20] Furthermore, the question of how to evaluate deprivation over distributions comprising different reference groups is still open.

The scope of this section is to introduce a class of indices of aggregate deprivation and satisfaction, which could be interpreted as obtained through additive aggregation of individual deprivation indices, as in Hey and Lambert (1980), which depend on the ranking of the individuals within the reference group as well as on the average income gaps. These indices exhibit, for a given average income gap, sensitivity to changes in the income ranking due to changes in income of the individual considered. The framework of analysis is made also sufficiently general to allow for evaluations over distributions containing multiple reference groups. As a result we will show that the SWF in (1) could be interpreted as a special case of the indices of aggregate deprivation (or satisfaction depending on the assumption on the weight functions)

[19]On the connections between deprivation and welfare analysis see Yitzhaki (1982).

[20]These aspects has been considered in Yitzhaki (1982), Chakravarty and Chakraborty (1984), Paul (1991), Chakravarty and Chattopadhyay (1994) and Ebert and Moyes (1998). See also Chakravarty (1990) for a survey of ethically based deprivation indices.

discussed in this section. Moreover, basic assumptions on the relation existing between envy parameters of individuals belonging to different groups will allow to obtain sequential satisfaction (and deprivation) criteria involving the use of Generalized Satisfaction Curves introduced by Chakravarty (1997) or respectively Relative Deprivation Curves of Kakwani (1984).

3.2 Notation, axioms and results

We will consider a set of individuals $S = \{1, 2, ..m\}$. The mth dimensional income profile is given by $\mathbf{y} \in \mathbf{Y}^m$, where the income of individual k is denoted y_k. The population is partitioned into n non overlapping and exhaustive reference groups S^i, $i \in \mathbf{N} = \{1, 2, ..n\}$ ($\mathbf{N}$ denotes the set of all types of reference groups) that is $\cup_i S^i = S, S^l \cap S^i = \emptyset \ \ \forall l, i \in \mathbf{N}$; if $n = 1$ then $S^i = S$. All reference subgroups are supposed to be *closed*, that is all deprivation cross comparisons are made within the reference group, or, in other terms, there is no pair a, b of individuals such that a is in the reference group of b but b is not in that of a. The index associated to the reference group may denote a set of common characteristics of the individuals, it could depend for instance on the ranking of needs within the population, on geographical locations, or income classes. In general it may depend on a partition based on variables considered ethically relevant for the measurement problem. We suppose that the set of reference group is exogenously determined, that is, there exist predefined rules specifying the groups, these rules are independent from the population to which they are applied, it could therefore happen that for some populations some reference groups are empty.

The index of individual deprivation is $d_k^i(\mathbf{y}) \geq 0$ which represents the individual deprivation of person k, belonging to set i, evaluated over $\mathbf{y}$. We will follows the suggestion of Hey and Lambert (1980) and consider individual deprivation as the aggregation of the feeling of deprivation felt by one individual with respect to all the other individuals belonging to the reference group. In this case $d_k^i(\mathbf{y}) = d_k^i(\mathbf{y}^i)$ that is individual deprivation is not affected by changes in the incomes distribution outside the reference group[21].

[21] In order to evaluate the individual deprivation the only relevant information is embodied into the income distribution of the reference group, any change in the distribution of incomes of other groups is not relevant. Even the identity of the individuals belonging to the reference group is not relevant, as long as they share the same relevant characteristics, in other terms, individuals are considered in an anonymous way except from what concerns their income and the ethically relevant characteristics. Moreover, the perception

We denote

$$d^i_{kj}(\mathbf{y}) = \begin{cases} y^i_j - y^i_k & if \quad y^i_j > y^i_k \\ 0 & if \quad y^i_j \leq y^i_k \end{cases} \tag{20}$$

the deprivation felt by individual k w.r.t. individual j within the same reference group i.

Total deprivation felt by an individual is evaluated as the average of cross individual deprivation comparisons[22], $d^i_k(\mathbf{y}) = \frac{1}{m_i}\sum_{j\in S^i} d^i_{kj}(\mathbf{y})$. Following Yitzhaki (1979) and Hey and Lambert (1980) we define the absolute level of individual *relative satisfaction* $s^i_k(\mathbf{y})$ as the complement with respect to maximal deprivation of $d^i_k(\mathbf{y})$, that is $s^i_k(\mathbf{y}) = \mu(\mathbf{y}^i) - d^i_k(\mathbf{y})$, where $\mu(\mathbf{y}^i)$ is the average income within the group i, but also measures the maximal possible deprivation felt within the group, that is the feeling of deprivation experienced by the poorest individual with no income.

According to the procedure suggested in Hey and Lambert (1980) we aggregate linearly the individual deprivation indices in order to derive a reference group index of deprivation (or satisfaction). Instead of simply averaging $d^i_k(\mathbf{y})$ we apply a rank-dependent aggregating function, where weights depend on the ranking within the individual deprivation distribution or equivalently within the income distribution[23]. For convenience suppose that individuals are ranked in ascending order within a reference group i, $0 < y^i_{(k)} \leq y^i_{(k+1)} \leq y^i_{(m_i)}$. At each position j is associated a weight $a^i(j, m_i) > 0$ which depends on the size of the group, and is independent from the income distribution, moreover $a^i(j, m_i)$ could, in general, be different between groups. In order to represent the weights directly as a function of the number of individuals poorer and with the same income as j, following Donaldson and Weymark (1980) we define $A^i(j, m_i) = \sum_{k=0}^{j} a^i(k, m_i)$, where $a^i(0, m_i)$ is set by definition equal 0. It follows $a^i(k, m_i) = A^i(k, m_i) - A^i(k-1, m_i)$. From which we

of individual deprivation is independent from changes in incomes of individuals not richer than the one considered.

[22]Ebert and Moyes (1998) provide an axiomatic characterization of this index of individual deprivation. Chakravarty and Chattopadhyay (1994) suggest an alternative individual deprivation index. For a general discussion of deprivation indices see Chakravarty (1990).

[23]Duclos (2000) and Duclos and Grégoire (1999) consider a similar approach using S-Gini aggregating functions. Chakravarty and Chakraborty (1984), Berrebi and Silber (1985) and Paul (1991) consider different aggregation function. Chakravarty and Mukherjee (1999) derives relative and absolute measures of deprivation making use of general social satisfaction functions depending on individual indices of satisfaction.

represent total deprivation within group i as

$$\mathcal{D}^i(\mathbf{y}^i) = \sum\nolimits_{k=1}^{m_i} \left[A^i(k, m_i) - A^i(k-1, m_i)\right] d^i_{(k)}(\mathbf{y}), \tag{21}$$

where $d^i_{(k)}(\mathbf{y})$ is the deprivation felt by the individual ranked k within the illfare ranked permutation of the deprivation indices (or equivalently of the incomes). Total satisfaction $\mathcal{S}^i(\mathbf{y}^i)$ is obtained substituting $s^i_{(k)}(\mathbf{y})$ to $d^i_{(k)}(\mathbf{y})$, where $s^i_{(k)}(\mathbf{y})$ is ranked according to illfare ranked income permutation.

The weighting function $a^i(j, m_i)$ could also be considered, if we concentrate on deprivation evaluations, as an "*envy coefficient*" of the individual with position j within the reference group i of size m_i. That is a coefficient which measures the extent to which for a given average income gap a change in the income ranking could affect the deprivation felt by an individual[24]. The procedure is consistent whenever all incomes are different, or if $a^i(j, m_i) = b^i(m_i)$ that is, the weight is independent from the income position of the individual. Otherwise, individuals with the same income but ranked differently within the income scale face different weights. In order to satisfy the *anonymity condition* requiring that individuals belonging to the same reference group and with identical income feel the same level of deprivation, we could attach them the average weight of all individuals with identical incomes. That is, suppose that ℓ individuals have the same income level, then $y^i_{(k)} = y^i_{(k-1)} = .. = y^i_{(k-\ell+1)}$, therefore according to the suggested specification of the weights $\tilde{a}^i(k, m_i) = .. = \tilde{a}^i(k-\ell+1, m_i) = \frac{1}{\ell}\sum\nolimits_{j=k-\ell+1}^{k} a^i(j, m_i)$, where if $\ell = 1$, that is $y^i_{(k-1)} < y^i_{(k)} < y^i_{(k+1)}$ we have $\tilde{a}^i(k, m_i) = a^i(k, m_i)$. It follows $\tilde{a}^i(k, m_i) = V^i(k, m_i) - V^i(k-1, m_i)$, if $\ell = 1$, and in general $\tilde{a}^i(k, m_i) = .. = \tilde{a}^i(k-\ell+1, m_i) = \frac{1}{\ell}\left[V^i(k, m_i) - V^i(k-\ell, m_i)\right]$ if $\ell \geq 1$.

The individual feeling of deprivation could therefore depend on the sum of cross income comparisons and on a weight $\frac{1}{\ell}\left[V^i(k, m_i) - V^i(k-\ell, m_i)\right]$ depending on the size of the reference group population. That is we can redefine cross individuals deprivation as

$$d^i_{kj}(\mathbf{y}) = \begin{cases} \varepsilon^i(m_i, L^i_k, E^i_k)\left[y_{(j)} - y_{(k)}\right] & if \quad j > k \\ 0 & if \quad j \leq k \end{cases}$$

[24]For instance, consider two income profiles where the same individual faces two identical distributions of incomes higher than his/her own, but two different distributions of poorer incomes within his/her reference group. When he/she compares him/herself with all richer individuals he/she may feel deprived, but the fact of being placed in a different position in the income ranking may play an effect in his/her evaluation.

where $\varepsilon^i(.) = \tilde{a}^i(k, m_i)$ is the function determining the envy coefficient and L_k^i and E_k^i are respectively the number of individuals within the reference group whose income is lower or equal than that of the individual considered. Personal total deprivation $d_k^i(\mathbf{y})$ will be therefore the sum of the cross deprivation comparisons

$$d_k^i(\mathbf{y}) = \sum_{j \in S^i} d_{kj}^i(\mathbf{y}) = \sum_{y_j > y_k} \varepsilon^i(m_i, L_k^i, E_k^i) \left[y_j - y_k \right],$$

that is $d_k^i(\mathbf{y}) = \frac{1}{E_k^i} \left[V^i(L_k^i + E_k^i, m_i) - V^i(L_k^i, m_i) \right] \sum_{y_j > y_k} (y_j - y_k)$, after we have rewritten $\varepsilon^i(m_i, L_k^i, E_k^i)$ as $\frac{1}{E_k^i} \left[V^i(L_k^i + E_k^i, m_i) - V^i(L_k^i, m_i) \right]$. The main result in this section will be obtained imposing conditions linking the $\varepsilon^i(.)$s of individuals within different groups.

We consider now the aggregation procedure leading to an overall societal deprivation index $\mathcal{D}(\mathbf{y}) \geq 0$ evaluated over a generic population exhaustively partitioned into a set of non-overlapping and closed subgroups.

We restrict our attention to *individualistic, monotonic deprivation (satisfaction) function* that is, $\mathcal{D}(\mathbf{y})$ is an increasing function of all individual deprivation indices $d_k^i(\mathbf{y})$, and therefore of all group deprivation indices $\mathcal{D}^i(\mathbf{y}^i)$. We make the simplifying assumption that

- the deprivation (satisfaction) evaluated over a set of closed reference subgroups is represented as a *additive function of the deprivation levels of the individuals belonging to each group aggregated according to* $\mathcal{D}^i(\mathbf{y}^i)$ (or $\mathcal{S}^i(\mathbf{y}^i)$) *; each function* $\mathcal{C}^i(.) \geq 0$, *evaluating the group deprivation (satisfaction) contribution to overall deprivation (satisfaction), could depend on the group considered and possibly on the sizes of all reference groups.*

The additivity condition imposed is more general than the usual ones because it allows for group evaluations conditional on the size of all reference groups. The reason for this simplifying choice is that a more general approach will go outside of the scope of this paragraph, moreover it is likely that opportune independence properties applied in a more general context will play a similar role in restricting the class of aggregation rules[25]. Moreover,

[25] Nevertheless even in this simplified framework the degree of generality we can reach in representing overall deprivation is to some extent higher than what is obtainable following the usual approaches to measurement of total deprivation.

the imposition of few properties on $\mathcal{D}^i(\mathbf{y})$ will make clear the link between the aggregate deprivation and satisfaction indices and the welfare representations in (1). Notice that we require the aggregating function $\mathcal{C}^i(.)$ being the same for deprivation and satisfaction indices.

According to the initial definition of aggregate deprivation we can write it as:

$$\mathcal{D}(\mathbf{y}) = \sum_{i=1}^{n} \mathcal{C}^i(\mathcal{D}^i(\mathbf{y}); m_1, m_2, ..m_n), \tag{22}$$

where $\mathcal{C}^i(\mathcal{D}^i(\mathbf{y}); m_1, m_2, ..m_n) \geq 0$ is the function transforming the group deprivation $\mathcal{D}^i(\mathbf{y})$ within the aggregate evaluation, with $\mathcal{C}^i = 0$ if $m_i = 0$. The function $\mathcal{C}^i$ is supposed to depend on the group i in order to allow for different social considerations of the deprivation felt within each group, and also to depend on the distribution of sizes of all reference groups, being therefore sensitive to composition of the society in terms of groups. The deprivation felt by individuals belonging to needier groups could, for instance, receive a higher weight in the aggregation procedure. Moreover, for political reasons the policy maker could be interested in reducing the deprivation felt within groups containing larger shares of the population, therefore implying higher concern for these groups' deprivation level in the aggregation procedure.

We will impose on (22) only two axioms specified in terms of changes in the distribution $\mathbf{y}$, without considering directly the distribution of the individual deprivation indices.

We suppose that deprivation (satisfaction) is evaluated in terms of real incomes, or, in general that there exists agreement on the unit of measure. If this is the case, scaling the income distribution is assumed to correspond to a proportional scaling of the individual deprivation $d_k^i(\mathbf{y})$, and, given the specification of $\mathcal{D}^i(\mathbf{y})$ also of group deprivation, what we require is also total deprivation (and satisfaction) being scaled proportionally.

Axiom 4 (LH: Linear Homogeneity) $\mathcal{D}(\lambda\mathbf{y}) = \lambda\mathcal{D}(\mathbf{y})$ *&* $\mathcal{S}(\lambda\mathbf{y}) = \lambda\mathcal{S}(\mathbf{y})$, $\forall\mathbf{y} \in \mathbf{Y}$, *and* $\lambda > 0$.

LH imposes that total deprivation should act in the same way as individual deprivation to the scaling of incomes[26].

[26] LH is a restrictive condition, not all indices suggested in the literature are linearly homogeneous, Yitzhaki (1979) and Hey and Lambert (1980) indices satisfy LH as well as indices in Chakravarty and Chakraborty (1984), Berrebi and Silber (1985) and in Chakravarty and Mukherjee (1999).

While LH provides a normative judgement of how deprivation should be evaluated when incomes are scaled, the next axiom provides a judgement on how deprivation (satisfaction) perception should change if we replicate the initial distribution. We consider a situation in which the whole population is replicated r times. Each person therefore compares him/herself with a larger set of reference individuals: an r times replication of individuals different from him/her and $r-1$ individuals similar to him/her in all respects. While total deprivation (satisfaction) is aggregated across the new replicated population. The change in the deprivation (satisfaction) felt is supposed to depend to some extent on the replication parameter r according to a function $\phi(r)$.

Let $\mathbf{y}_r$ be the r times replication of distribution $\mathbf{y}$, and S_r^i the r times replication of S^i where $r \in \mathbb{I}_+$ ($\mathbb{I}_+$ is the set of positive integers: 1,2,3,...). Notice: $\mathbf{y}_1 \equiv \mathbf{y}$. The replicated individuals are supposed to be identical in all respects to the original ones. Moreover, the function $\phi(.)$ is defined *weakly monotonic* if it is either non decreasing or non increasing, that is iff either $r' > r \rightarrow \phi(r') \geq \phi(r)$, or $r' > r \rightarrow \phi(r') \leq \phi(r)$ for all r', r belonging to the domain of $\phi(.)$.

Axiom 5 (PR: Population Replication) *There exists a weakly monotonic function* $\phi : \mathbb{I}_+ \rightarrow]0, +\infty[$, *such that* $\mathcal{D}(\mathbf{y}_r) = \phi(r)\mathcal{D}(\mathbf{y})$ (*&* $\mathcal{S}(\mathbf{y}_r) = \phi(r)\mathcal{S}(\mathbf{y})$), $\forall \mathbf{y} \in \mathbf{Y}$ *and* $\forall r \in \mathbb{I}_+$.

The replication axiom is weaker than the usual one applied in variable population comparisons, in that the function $\phi(r)$ is not specified. The only requirement is an unconstrained positive multiplicative size effect of the replication parameter, and weak monotonicity of the function. The intuition behind this latter restriction is that there is no reason for changes in trends of the evaluation as the number of replications increase. Notice that the usual replication invariance property requires $\phi(r) = 1$.

Although the indices $d_k^i(\mathbf{y})$ and $s_k^i(\mathbf{y})$ are replication invariant the aggregating procedure does not ensure that group and population deprivation indices are.

If we require deprivation and satisfaction social indices being invariant with respect to replication of the population, then $A^i(k, m_i)$ should be expressed as a function of the relative position of the individuals in the income scale, i.e. $A^i(k, m_i) = V^i(k/m_i)$.

The following proposition introduces the class of aggregate indices in (22) consistent with LH and PR. Let $\mathbb{I}_+$ the set of positive integers, $\mathbb{Z}_+$ the set of positive rationals considering also zero, $\mathbb{Z}_+^n$ is the n-th fold Cartesian product of $\mathbb{Z}_+$, and $\mathcal{Z}_1 = \{x \in \mathbb{Z}_+ : 0 \leq x \leq 1\}$. Moreover, let $\mathcal{M}^n = \{\mathbf{m} \in \mathbb{I}_+^n/\mathbf{0}\}$, denote $\mathcal{Z}^n = \{\mathbf{q} \in \mathbb{Z}_+^n/\mathbf{0}\}$, while $\mathcal{Z}_1^n = \{\mathbf{q} \in \mathcal{Z}^n : \sum_i q_i = 1\}$ the set of all vectors in $\mathcal{Z}^n$ whose elements sum to 1.

Proposition 5 *$\mathcal{D}(\mathbf{y})$ and $\mathcal{S}(\mathbf{y})$ in* (22) *satisfy LH and PR if and only if there exist not-decreasing functions $\mathcal{H}^i : \mathcal{Z}_1^n \to \mathbb{R}_+$, $V^i : \mathcal{Z}_1 \to \mathbb{R}_+$, and constant $\alpha \in \mathbb{R}$ such that*

$$\mathcal{D}(\mathbf{y}) = \sum_{i=1}^{n} \mathcal{H}^i(\mathbf{q}) m_i^\alpha \sum\nolimits_{k=1}^{m_i} \left[V^i(k/m_i) - V^i((k-1)/m_i)\right] d^i_{(k)}(\mathbf{y}),$$

and similarly

$$\mathcal{S}(\mathbf{y}) = \sum_{i=1}^{n} \mathcal{H}^i(\mathbf{q}) m_i^\alpha \sum\nolimits_{k=1}^{m_i} \left[V^i(k/m_i) - V^i((k-1)/m_i)\right] s^i_{(k)}(\mathbf{y}).$$

Proof: See appendix.

An interesting insight from this result is given by Lemma 5 in the appendix, where it is shown that the general Population Replication condition specifies $\phi(r) = r^\alpha$, $\alpha \in \mathbb{R}$, from where the final aggregate evaluation depends on m_i^α. This result could provide a new starting point of discussion for analyzing social evaluations made over variable population size distributions[27].

We now highlight the similarities between the suggested deprivation (satisfaction) indices and those introduced in (1). If we consider $F_i(y)$ the cumulative distribution of group i, and $F(y)$ the whole population distribution (both with support $0, \infty$), then the individual deprivation $d^i_{(k)}(\mathbf{y})$ in (20) associated to the individual with income y belonging to the reference group i

[27] Blackorby and Donaldson (1984) show that under weak conditions the index of total welfare of a distribution is separable between the welfare of a representative individual (which is population size independent) and the size of the distribution. The *population replication invariance* conditions, usually applied within the context of inequality, welfare and poverty measurement rule out any role for the size of the distribution. From Lemma 5 we know that the generalization of this property is parameterized by α. An equivalent of Blackorby and Donaldson's (1984) result satisfying the generalized version of the replication principle could provide a multiplicative specification of the separable function they suggested. The aggregate welfare could be represented as the product of the representative individual welfare times m^α, $\alpha \in \mathbb{R}$, where m is the size of the distribution. For $\alpha = 0$, the usual result connected to the Population Replication Invariance condition holds, while for $\alpha = 1$ we get evaluations based on the total amount of welfare distributed.

is

$$\tilde{\delta}^i_F(y) = \int_y^\infty (x-y)\, f_i(x)dx,$$

or alternatively

$$\delta^i_F(p) = \int_p^1 \{F_i^{-1}(t) - F_i^{-1}(p)\}dt$$

where $p = F_i(y) = k/m_i$, and $F_i^{-1}(p)$ is the usual left continuous inverse distribution function of F_i, while $\delta^i_F(p)$ is the deprivation felt by the portion of population at the $100p^{th}$ percentile of group i income distribution.

Letting $V^i(k/m_i) = \int_0^{k/m_i} v_i(p)dp$, and considering $\delta^i_F(p)$, then aggregate deprivation is given by

$$D(F) = \sum_{i=1}^n \mathcal{H}^i(\mathbf{q})m_i^\alpha \int_0^1 v_i(p)\delta^i_F(p)dp, \qquad (23)$$

which could be rearranged such that:

Proposition 6 *The social deprivation index in* (23) *is*

$$D(F) = \sum_{i=1}^n \mathcal{H}^i\left(\mathbf{q}\right) m_i^\alpha \int_0^1 \varpi_i(p)F_i^{-1}(p)dp \qquad (24)$$

where $\varpi_i(p) = \int_0^p v_i(t)dt - v_i(p)(1-p)$.

See appendix for the proof.

Notice that $\int_0^1 \varpi_i(p)dp = 0$ for all $i = 1, 2, ..n$. This result is consistent with our expectations, if incomes approach the equal distribution then deprivation disappears: in this case $\int_0^1 \varpi_i(p)F_i^{-1}(p)dp$ tends to $\int_0^1 \varpi_i(p)\mu(F_i)dp = \mu(F_i)\int_0^1 \varpi_i(p)dp = 0$. $D(F)$ could therefore be considered as a linear transformation of the linear inequality measures investigated in Mehran (1976) where the deprivation within each subgroup is aggregated through a weighted average where weights depend on the shares of population in each subgroup transformed according to the functions $\mathcal{H}^i\left(.\right)$, and the total population in each group whose contribution depends on the population replication coefficient α.

The function $D(F)$ in (24) could be interpreted as the *needs based deprivation function* associated to $W(F)$ in (1). The original function in (1) is generalized in order to include an extra weighting factor $\mathcal{H}^i(\mathbf{q})\, m_i^{\alpha}$. All results concerning sequential inverse stochastic dominance could be appropriately rearranged in order to obtain the set of *deprivation partial orderings* associated to $D(F)$ once we consider $\theta_i^F = \mathcal{H}^i(\mathbf{q})\, m_i^{\alpha}$ instead of the population shares of each group, and define properties equivalent to A, A1, and A2 taking into account that $\int_0^1 \varpi_i(p)dp = 0$.[28]

Furthermore if we consider the *satisfaction index*

$$S(F) = \sum_{i=1}^{n} \theta_i^F \int_0^1 v_i(p)[\mu(F_i) - \delta_F^i(p)]dp$$

associated to $D(F)$ the similarity with $W(F)$ is even more evident. The satisfaction orderings could be obtained substituting to $\delta_F^i(p)$ the individual average satisfaction index, or alternatively as the complement to the average income of a reference group of the overall group deprivation[29]. When deprivation is maximal $D_i(F)$ reaches the average income level (once it is evaluated considering only the income distances between individuals), average income is the highest equally distributed equivalent level of satisfaction in the absence of deprivation. But, what is the appropriate maximum level of satisfaction if we weight all incomes using $\varpi_i(p)$? Is it reasonable to consider the same indicator of total satisfaction for all reference groups if we extend the analysis to populations partitioned into various closed groups?

Consider the average deprivation indicator $D_i(F) = \int_0^1 \varpi_i(p)F_i^{-1}(p)dp$ associated to group i, integrating it by parts we get

$$D_i(F) = \mu(F_i)\left\{\varpi_i(1) - \int_0^1 \varpi_i'(p)L_i(p)dp\right\}$$

where $L_i(p)$ is the Lorenz curve of group i. When there is complete inequality the deprivation reaches its maximum level. Notice that if this is the case, $L_i(p)$ tends to zero for all $p \in [0,1)$, therefore the weighted integral of $L_i(p)$ also tends to zero. It follows that $\max D_i(F) = \mu(F_i)\varpi_i(1)$. Therefore if we require average satisfaction to go to 0 in this extreme case we obtain the

[28] The function simplifies to (1) once we consider $\alpha = 0$, that is, we suppose that deprivation satisfies the *Population Replication Invariance Condition.* Moreover, we need to let $\mathcal{H}^i(\mathbf{q}) = q_i$. That is eliminate any distributive concern from $\mathcal{H}^i$ which is independent from any movement of population shares between groups other than i, and evaluates each group contribution in an anonymous way, without discriminating between groups.

[29] See Yitzhaki (1979, 1982). See also Chakravarty and Mukherjee (1999).

upper bound for satisfaction. It follows that $S_i(F)$, the *average satisfaction for group* i, could be quantified as

$$S_i(F) = \mu(F_i)\varpi_i(1) - D_i(F)$$

If income is equally distributed, there is no deprivation and $S_i(F) = \mu(F_i)\varpi_i(1)$.

Notice that for the Yitzhaki (1979) index $\varpi_i(p) = (2p-1)$, therefore $\varpi_i(1) = 1$ and consistently $S_i(F) = \mu(F_i) - D_i(F)$.

Following the Yitzhaki (1979) and Hey and Lambert (1980) procedure for deriving aggregate satisfaction from additive aggregation of individual levels, it could be checked that the $S_i(F)$ formula introduced could be derived aggregating the individual satisfactions that is $S_i(F) = \int_0^1 v_i(p)\sigma_F^i(p)dp$ where $\sigma_F^i(p) = \mu(F_i) - \delta_F^i(p) = \int_0^p F_i^{-1}(t)dt + (1-p)F_i^{-1}(p)$. This formula is obtained subtracting from $\mu(F_i)$

$$\delta_F^i(p) = \mu(F_i)\left(1 - L_i(p) - (1-p)\frac{F_i^{-1}(p)}{\mu(F_i)}\right)$$

obtained in (62) ,where the term within brackets is the Kakwani *Relative Deprivation Curve* (see Kakwani, 1984 and Chakravarty, Chattopadhyay and Majumdar, 1995), defined for deprivation orderings which are invariant w.r.t. income scaling.

Summarizing, within our context

$$S_i(F) = \int_0^1 (\varpi_i(1) - \varpi_i(p)) F_i^{-1}(p)dp.$$

Letting $\varphi_i(p) = (\varpi_i(1) - \varpi_i(p))$, in terms of the original weighting parameters we get $\varphi_i(p) = \int_p^1 v_i(t)dt + v_i(p)(1-p)$, and the total satisfaction is given by

$$S(F) = \sum_{i=1}^{n} \theta_i^F \int_0^1 \varphi_i(p)F_i^{-1}(p)dp \tag{25}$$

where $\varphi_i(1) = 0\ \forall i$ (that is Property A* is satisfied). For all other properties A, A1, A2 it is possible to provide restrictions on $v_i(p)$. Notice that if $v_i(p) = \gamma_i\ \forall i$ then

$$S(F) = \sum_{i=1}^{n} \theta_i^F \gamma_i \left[\mu(F_i)\left(1 - \Gamma(F_i)\right)\right]$$

which represents the class of rank-dependent evaluation functions with linear weights (with the additional requirement $\varphi_i(1) = 0\ \forall i$) discussed in remark 1.

Finally we highlight that the dominance conditions obtained in the first paragraph are built on the characterization of (1) in terms of transfers satisfying the Principle of Transfers and its Positional sensitivity extension. It is not surprising therefore that all criteria obtained are specified in terms of curves associated to the inverse stochastic dominance conditions of different degrees. The analogies between the representation in (1) and relative satisfaction orderings make more plausible the specification of criteria based on comparisons involving the *Generalized Satisfaction Curves* (see Chakravarty, 1997)

$$GSC(F_i, p) = GL_i(p) + (1 - p)F_i^{-1}(p)$$

obtained considering the satisfaction counterpart of the Kakwani *Relative Deprivation Curve.* The partial ordering associated to such curves is less fine than that associated to generalized Lorenz curves, which is therefore implied by the first. Chakravarty, Chattopadhyay and Majumdar (1995) show that the ranking associated to relative deprivation curves is consistent with a sequence of progressive transfers satisfying an additional restriction, namely that the transfer made by an individual (with negative sign if the individual is the recipient) should be lower than the average transfers of the richer individuals.

Coming back to the interpretation of individual deprivation weighted according to the "envy parameter", notice that for the individual whose income is ranked k within subgroup i, the envy coefficient $\tilde{a}^i(k, m_i)$ previously discussed is given by $\frac{1}{\ell}\int_{(k-\ell)/m_i}^{k/m_i} v_i(p)dp > 0$. In what follows we simply suppose that the envy coefficient is positively related to the level of needs of the individual independently from his/her position in the income ranking and the size of the population in the group.

Axiom 6 (Needs based Positional Envy) $v_i(p) \geq v_{i+1}(p)$ *for all* p, i.

That is $v_i(p)$ satisfies Property A. At a given percentile of the income distribution, the envy factor is higher the higher is le level of needs of the associated portion of the population.

Denote with $\mathcal{S}$ the set of satisfaction indices $S(F)$ satisfying the above property, and suppose, with analogy to the results in the previous section, that we restrict $\theta_i^F = q_i^F$, it follows:

Proposition 7 *Given two distributions F, $G \in \mathcal{F}$, $S(F) \geq S(G)$ for all Satisfaction indices in $\mathcal{S}$ if and only if*

$$\sum_{i=1}^{k} \left[q_i^F GSC(F_i, p) - q_i^G GSC(G_i, p)\right] \geq 0$$

for any $k = 1, 2, ...n$ and every $p \in [0, 1]$.

Proof:

Consider $\Delta S = S(F) - S(G)$ according to the satisfaction index in (25), the change in satisfaction could be written as it could be written as $\varphi_i(p) = \int_p^1 v_i(t)dt + v_i(p)(1-p)$, and the total satisfaction is given by

$$\Delta S = \sum_{i=1}^{n} \left\{ \int_0^1 \left(\int_p^1 v_i(t)dt + v_i(p)(1-p) \right) \left[q_i^F F_i^{-1}(p) - q_i^G G_i^{-1}(p)\right] dp \right\}$$

denoting, as in Proposition 1, $\phi_i(p) = q_i^F F_i^{-1}(p) - q_i^G G_i^{-1}(p)$ we can rewrite

$$\Delta S = \sum_{i=1}^{n} \left\{ \int_0^1 \left(\int_p^1 v_i(t)dt \right) \phi_i(p)dp + \int_0^1 v_i(p)(1-p)\phi_i(p)dp \right\}.$$

Integrating by parts, $\int_0^1 \left(\int_p^1 v_i(t)dt \right) \phi_i(p)dp = \int_0^1 v_i(p) \int_0^p \phi_i(t)dtdp$, it follows

$$\Delta S = \sum_{i=1}^{n} \left\{ \int_0^1 v_i(p) \left[\psi_i(p) + (1-p)\phi_i(p)\right] dp \right\}. \tag{26}$$

Notice that $\psi_i(p) + (1-p)\phi_i(p) = q_i^F GSC(F_i, p) - q_i^G GSC(G_i, p)$, which we will denote as satisfaction gap $\Delta\mathcal{S}_i(p)$, then $\Delta S = \sum_{i=1}^n \int_0^1 v_i(p)\Delta\mathcal{S}_i(p)dp$. Since $v_i(p)$ satisfies Property A, substituting for $v_i(p) = \sum_{j=i}^n w_j(p)$, and rearranging we have, as in proof of Proposition 1:

$$\Delta S = \int_0^1 \sum_{i=1}^{n} \left(w_i(p) \sum_{j=1}^{i} \Delta\mathcal{S}_j(p) \right) dp.$$

Then, since $w_i(p) \geq 0 \, \forall i, p$, the condition $\sum_{j=1}^{i} \Delta\mathcal{S}_j(p) \geq 0$ for any $i = 1, 2, ..n$ is sufficient for ΔW being non-negative.

The necessity part follows from the proof of Proposition 1 once we substitute $\Delta\mathcal{S}_i(p)$ for $\phi_i(p)$. ■

We have therefore obtained a sequential dominance result in terms of combinations of Generalized Satisfaction Curves.

The dominance condition in terms of satisfaction indices is a combination of the sequential inverse dominance conditions. Let $\Sigma_k(p) = \sum_{j=1}^{k} \Delta\mathcal{S}_j(p)$, $\Psi_k(p) = \sum_{j=1}^{k} \psi_j(p)$ and $\Phi_k(p) = \sum_{j=1}^{k} \phi_j(p)$, then the sequential satisfaction condition is $\Sigma_k(p) \geq 0 \Leftrightarrow \Psi_k(p) + (1-p)\Phi_k(p) \geq 0$, from which, given the definitions of $\Psi_k(p) = \int_0^p \Phi_k(t)dt$, follows $\Phi_k(p) \geq 0 \Rightarrow \Sigma_k(p) \geq 0 \Rightarrow \Psi_k(p) \geq 0$ which provide the degree of fineness of the sequential ordering associated.

Suppose now that the envy coefficient $v_i(p)$ is *decreasing with respect to the position of the individual within the income ranking*, that is for a given level of interpersonal income gap comparisons the perception of deprivation is decreasing with the position in the income ranking. Furthermore we suppose that the way in which deprivation changes between groups is such that a movement within the income scale for a given level of interpersonal income gaps has a bigger effect the needier is the group. This property leads to a characterization of the envy weight equivalent to that in Property A1, that is $w_i'(p) \leq 0$. We can apply the result in Proposition 2 in order to get the satisfaction ordering obeying the above condition.

It follows that $S(F) \geq S(G)$ for all Satisfaction indices in $\mathcal{S}$ satisfying $w_i'(p) \leq 0 \ \forall p, i$ if and only if

$$\sum_{i=1}^{k} \int_0^p \left[q_i^F GSC(F_i,t) - q_i^G GSC(G_i,t)\right] dt \geq 0$$

for any $k = 1, 2, ...n$ and every $p \in [0,1]$.

According to our definition $\int_0^p \Delta\mathcal{S}_i(t)dt = \int_0^p \psi_i(t) + (1-t)\phi_i(t)dt$, which applying integration by parts leads to $\int_0^p \Delta\mathcal{S}_i(t)dt = 2\int_0^p \psi_i(t)dt + (1-p)\psi_i(p)$, from which

$$\sum_{i=1}^{k} \int_0^p \psi_i(t)dt + \frac{(1-p)}{2}\psi_i(p) = \sum_{i=1}^{k} \left[\tau_i(p) + \frac{(1-p)}{2}\psi_i(p)\right] \geq 0$$

which is an averaged condition between second and third degree sequential inverse dominance introduced in Propositions 2 and 3. Notice that, when we consider comparisons of satisfaction orderings over a single reference group the condition boils down to combinations of standard second and third degree inverse stochastic dominance, once we consider the set of all positive and non-increasing envy parameters.

All the previous results could easily be extended to deprivation comparisons simply recalling that $D_i(F) = \mu(F_i)\varpi_i(1) - S_i(F)$ and therefore realizing that $\Delta D = D(G) - D(F) = \sum_{i=1}^{n}\left[q_i^G\mu(G_i) - q_i^F\mu(F_i)\right]\varpi_i(1) + \Delta S$, or in other terms

$$\Delta D = \sum_{i=1}^{n}\left(-\int_0^1 v_i(t)dt\right)\int_0^1 \phi_i(p)dp + \Delta S.$$

After substituting from (26) and rearranging we get

$$\Delta D = \int_0^1 \sum_{i=1}^{n}\left\{-v_i(p)\left[\psi_i(1) - \psi_i(p) - (1-p)\phi_i(p)\right]\right\}dp.$$

It follows that for all deprivation indices satisfying Property A $\Delta D \geq 0$ iff $\sum_{i=1}^{k}\left[\psi_i(1) - \psi_i(p) - (1-p)\phi_i(p)\right] \leq 0$ for all p and k. But $\psi_i(1) - \psi_i(p) - (1-p)\phi_i(p) = q_i^F\mu(F_i)\,RDC(F_i,p) - q_i^G\mu(G_i)\,RDC(G_i,p)$ where $RSC(F_i,p)$ is the Kakwani Relative Deprivation Curve evaluated at percentile p of group i. That is the dominance condition requires sequential comparisons of the averaged absolute versions of the RDCs

$$\sum_{i=1}^{k}\left[q_i^F\mu(F_i)\,RDC(F_i,p) - q_i^G\mu(G_i)\,RDC(G_i,p)\right] \leq 0.$$

Notice that $RDC(F_i,p) \leq RDC(G_i,p)$ for all p is the dominance condition of distribution F_i over G_i in terms of relative deprivation [30].

4 Rank-dependent poverty measurement

Within the context of poverty measurement two classes of indices play a major role. The first is the *additively decomposable* class of indices which aggregate additively individual indicators of poverty, that is functions defined over individual incomes and poverty lines. The other class comprises the *rank-dependent indices* associate to the poverty line z, where either the

[30] See Chakravarty, Chattopadhyay and Majumdar (1995).

incomes of poor individuals (Sen, 1976) or censored incomes ${x_i}^* = \min(x_i, z)$ (Chakravarty, 1983) or poverty gaps both absolute $\tilde{g}_i = z - x_i^*$, and relative $g_i = \tilde{g}_i/z$ (Shorrocks, 1995, 1998), are aggregated as weighted averages where the weights depend on their ranking either within the population of the poor individuals or the whole population.[31]

Some partial ranking criteria, which are consistent with unanimous evaluation for a large set of indices, are available in the poverty measurement context. The most relevant are those associated to the curves aggregating the poverty gaps at each percentile of the population starting from the poorest individuals, (Spencer and Fisher, 1992, Shorrocks, 1998 and Jenkins and Lambert, 1997). Unfortunately the class of indices in Sen (1976) is not included within the set of indices whose ranking is consistent with the one induced by these curves.

Moreover, attempts have been made in order to specify criteria consistent with poverty dominance for a wide range of poverty lines, (Atkinson 1987, Foster and Shorrocks, 1988,a,b,c, Zheng, 1999). All these criteria are associated to poverty dominance according to the additively decomposable indices.

Recently some attempts have been made also to incorporate, within the poverty evaluation, judgements concerning differences in needs (Atkinson, 1992, Jenkins and Lambert, 1993 and Chambaz and Maurin, 1998). Even in this case the results available concern poverty dominance associated to additively decomposable poverty indices.

It seems therefore interesting, or at least theoretically challenging, to provide some results concerning rank-dependent poverty indices. In what follows we will discuss the connections between welfare dominance associated to either the Yaari SWF or needs based welfare functions in (1) and poverty dominance for classes of rank-dependent poverty indices.

The next subsection we will discuss dominance conditions, for given poverty lines, associated to a class of indices generalizing the Sen index. It will follow an extension to the class of *revised Sen* indices suggested by Shorrocks (1995). Finally a possible implication of the results introduced in the initial section of the paper, for needs based poverty dominance associated to rank-dependent indices is discussed. All results specify dominance criteria related to transformations of the income gap curves.

[31] For surveys on poverty measurement see among others Sen (1973), Foster (1984), Seidl (1988), Chakravarty (1990) and Zheng (1998).

4.1 Generalized Sen indices

Let $F(x)$ the income distribution function with density $f(x)$, denote by $F_{F(z)}$ the distribution of income of poor individuals according to the poverty line z, that is $F_{F(z)}(x) = F(x)/F(z)$, if $x \leq z$, $F_{F(z)}(x) = 1$ if $x > z$. The asymptotic version of Sen's poverty index $P(F, z)$ can be written

$$P(F, z) = 2\int_0^z \frac{z-x}{z}\left[1 - \frac{F(x)}{F(z)}\right] f(x)dx,$$

or equivalently, rearranging

$$\begin{aligned} P(F, z) &= H(F, z)\left[I(F, z) + (1 - I(F, z))\Gamma(F_{F(z)})\right] = \\ &= H(F, z)\left[1 - \frac{\mu(F_{F(z)})}{z}\left[1 - \Gamma(F_{F(z)})\right]\right]. \end{aligned}$$

Where $H(F, z) = F(z)$ is the *Head count ratio*, the proportion of poor individuals within the society, $I(F) = \left[z - \mu(F_{F(z)})\right]/z$ is the *Income gap ratio,* the average income gap (relative) for the distribution of poor individuals, and $\Gamma(F_{F(z)})$ is the *Gini coefficient evaluated over the distribution of poor* individuals. As noticed by Clark et al. (1981) the Sen index could be rewritten as a function of the Gini coefficient of the distribution of the poverty gaps. Let

$$\Pi_{F,z}^{-1}(p) = \left[z - F_{F(z)}^{-1}(1-p)\right]/z,$$

be the right continuous inverse distribution of the (relative) poverty gaps of distribution F conditional on poverty line z, where $\Pi_{F,z}(g) = 1 - F_{F(z)}(z(1-g))$ is the cumulative distribution function of the poverty gap g. Notice that the ranking of the poverty gaps (in terms of percentiles) is the complement to 1 of the ranking of the incomes of poor individuals. It follows:

$$P(F, z) = H(F, z)I(F, z)\left[1 + \Gamma(\Pi_{F,z})\right]. \tag{27}$$

Kakwani (1980) provided a parametrized version of $P(F, z)$ which could be connected to the class of Extended Gini indices[32],

$$P_\theta(F, z) = (\theta + 1)\int_0^z \frac{z-x}{z}\left[1 - \frac{F(x)}{F(z)}\right]^\theta f(x)dx, \qquad \theta \geq 0.$$

[32]See Donaldson and Weymark (1980, 1983) and Yitzhaki (1983).

Moving from incomes to percentiles $p = F_{F(z)}(x)$ in considering the integration variable, the index could be rewritten as

$$P_\theta(F, z) = F(z)\,(\theta + 1) \int_0^1 \frac{z - F_{F(z)}^{-1}(p)}{z} [1 - p]^\theta \, dp, \qquad \theta \geq 0. \tag{28}$$

Denoting $\tilde{\Pi}_{F,z}^{-1}(p) = \gamma_F^z(p) = \frac{z - F_{F(z)}^{-1}(p)}{z} = \Pi_{F,z}^{-1}(1-p)$, so that $\tilde{\Pi}_{F,z}^{-1}(p)$ is the left continuous inverse distribution of the poverty gap ranked in *reverse* order, that is the poverty gap is not increasing in p, it follows that $P_\theta(F, z) = F(z)\Xi_\theta(\tilde{\Pi}_{F,z})$, that is, the Kakwani poverty index is obtained multiplying by the Head count ratio the welfare function associated to the extended Gini index of the distribution of the poverty gaps ranked in reverse order. Similarly, if we consider the distribution of poverty gaps, we have instead:

$$P_\theta(F, z) = F(z)\,(\theta + 1) \int_0^1 \Pi_{F,z}^{-1}(p) p^\theta dp, \qquad \theta \geq 0, \tag{29}$$

which is $F(z)$ times the welfare function $\tilde{\Xi}_\theta(\Pi_{F,z})$ associated to the extended Gini index of the *illfare ranked* permuted distribution of poor incomes introduced in Donaldson and Weymark (1980, 1983)[33]. For $\theta = 1$, the index becomes $P_1(F, z) = H(F, z)I(F, z)\left[1 - \tilde{\Gamma}_1(\Pi_{F,z})\right]$, which is also $P_1(F, z) = H(F, z)I(F, z)\left[1 + \Gamma(\Pi_{F,z})\right] = P(F, z)$. When $\theta \neq 1$ the extended poverty index cannot be expressed in terms of the welfare ranked extended Gini index of the poverty gap distribution, only the illfare ranked version:

$$P_\theta(F, z) = H(F, z)I(F, z)\left[1 - \tilde{\Gamma}_\theta(\Pi_{F,z})\right] \tag{30}$$

is relevant.

It could be argued that in the same way as the welfare functions based on the illfare ranked extend Ginis are a subset of the class of the Yaari Social Welfare Functions (YSWFs), the class of indices $P_\theta(F, z)$ could be extended to the illfare ranked Yaari SWFs over the poverty gaps distribution. This extension allows to provide a *general version of the Sen index* which could

[33] See also Bossert (1990) for a discussion of the merits of the illfare ranked extended Ginis with respect to the welfare ranked version especially in the context of poverty measurement. As Bossert pointed out, the illfare and welfare ranked extended Gini classes do not coincide, moreover the Gini coefficient belongs to the second class of indices.

be consistent with a wider set of assumptions on the sensitivity to transfers of the poverty ordering.

In order to highlight the similarities between the characterization of the YSWF weights provided in Yaari (1987,1988) and Mehran (1976) and the characterization of the weights of this poverty index, we could write the index

$$P_v(F,z) = F(z)\int_0^1 v(p)\tilde{\Pi}^{-1}_{F,z}(p)dp, \tag{31}$$

where $\tilde{\Pi}^{-1}_{F,z}(p)$ is the left continuous inverse distribution function of the poverty gaps ranked in reverse order (that is ranked according to the welfare ranking procedure), and $v(p)$ is the weight function embodying the normative judgements of the policy maker w.r.t. poverty perception.

Notice that $P_v(F,z)$ satisfies the traditional poverty axioms of *Focus* because it is independent from the distribution of incomes above the poverty line, and *Replication Invariance* because the weights depend on the relative ranking of poor people. Moreover $P_v(F,z)$ is *Scale Invariant* since the (relative) poverty gap is invariant w.r.t. a change in the scale of measure of the incomes.

In addition to these properties we will consider extra axioms characterizing the poverty index.

Axiom 7 (M: Monotonicity (Sen, 1976)) *Given other things, a reduction in income of a person below the poverty line must increase poverty.*

The interpretation of M is straightforward. The next axiom is a weaker version of the *Transfer axiom* suggested in Sen (1976). Its scope is to introduce inequality aversion within the poverty evaluation, which captures the relative deprivation content of poverty. Widening income differences among poor persons, or in other terms, transferring money from more deprived person to less deprived, must lead to an increase in the overall deprivation.

Axiom 8 (WT: Weak Transfer (Donaldson and Weymark, 1986)) *Other things given, a transfer of income from a person below the poverty line to anyone richer, with no one crossing the poverty line as a consequence of the transfer, must not decrease poverty.*

Axiom WT is weaker than the original one in Sen (1976) not only because it does not consider situations in which the receiver crosses the poverty line as the result of the transfer, which could lead to some problem for these rank-dependent indices, but also because it does not rule out the class of indices which are insensitive to transfers, like the Head count ratio and the Income gap ratio.

Next axiom specifies the way in which transfers could affect poverty. If we follow the relative deprivation interpretation of poverty then differences in ranking also become important, rather than only income differences. It seems reasonable that a transfer could play a higher effect if experienced at the lower end of the distribution, when the amount of the transfer and the distance in the ranking of the individuals involved is not changed. Let $\Delta P(p, \rho, \delta)$ be the increase in poverty due to a transfer of income from a tiny fraction dp of the population at the $100p^{th}$ percentile to a fraction dp at the $100(p+\rho)^{th}$ percentile $\rho \geq 0$.

Axiom 9 (PTS: Positional Transfer Sensitivity (Kakwani, 1980)) *For all $p, q, \rho \in [0,1]$, such that $p+\rho, q+\rho \in [0,1]$, and $\delta > 0$, if $p < q$ then $\Delta P(p, \rho, \delta) \geq \Delta P(q, \rho, \delta)$.*

Consistently to what is know about YSWFs[34], it is straightforward to check that $P_v(F, z)$ satisfies M, WT and PTS if and only if $v(p) > 0, v(p)' \leq 0$ and $v''(p) \geq 0$ respectively.

We will consider now the partial ordering associated to the suggested class of poverty indices satisfying these axioms. Only the sufficiency part of the proof will be provided, the necessity part could be obtained adapting appropriately the results on welfare dominance associated to the YSWFs.[35]

Since most results provide a rationale for dominance criteria related to transformations of the *poverty gap profile curve* introduced by Spencer and Fisher (1992), Jenkins and Lambert (1997) and Shorrocks (1998), it is worth to introduce its formalization in the present context.

Let $\Omega_F^*(p, z)$ be the poverty gap profile of distribution F evaluated at the percentile $100(1-p)^{th}$ of the income distribution of the **total population**, given a poverty line z. Notice that $\Omega_F^*(p, z)$ is ranked in decreasing order and $\Omega_F^*(p, z) = 0$ if $p \geq F(z)$.

[34] See Mehran (1976), Yaari (1987) and Zoli (1999).

[35] See Zoli (1999), or consider Propositions 1, 2, 3 in the first section, adapted for the special case in which the whole population is homogeneous in non income ethically relevant characteristics.

Definition 4 (PGPC: Poverty Gap Profile Curve)

$$\mathcal{PG}_F(p,z) = \int_0^p \Omega_F^*(q,z)dq. \tag{32}$$

Notice that $\mathcal{PG}_F(t,z) = \mathcal{PG}_F(F(z),z) = H(F,z)I(F,z)$ for all $t \geq F(z)$.

We are ready now to discuss dominance conditions associated to $P_v(F,z)$. Let $\Omega_F(p,z) = F(z)\tilde{\Pi}_{F,z}^{-1}(p)$, that is $\Omega_F(i/\pi,z) = \frac{\pi}{m}\frac{z-x_i}{z}$, where $x_i \geq x_{i-1}$ and π is the number of poor individuals, while m is the size of the whole population. That is $\Omega_F(p,z)$ is the welfare ranked (relative) poverty gap at the percentile $100(1-p)^{th}$ of the income **distribution of poor** individuals, weighted by the Head count ratio. It follows that

$$P_v(F,z) = \int_0^1 v(p)\Omega_F(p,z)dp. \tag{33}$$

Proposition 8 *$P_v(F,z) \geq P_v(G,z)$ given z, for all $P_v(.,z)$ satisfying M if and only if $\Omega_F(p,z) \geq \Omega_G(p,z)$ for all $p \in [0,1]$.*

Proof.

$P_v(F,z) - P_v(G,z) \geq 0 \longleftrightarrow \int_0^1 v(p)\left[\Omega_F(p,z) - \Omega_G(p,z)\right]dp \geq 0$. Since, given M, $v(p) > 0$, $\Omega_F(p,z) \geq \Omega_G(p,z)$ for all p is a sufficient condition for $P_v(F,z) \geq P_v(G,z)$. ■

Proposition 9 *$P_v(F,z) \geq P_v(G,z)$ given z, for all $P_v(.,z)$ satisfying M and WT if and only if $\int_0^p \left[\Omega_F(q,z) - \Omega_G(q,z)\right]dq \geq 0$ for all $p \in [0,1]$.*

Proof:

Let $\Delta\Omega(p,z) = \left[\Omega_F(p,z) - \Omega_G(p,z)\right]$. Integrating by parts $\Delta P_v = P_v(F,z) - P_v(G,z)$ in (33), we get

$$\Delta P_v = v(1)\int_0^1 \Delta\Omega(p,z)dp - \int_0^1 v'(p)\left(\int_0^p \Delta\Omega(t,z)dt\right)dp. \tag{34}$$

Since $v(p)' \leq 0$ consistently with WT, $\int_0^p \Delta\Omega(t,z)dt \geq 0 \quad \forall p$ is a sufficient condition for $\Delta P_v \geq 0$. ■

Notice that

$$\mathcal{PG}_F^{\mathcal{N}}(p,z) = \int_0^p \Omega_F(q,z)dq$$

is the *(relative) poverty gap curve for the distribution of poor* individuals (the superscript $\mathcal{N}$ stands for normalized to the distribution of the poor), scaled by the Head count ratio. $\mathcal{PG}_F^{\mathcal{N}}(p,z)$ differs from the PGPC $\mathcal{PG}_F(p,z)$ because it is evaluated only over the distribution of the poor individuals, therefore it reaches its maximum when $p=1$, which is $\int_0^1 \Omega_F(q,z)dq = H(F,z)I(F,z) = \mathcal{PG}_F^{\mathcal{N}}(1,z)$. It is possible to write the dominance condition in terms of $\Omega_F^*(q,z)$, simply changing the variable of integration, moving from the distribution of incomes of poor individuals to the income distribution of the whole population. Let $q = pF(z)$ where $100q$ denotes the percentile in the whole distribution associated to the $100p^{th}$ percentile in the distribution of the poor individuals. Recalling that $\int_0^p \Omega_F(p,z)dp = F(z)\int_0^p [1 - F_{F(z)}^{-1}(p)/z]dp$, substituting we get

$$\mathcal{PG}_F^{\mathcal{N}}(p,z) = \int_0^p \Omega_F(t,z)dt = \int_0^{pF(z)} \Omega_F^*(q,z)dq = \mathcal{PG}_F(pF(z),z).$$

It follows that the previous dominance condition requires

$$\int_0^{pF(z)} \Omega_F^*(q,z)dq \geq \int_0^{pG(z)} \Omega_G^*(q,z)dq \quad \text{for all } p \in [0,1]$$

while the usual dominance in terms of PGPCs is $\int_0^p \Delta\Omega_F^*(q,z)dq \geq 0$ for all $p \in [0,1]$.

We consider now the refinement of the previous criterion associated to the $P_v(.,z)$ satisfying in addition PTS.

Proposition 10 *$P_v(F,z) \geq P_v(G,z)$ given z for all $P_v(.,z)$ satisfying M, WT and PTS if and only if $\int_0^p \left[\mathcal{PG}_F^{\mathcal{N}}(t,z) - \mathcal{PG}_G^{\mathcal{N}}(t,z)\right] dt \geq 0$ for all $p \in [0,1]$ and $\mathcal{PG}_F^{\mathcal{N}}(1,z) \geq \mathcal{PG}_G^{\mathcal{N}}(1,z)$ (i.e. $H(F,z)I(F,z) \geq H(G,z)I(G,z)$).*

Proof:

Denote $\Delta\mathcal{PG}^{\mathcal{N}}(t,z) = \mathcal{PG}_F^{\mathcal{N}}(t,z) - \mathcal{PG}_G^{\mathcal{N}}(t,z)$. Integrating by parts (34) we get

$$\begin{aligned}\Delta P_v &= v(1)\Delta\mathcal{PG}^{\mathcal{N}}(1,z) - v'(1)\int_0^1 \Delta\mathcal{PG}^{\mathcal{N}}(p,z)dp \\ &\quad + \int_0^1 v''(p)\left(\int_0^p \Delta\mathcal{PG}^{\mathcal{N}}(t,z)dt\right)dp.\end{aligned}$$

Given PTS $v''(p) \geq 0$, therefore $\Delta\mathcal{PG}^{\mathcal{N}}(1,z) \geq 0$ and $\int_0^p \Delta\mathcal{PG}^{\mathcal{N}}(t,z)dt \geq 0$ for all p are sufficient conditions for poverty dominance. ■

That is, at every percentile the area under the normalized poverty gap curves is compared, and dominance for the poverty index $H(.,z)I(.,z)$ is required. The first condition could be seen as a result of the inequality aversion introduced within the evaluation of poverty, the second condition is connected to the relative extent of the poverty within the population, what Jenkins and Lambert (1997) call the *Intensity* of the poverty. Notice that $\int_0^1 \mathcal{PG}_F^{\mathcal{N}}(p,z)dt = \frac{1}{2}P(F,z)$ that is one half of the Sen poverty index. This could be shown writing $\int_0^1 \mathcal{PG}_F^{\mathcal{N}}(p,z)dt = F(z)\int_0^1\{\int_0^p[1-[F_{F(z)}^{-1}(t)/z]]dt\}dp$. Using integration by parts we obtain $\int_0^1 \mathcal{PG}_F^{\mathcal{N}}(p,z)dt = \frac{1}{2}F(z)\Xi_1(\tilde{\Pi}_{F,z})$ which is defined in (29) , from which follows the equivalence with Sen poverty index.

The analogies with the dominance conditions valid for all increasing and inequality averse YSWFs satisfying the Principle of Positional Transfer Sensitivity is evident.

The following remark is therefore straightforward, and could be proved following the same line of reasoning as in the proof of Proposition 4. (See also Zoli, 1999). Recall that $G \succ_R F$ denotes leximin dominance of G over F.

Remark 3 *If $\mathcal{PG}_F^{\mathcal{N}}(p,z)$ and $\mathcal{PG}_G^{\mathcal{N}}(p,z)$ cross once, $G \succ_R F$, and $H(F,z)I(F,z)$ $= H(G,z)I(G,z)$, then $P_v(F,z) \geq P_v(G,z)$, given z, for all $P_v(.,z)$ satisfying M, WT and PTS if and only if $\Gamma(\Pi_{F,z}) \geq \Gamma(\Pi_{G,z})$.*

Again the Gini coefficient associated to the distribution of the poverty gaps within the population of poor individuals becomes normatively relevant. In the general case when $H(.,z)I(.,z)$ differs between distributions, is the Sen index which becomes relevant (it is only a sufficient condition) but we have to restrict the set of poverty indices in order to eliminate the less inequality averse.

We will discuss now briefly the implication of these results applied to the extension of the Sen index suggested in Shorrocks (1995).

4.2 Generalized Shorrocks indices

The Sen index shows a serious drawback: it fails in satisfying the Principle of Transfers which states that rank preserving transfers from rich to poor persons should not increase poverty. The reason is that as a result of a transfer the income of the recipient could cross the poverty line; since the ranking of individuals is based on the distribution of poor, this action could lead to a change in the rankings and a reduction in poverty is not guaranteed as a result of the transfer. In order to conform the Sen index to the Principle of Transfers (PT), Shorrocks (1995) suggests a modification of the index. He considers the distribution of the income gaps but in this case the weights associated to them are defined in terms of the ranking in the income distribution of the whole population. The index could therefore be written as

$$P^*(F,z) = 2\int_0^{\bar{x}} \frac{z - x^*}{z} \left[1 - F(x)\right] f(x)dx,$$

where x^* are censored incomes, i.e. $x^* = z$ if $x \geq z$, otherwise $x^* = x$. Alternatively we can write

$$P^*(F,z) = H(F,z)I(F,z)\left[1 + \Gamma(\Pi^*_{F,z})\right] = \Xi_1(\tilde{\Pi}^*_{F,z}),$$

where $\Pi^*_{F,z}$ is the distributions of poverty gap profiles for all the population, and $\tilde{\Pi}^*_{F,z}$ is the same distribution ranked in reverse order, notice that $\tilde{\Pi}^{*-1}_{F,z}(q) = \Omega^*_F(q,z)$. The connection with the Sen index is evident, the only difference is that, for the $P^*(F,z)$ is the Gini index of the gaps (comprising therefore also the zero values associated to all non poor individuals) which becomes relevant. Notice that the modified Sen index could be also rewritten as

$$P^*(F,z) = 2\int_0^1 (1-p)\Omega^*_F(p,z)dp.$$

Following the same line of reasoning as for Sen index we consider the generalization of $P^*(F,z)$,

$$P^*_v(F,z) = \int_0^1 v(p)\Omega^*_F(q,z)dp, \tag{35}$$

where $v(p)$ is continuous twice differentiable and $P^*_v(F,z)$ is supposed to satisfy Monotonicity $v(p) > 0$, Principle of Transfers $v'(p) \leq 0$ and Positional Sensitivity $v''(p) \geq 0$.[36]

As discussed in Shorrocks (1998) (and could be checked appropriately modifying the Proposition 9) the dominance for all $P^*_v(.,z)$ satisfying M

[36]A similar index has been also discussed in Hagenaars (1986). Duclos and Grégoire (1999) consider the extended Ginis version.

and PT is both necessary and sufficient for ensuring dominance in terms of PGPCs $\mathcal{PG}(p,z)$. This is consistent with the result in Jenkins and Lambert (1997) which proves that dominance in terms of PGPCs is equivalent to poverty dominance for the wider class of poverty indices satisfying M and PT (comprising therefore also most of the additively decomposable indices, and of course also $P_v^*(.,z)$). The extension of Proposition 10 to the poverty dominance for $P_v^*(.,z)$ where $v''(p) \geq 0$ is straightforward (the proof is omitted)

Proposition 11 *$P_v^*(F,z) \geq P_v^*(G,z)$ given z for all $P_v^*(.,z)$ satisfying M, PT and PTS if and only if $\int_0^p [\mathcal{PG}_F(t,z) - \mathcal{PG}_G(t,z)]\, dt \geq 0$ for all $p \in [0,1]$ and $H(F,z)I(F,z) \geq H(G,z)I(G,z)$.*

This result requires comparisons of the areas under the PGPCs at each percentile. Notice that, as pointed out by Shorrocks (1998), $\int_0^1 \mathcal{PG}_F(t,z)dt = P^*(F,z)/2$. The following remark is equivalent to Remark 3 when poverty is evaluated in terms of $P_v^*(.,z)$

Remark 4 *If $\mathcal{PG}_F(p,z)$ and $\mathcal{PG}_G(p,z)$ cross once, $G \succ_R F$, and $H(F,z)I(F,z) = H(G,z)I(G,z)$, then $P_v^*(F,z) \geq P_v^*(G,z)$, given z, for all $P_v^*(.,z)$ satisfying M, PT and PTS if and only if $\Gamma(\Pi_{F,z}^*) \geq \Gamma(\Pi_{G,z}^*)$.*

In this case the Gini coefficient of the distribution of poverty gaps over the whole population is normatively relevant. When $H(F,z)I(F,z) \neq H(G,z)I(G,z)$ then $P^*(F,z)$ becomes relevant in order to provide a poverty ranking of the distributions.

Moreover the classes of poverty indices $P(F,z)$ and $P^*(F,z)$ could be used in order to specify poverty dominance conditions over distributions where individuals differ in needs. If aggregating conditions discussed in the previous sections hold, it could be possible to provide an interpretation in terms of poverty perception of Properties A, A1 and A2 discussed in the first section.

In this case the extension of the results in the first sections to poverty measurement is straightforward. Consider the overall poverty evaluation function

$$\mathcal{P}(F,\mathbf{z}) = \sum_{i=1}^{n} \theta_i^F \int_0^1 v_i(p)\Omega_{F_i}^*(p,z_i)dp,$$

where $\mathbf{z}$ is the vector of appropriate poverty lines z_i of each subgroup, which may differ, and θ_i^F is the weight associated to the poverty in group i with the aggregate evaluation[37]. We will provide, without proof, (which could be obtained rearranging those of Proposition 2) the dominance condition for all $\mathcal{P}(F, \mathbf{z})$ satisfying the poverty equivalent of properties A, A1. We denote this class of poverty indices $\mathcal{P}_1$.

Proposition 12 *Given two distributions F and G, given $\mathbf{z}$, then $\mathcal{P}(F, \mathbf{z}) \geq \mathcal{P}(G, \mathbf{z})$ for all $\mathcal{P}(F, \mathbf{z})$ in $\mathcal{P}_1$ if and only if*

$$\sum_{i=1}^{k} \left[\theta_i^F \mathcal{PG}_{F_i}(p, z_i) - \theta_i^G \mathcal{PG}_{G_i}(p, z_i) \right] \geq 0$$

for any $k = 1, 2, ...n$ and every $p \in [0, 1]$.

We get therefore sequential dominance in terms of weighted averages of the PGPCs of each subgroup. If we consider comparison of aggregate indices of poverty based on absolute poverty gaps, then it is possible to provide an immediate interpretation of properties A and A1. Property A requires that for the indices satisfying M, at any percentile a decrease in the poverty gap of a poor individual has a bigger impact on poverty evaluation the needier is the individual. While Property A1, given PT, is obtained imposing that in comparing the effect of a Progressive Transfer occurring between individuals at given percentiles in the income distributions, the needier is the reference group to whom they belong the higher is the impact of such transfer on poverty reduction. Notice that since the weights $v_i(p)$ are supposed not to depend on the income distribution, then, in order to satisfy the previous requirements, the weights should be characterized as in the rank-dependent welfare evaluation framework, without taking into account the information on the poverty line. Within this framework the poverty dominance criteria obtained are based on sequential comparison of averaged Absolute Poverty Gap Profiles.

All the results equivalent to those discussed in the first section follow intuitively, as well as all those associated to dominance in terms of generalized Sen indices.

[37]Duclos and Grégoire (1999) suggest a similar aggregate poverty index based on extended Ginis version of $v_i(p)$ independent from i.

5 Conclusions

In this paper we have discussed a simple extension of the rank dependent welfare representation model to evaluations over distributions of individuals differing in needs. The major point of weakness of the approach is that given its own nature the rank-dependent model is not additively decomposable over subgroups. It becomes therefore necessary to specify some aggregating assumptions of the groups evaluation. What is suggested is a very simple additivity condition, which is far from being indisputable. Nevertheless, even in a very simplified context the result obtained seems to match the intuition with the standard Atkinson-Bourguignon dominance condition associated to the utilitarian framework. What is obtained is a *dual version* of the above mentioned sequential dominance criteria: instead of considering direct dominance, the results obtained are based on inverse sequential dominance of different degrees.

An interpretation in terms of relative deprivation of the imposed welfare characterization is provided. Sequential orderings associated to generalized satisfaction curves or deprivation curves are introduced. Finally, the implications for poverty measurement, based on general classes of rank-dependent indices are discussed. Some simple extensions of existing results are provided as well as a possible specification of ranking criteria over distributions with individuals exhibiting different needs consistent with rank-dependent poverty orderings.

6 Appendix

6.1 Proof Proposition 2:

Necessity Part: Making use of reduction to absurd arguments we will show that it is impossible that $\sum_{i=1}^{k}\psi_i(p)$ being negative for some k when $\Delta W \geq 0$ for any $W \in \mathcal{Y}_1$.

Denote with $\Psi_k(p) = \sum_{j=1}^{k}\psi_j(p)$, then (9) becomes

$$\Delta W = -\int_0^1 \sum_{i=1}^{n} \left(w_i'(p)\Psi_i(p)\right) dp + \sum_{i=1}^{n} \left(w_i(1)\Psi_i(1)\right) \qquad (36)$$

Suppose there exists a k such that $\Psi_k(p) < 0$ for some $p \in [0,1]$ then, following Lemma 1, there exist an interval $\mathcal{J} \subset [0,1]$ and a set of functions

$\hat{\varepsilon}_1(p),..\hat{\varepsilon}_i(p),...\hat{\varepsilon}_n(p) \in V^+$; such that $\sum_{i=1}^n \hat{\varepsilon}_i(p)\Psi_i(p) < 0$ for any $p \in \mathcal{J}$. Following Lemma 2 we obtain:

$$\int_0^1 z(p)\left(\sum_{i=1}^n \hat{\varepsilon}_i(p)\Psi_i(p)\right)dp < 0. \tag{37}$$

Denote $v_i(p)$ with:

$$\begin{aligned} v_i(p) &= \sum_{j=i}^n \left[\int_0^1 \hat{\varepsilon}_j(p)z(p)dp - \int_0^p \hat{\varepsilon}_j(p)z(p)dp + \zeta_j\right] = \\ &= \sum_{j=i}^n \int_p^1 \hat{\varepsilon}_j(p)z(p)dp + \sum_{j=i}^n \zeta_j \end{aligned} \tag{38}$$

where $z(p) \in V^+$, $\zeta_j \geq 0$ and $\hat{\varepsilon}_i(p) \in V^+$ $\forall i = 1, 2, ..n$, then $v_i(p) \geq 0$. Given Property A:

$$v_i(p) - v_{i+1}(p) = w_i(p) = \int_p^1 \hat{\varepsilon}_i(p)z(p)dp + \zeta_i \geq 0, \tag{39}$$

thus $w_i'(p) = -\hat{\varepsilon}_i(p)z(p)$. The obtained $w_i'(p)$ satisfy Property A1, moreover notice that $w_i(1) = \zeta_i \geq 0$. Substituting for $w_i'(p)$ into (36) we obtain:

$$\Delta W = \int_0^1 z(p)\left(\sum_{i=1}^n \hat{\varepsilon}_i(p)\Psi_i(p)\right)dp + \sum_{i=1}^n \left(\zeta_i \Psi_i(1)\right). \tag{40}$$

Recall that $\Delta W \geq 0$ for any $W \in \mathcal{Y}_1$ implies that the integral in (40) is not negative.

The necessity part of the proof is obtained following the same line of reasoning of the necessity part of Proposition 1 proof. Notice that ζ_i is independent from the choice of $z(p)$ and $\hat{\varepsilon}_i(p)$ $\forall i, p$, then they could be chosen such that if $\Psi_k(p) < 0$ for some $p \in [0, 1]$ at least one of the elements in the r.h.s. of (40) is negative. Since ζ_i is independent from $z(p)$ and $\hat{\varepsilon}_i(p)$ it could always be set in such a way to contradict (40). ■

6.2 Proof Proposition 3:

Necessity Part: Making use of reduction to absurd arguments we will show that it is impossible that $\Psi_i(1)$, and or $\Upsilon_i(p)$ being negative for some i when $\Delta W \geq 0$ for any $W \in \mathcal{Y}_2$. Consider (19):

$$\Delta W = \sum_{i=1}^{n} [w_i(1)\Psi_i(1) - w_i'(1)\Upsilon_i(1)] + \int_0^1 \sum_{i=1}^{n} (w_i''(p)\Upsilon_i(p))\, dp \qquad (41)$$

Suppose there exists a k such that $\Upsilon_k(p) < 0$ for some $p \in [0,1]$ then, following Lemma 1, there exist an interval $\mathcal{J} \subset [0,1]$ and a set of $\tilde{e}_1(p), ..\tilde{e}_i(p), ...\tilde{e}_n(p) \in V^+$; such that $\sum_{i=1}^{n} \tilde{e}_i(p)\Upsilon_i(p) < 0$ for any $p \in \mathcal{J}$. Following Lemma 2 we obtain:

$$\int_0^1 z(p) \left(\sum_{i=1}^{n} \tilde{e}_i(p)\Upsilon_i(p) \right) dp < 0. \qquad (42)$$

We follow the same line of reasoning as in the proof of the previous proposition. As in (38) we let

$$v_i(p) = \sum_{j=i}^{n} \left[\int_p^1 \left(\int_q^1 \tilde{\varepsilon}_j(t)z(t)dt \right) dq + (1-p)\breve{\varepsilon}_j + \vartheta_j \right] \qquad (43)$$

where $z(p) \in V^+$ and $\tilde{\varepsilon}_i(p) \in V^+$ $\forall i = 1, 2, ..n$, $\breve{\varepsilon}_i \geq 0$, $\vartheta_i \geq 0$, notice that $v_i(1) = \sum_{j=i}^{n} \vartheta_j$. We now have to check that the representation of $v_i(p)$ in (43) satisfies properties A, A1 and A2. According to Property A:

$$v_i(p) - v_{i+1}(p) = w_i(p) = \int_p^1 \left(\int_q^1 \tilde{\varepsilon}_i(t)z(t)dt \right) dq + (1-p)\breve{\varepsilon}_i + \vartheta_i, \qquad (44)$$

it is evident that $w_i(p) \geq 0$, and $w_i(p)$ reaches its minimum for $p = 1$: $w_i(1) = \vartheta_i$. Taking the derivative of $w_i(p)$ we obtain

$$w_i'(p) = -\int_p^1 \tilde{\varepsilon}_i(t)z(t)dt - \breve{\varepsilon}_i, \qquad (45)$$

which is consistent with A1 because $w_i'(p) \leq 0, w_i'(1) = -\breve{\varepsilon}_i \leq 0$. Consider now the second derivative, $w_i''(p) = \tilde{\varepsilon}_i(p)z(p) \geq 0$, therefore $v_i(p)$ in (43) is consistent with properties A, A1 & A2. After substituting for $w_i(1), w_i'(1)$ and $w_i''(p)$ into (41) we obtain:

$$\Delta W = \sum_{i=1}^{n} \vartheta_i \Psi_i(1) + \sum_{i=1}^{n} \breve{\varepsilon}_i \Upsilon_i(1) + \int_0^1 z(p) \left(\sum_{i=1}^{n} \tilde{\varepsilon}_i(p) \Upsilon_i(p) \right) dp. \quad (46)$$

Suppose now that $\Delta W \geq 0$ for any $W \in \mathcal{Y}_2$. This implies that all three terms should not be negative, if even one of them is negative, it will always be possible to select appropriate values for $\vartheta_i, \breve{\varepsilon}_i$ and for the functions $z(p)$ and $\tilde{\varepsilon}_i(p)$ characterizing the two remaining terms (which are independent each other), in order to obtain $\Delta W < 0$. Therefore $\Delta W \geq 0$ implies $\sum_{i=1}^{n} \vartheta_i \Psi_i(1) \geq 0$, $\sum_{i=1}^{n} \breve{\varepsilon}_i \Upsilon_i(1) \geq 0$, and $\int_0^1 z(p) \left(\sum_{i=1}^{n} \tilde{\varepsilon}_i(p) \Upsilon_i(p) \right) dp \geq 0$, from which it follows that $\Psi_i(1) \geq 0$, $\Upsilon_i(p) \geq 0$ for all i, p. Otherwise $\vartheta_i, \breve{\varepsilon}_i, z(p)$ and $\tilde{\varepsilon}_i(p)$ could be chosen such that to obtain at least a negative term in (46). ■

6.3 Proof Proposition 5:

Proposition $\mathcal{D}(\mathbf{y})$ and $\mathcal{S}(\mathbf{y})$ in (22) satisfy LH and PR if and only if there exist not-decreasing functions $\mathcal{H}^i : \mathcal{Z}_1^n \to \mathbb{R}_+$, $V^i : \mathcal{Z}_1 \to \mathbb{R}_+$, , and constant $\alpha \in \mathbb{R}$ such that

$$\mathcal{D}(\mathbf{y}) = \sum_{i=1}^{n} \mathcal{H}^i(\mathbf{q}) m_i^{\alpha} \sum\nolimits_{k=1}^{m_i} \left[V^i(k/m_i) - V^i((k-1)/m_i) \right] d^i_{(k)}(\mathbf{y}),$$

and similarly

$$\mathcal{S}(\mathbf{y}) = \sum_{i=1}^{n} \mathcal{H}^i(\mathbf{q}) m_i^{\alpha} \sum\nolimits_{k=1}^{m_i} \left[V^i(k/m_i) - V^i((k-1)/m_i) \right] s^i_{(k)}(\mathbf{y}).$$

In order to prove Proposition 5 we follow a sequence of steps represented by the following lemmata.

Lemma 3 *If* (22) *satisfies LH then:*
a) $\mathcal{C}^i(0; m_1, m_2, ..m_n) = 0$ *and*
b) $\mathcal{C}^i(\mathcal{D}^i(\mathbf{y}); m_1, m_2, ..m_n) = \mathcal{F}^i(m_1, m_2, ..m_n)\mathcal{D}^i(\mathbf{y})$ *if* $\mathcal{D}^i(\mathbf{y}) > 0$.

Proof:

According to the initial definition of aggregate deprivation we can write it as in (22):

$$\mathcal{D}(\mathbf{y}) = \sum_{i=1}^{n} \mathcal{C}^i(\mathcal{D}^i(\mathbf{y}); m_1, m_2, ..m_n), \tag{47}$$

Let, for convenience, $\mathbf{m}$ the vector of cardinalities of all reference groups i.e. $\mathbf{m} = (m_1, m_2, ..m_i, .m_n)$, we denote the set of all n^{th}-dimensional vectors $\mathbf{m}$ such that each element is either a positive integer or zero, and at least a m_i is positive by $\mathcal{M}^n$. In order to apply LH we consider $D(\lambda\mathbf{y}) = \sum_{i=1}^{n} \mathcal{C}^i(\mathcal{D}^i(\lambda\mathbf{y}); \mathbf{m})$; and $\lambda D(\mathbf{y}) = \lambda \sum_{i=1}^{n} \mathcal{C}^i(\mathcal{D}^i(\mathbf{y}); \mathbf{m})$, where for all indices in (21) $\mathcal{D}^i(\lambda\mathbf{y}) = \lambda\mathcal{D}^i(\mathbf{y})$. According to LH after substituting for $\mathcal{D}^i(\mathbf{y})$ in (21) we get,

$$\sum_{i=1}^{n} \mathcal{C}^i(\lambda\mathcal{D}^i(\mathbf{y}); \mathbf{m}) = \lambda \sum_{i=1}^{n} \mathcal{C}^i(\mathcal{D}^i(\mathbf{y}); \mathbf{m}) \quad \forall\lambda > 0, \mathcal{D}^i(\mathbf{y}) \geq 0, \mathbf{m} \in \mathcal{M}^n \tag{48}$$

which is equivalent to

$$\mathcal{C}^i(\lambda\mathcal{D}^i(\mathbf{y}); \mathbf{m}) = \lambda\mathcal{C}^i(\mathcal{D}^i(\mathbf{y}); \mathbf{m}) \quad \text{for all } \lambda > 0, i, \mathcal{D}^i(\mathbf{y}) \geq 0, \mathbf{m} \in \mathcal{M}^n. \tag{49}$$

This condition implies (48), the reverse could be checked making use of a counter-example. Consider for instance a distribution where within all the reference groups income is shared equally, it follows that deprivation is 0 in all reference groups. According to (48) we have $\sum_i \mathcal{C}^i(\lambda 0; \mathbf{m}) = \lambda \sum_i \mathcal{C}^i(0; \mathbf{m})$, which is satisfied iff $\sum_i \mathcal{C}^i(0; \mathbf{m}) = 0$. Given that $\mathcal{C}^i(.; \mathbf{m}) \geq 0$, it follows $\mathcal{C}^i(0; \mathbf{m}) = 0$ for all i and all $\mathbf{m} \in \mathcal{M}^n$, which gives statement a) in the lemma.

Consider now an income distribution in which in group j income is shared unequally, while within all the other subgroups income is equally distributed. From (48) we have $\sum_{i \neq j} \mathcal{C}^i(\lambda 0; \mathbf{m}) + \mathcal{C}^j(\lambda\mathcal{D}^j(\mathbf{y}^j); \mathbf{m}) = \lambda \sum_{i \neq j} \mathcal{C}^i(0; \mathbf{m}) + \lambda\mathcal{C}^j(\mathcal{D}^j(\mathbf{y}^j); \mathbf{m})$ it follows $\mathcal{C}^j(\lambda\mathcal{D}^j(\mathbf{y}^j); \mathbf{m}) = \lambda\mathcal{C}^j(\mathcal{D}^j(\mathbf{y}^j); \mathbf{m})$ for all $\lambda > 0, j, \mathcal{D}^j(\mathbf{y}^j) > 0, \mathbf{m} \in \mathcal{M}^n$, which together with $\mathcal{C}^i(0; \mathbf{m}) = 0$ gives (49).

The solution of the functional equation in (49) could be obtained letting $\mathcal{D}^i(\mathbf{y}) = 1$, then $\mathcal{C}^i(\lambda; \mathbf{m}) = \lambda\mathcal{C}^i(1; \mathbf{m})$, from which setting $\mathcal{D}^i(\mathbf{y}) = \lambda > 0$:

$$\mathcal{C}^i(\mathcal{D}^i(\mathbf{y}); \mathbf{m}) = \mathcal{D}^i(\mathbf{y})\mathcal{C}^i(1; \mathbf{m}) = \mathcal{D}^i(\mathbf{y})\mathcal{F}^i(\mathbf{m}),$$

as in statement b). It follows

$$\mathcal{D}(\mathbf{y}) = \sum_{i=1}^{n} \mathcal{F}^i(\mathbf{m})\mathcal{D}^i(\mathbf{y}), \; \mathcal{S}(\mathbf{y}) = \sum_{i=1}^{n} \mathcal{F}^i(\mathbf{m})\mathcal{S}^i(\mathbf{y}). \tag{50}$$

■

Proof of Proposition 5 (continued).

Let $\mathbb{I}_+$ the set of positive integers, $\mathbb{Z}_+$ the set of positive rationals considering also zero, $\mathbb{Z}_+^n$ is the n-th fold Cartesian product of $\mathbb{Z}_+$, and $\mathcal{Z}_1 = \{x \in \mathbb{Z}_+ : 0 \leq x \leq 1\}$. Moreover, recall that $\mathcal{M}^n = \{\mathbf{m} \in \mathbb{I}_+^n/\mathbf{0}\}$, denote $\mathcal{Z}^n = \{\mathbf{q} \in \mathbb{Z}_+^n/\mathbf{0}\}$, while $\mathcal{Z}_1^n = \{\mathbf{q} \in \mathcal{Z}^n : \sum_i q_i = 1\}$ the set of all vectors in $\mathcal{Z}^n$ whose elements sum to 1.

Lemma 4 *If* (50) *satisfies PR then:*
A) There exist functions: $B^i : \mathbb{I}_+ \rightarrow \mathbb{R}_+$, *where* $B^i(m_i) := A^i(m_i, m_i)$ *and* $Q^i : \mathcal{Z}_1 \rightarrow \mathbb{R}_+$, *with* $Q^i(1) = 1$, *such that*

$$A^i(k, m_i) = Q^i(\frac{k}{m_i})B^i(m_i)$$

for all $k, m_i \in \mathbb{I}_+$, $k \leq m_i$, $i = 1, 2, ..n$.
B)There exists a function $\mathcal{H}^i : \mathcal{Z}_1^n \rightarrow \mathbb{R}_+$ *such that*

$$\mathcal{F}^i(\mathbf{m}) = \mathcal{H}^i(\mathbf{q})\mathcal{F}_0^i(m_i),$$

where $\mathbf{q} = \mathbf{m}/\sum_i m_i$, *and* $\mathcal{F}_0^i(m_i) := \mathcal{F}^i(0, 0, 0, ..m_i, ..0, 0)$.
C) Functions $\phi(r)$, $\mathcal{F}_0^i(m_i)$, $B^i(m_i)$ *should satisfy the conditions:* $\mathcal{F}_0^i(rm_i)B^i(rm_i) = \phi(r)\mathcal{F}_0^i(m_i)B^i(m_i)$ *for all* $r, m_i \in \mathbb{I}_+$, *and* $\phi(rt) = \phi(r)\phi(t)$ *for all* $r, t \in \mathbb{I}_+$.

Proof:

Part A) Apply PR to (50) , it derives

$$\begin{aligned} D(\mathbf{y}_r) &= \sum_{i=1}^{n} \mathcal{F}^i(r\mathbf{m})\mathcal{D}^i(\mathbf{y}_r) = \phi(r)D(\mathbf{y}) \\ &= \phi(r)\sum_{i=1}^{n} \mathcal{F}^i(\mathbf{m})\mathcal{D}^i(\mathbf{y}). \end{aligned}$$

Suppose $\mathcal{D}^i(\mathbf{y}) = 0$ for all $i \neq j$, then for any j we have

$$\mathcal{F}^j(r\mathbf{m})\mathcal{D}^j(\mathbf{y}_r) = \phi(r)\mathcal{F}^j(\mathbf{m})\mathcal{D}^j(\mathbf{y}). \tag{51}$$

Recalling that $\mathcal{D}^i(\mathbf{y}) = \sum_{k=1}^{m_i}[A^i(k, m_i) - A^i(k-1, m_i)]\, d^i_{(k)}(\mathbf{y})$ where $d^i_{(k)}(\mathbf{y}) = \frac{1}{m_i}\left(\sum_{j \geq k} y_{(j)} - y_{(k)}\right)$, noticing that $d^i_{(k)}(\mathbf{y}) = d^i_{(h)}(\mathbf{y}_r)$ for all $h = r(k-1) + 1, r(k-1) + 2, ..rk$, it follows

$$\mathcal{D}^i(\mathbf{y}_r) = \sum_{k=1}^{m_i}\left[A^i(rk, rm_i) - A^i(r(k-1), rm_i)\right] d^i_{(k)}(\mathbf{y}).$$

From which (51) becomes

$$\begin{aligned} & \mathcal{F}^i(r\mathbf{m}) \sum_{k=1}^{m_i} \left[A^i(rk, rm_i) - A^i(r(k-1), rm_i) \right] d^i_{(k)}(\mathbf{y}) \\ = & \ \phi(r)\mathcal{F}^i(\mathbf{m}) \sum_{k=1}^{m_i} \left[A^i(k, m_i) - A^i(k-1, m_i) \right] d^i_{(k)}(\mathbf{y}). \end{aligned}$$

Let $d^i_{(k)}(\mathbf{y}) = 0$ for all $k \geq l$, with $l \leq m_i$ and $d^i_{(k)}(\mathbf{y}) = x > 0$ for all $k < l$, consider $l = 2$, then

$$\mathcal{F}^i(r\mathbf{m}) \left[A^i(r1, rm_i) - A^i(r0, rm_i) \right] = \phi(r)\mathcal{F}^i(\mathbf{m}) \left[A^i(1, m_i) - A^i(0, m_i) \right],$$

that is, since $A^i(0, m) = 0$,

$$\mathcal{F}^i(r\mathbf{m}) A^i(r1, rm_i) = \phi(r)\mathcal{F}^i(\mathbf{m}) A^i(1, m_i).$$

Now, suppose that $l = 3$, then

$$\begin{aligned} & \mathcal{F}^i(r\mathbf{m}) \sum_{k=1}^{2} \left[A^i(rk, rm_i) - A^i(r(k-1), rm_i) \right] x \\ = & \ \phi(r)\mathcal{F}^i(\mathbf{m}) \sum_{k=1}^{2} \left[A^i(k, m_i) - A^i(k-1, m_i) \right] x \end{aligned}$$

which together with the previous condition leads to

$$\mathcal{F}^i(r\mathbf{m}) \left[A^i(r2, rm_i) - A^i(r1, rm_i) \right] = \phi(r)\mathcal{F}^i(\mathbf{m}) \left[A^i(2, m_i) - A^i(1, m_i) \right].$$

Following the same line of reasoning we get

$$\begin{aligned} & \mathcal{F}^i(r\mathbf{m}) \left[A^i(rk, rm_i) - A^i(r(k-1), rm_i) \right] \\ = & \ \phi(r)\mathcal{F}^i(\mathbf{m}) \left[A^i(k, m_i) - a^i(k-1, m_i) \right] \end{aligned} \quad (52)$$

for all $k < m_i$. We now check that the condition could be extended also to $k = m_i$.

Recall that property PR holds also for satisfaction indices, that is PR should be satisfied also if we substitute $s^i_{(k)}(\mathbf{y})$ to $d^i_{(k)}(\mathbf{y})$. Consider a distribution where total income is equally shared between individuals, then

$s^i_{(k)}(\mathbf{y}) = \mu(\mathbf{y}^i)$, since average income is replication invariant, PR requires for this distribution

$$\begin{aligned} & \mathcal{F}^i(r\mathbf{m}) \sum_{k=1}^{m_i} \left[A^i(rk, rm_i) - A^i(r(k-1), rm_i) \right] \mu(\mathbf{y}^i) \\ = \; & \phi(r)\mathcal{F}^i(\mathbf{m}) \sum_{k=1}^{m_i} \left[A^i(k, m_i) - A^i(k-1, m_i) \right] \mu(\mathbf{y}^i). \end{aligned}$$

Given (52), the condition boils down to

$$\begin{aligned} & \mathcal{F}^i(r\mathbf{m}) \left[A^i(rm_i, rm_i) - A^i(r(m_i-1), rm_i) \right] \\ = \; & \phi(r)\mathcal{F}^i(\mathbf{m}) \left[A^i(m_i, m_i) - A^i(m_i-1, m_i) \right]. \end{aligned}$$

Notice that $A^i(k, m_i)$ is independent from the size of all the reference groups different from i. Let $\mathbf{m}^i_0 = (0,0,0,m_i,0,0,0)$, and denote $\mathcal{F}^i(\mathbf{m}^i_0) = \mathcal{F}^i_0(m_i)$, then

$$\begin{aligned} & \mathcal{F}^i_0(rm_i) \left[A^i(rk, rm_i) - A^i(r(k-1), rm_i) \right] \\ = \; & \phi(r)\mathcal{F}^i_0(m_i) \left[A^i(k, m_i) - A^i(k-1, m_i) \right] \end{aligned}$$

for all $k \leq m_i$. Letting $k = 1$ we get

$$\mathcal{F}^i_0(rm_i) A^i(r, rm_i) = \phi(r)\mathcal{F}^i_0(m_i) A^i(1, m_i),$$

from which, if $A^i(1, m_i)$ and $\mathcal{F}^i_0(m_i)$ are different from 0 for all $m_i \in \mathbb{I}_+$:

$$\frac{A^i(r, rm_i)}{A^i(1, m_i)} = \phi(r) \frac{\mathcal{F}^i_0(m_i)}{\mathcal{F}^i_0(rm_i)}.$$

Substituting into the previous functional equation:

$$\begin{aligned} & \left[A^i(rk, rm_i) - A^i(r(k-1), rm_i) \right] \\ = \; & \frac{A^i(r, rm_i)}{A^i(1, m_i)} \left[A^i(k, m_i) - A^i(k-1, m_i) \right], \end{aligned} \tag{53}$$

which, after rearranging gives

$$\begin{aligned} & \frac{A^i(rk, rm_i)}{A^i(r, rm_i)} - \frac{A^i(k, m_i)}{A^i(1, m_i)} \\ = \; & \frac{A^i(k-1, m_i)}{A^i(1, m_i)} - \frac{A^i(r(k-1), rm_i)}{A^i(r, rm_i)} \end{aligned}$$

for all $k, r, m_i \in \mathbb{I}_+$ and for all groups i. From this condition it is evident that $\frac{A^i(rk,rm_i)}{A^i(r,rm_i)} - \frac{A^i(k,m_i)}{A^i(1,m_i)}$ is independent from k for all $1 \leq k \leq m_i$, Letting $k = 1$ we have $\frac{A^i(k-1,m_i)}{A^i(1,m_i)} - \frac{A^i(r(k-1),rm_i)}{A^i(r,rm_i)} = 0$, therefore

$$\frac{A^i(rk, rm_i)}{A^i(r, rm_i)} = \frac{A^i(k, m_i)}{A^i(1, m_i)} \tag{54}$$

for all $k, r, m_i \in \mathbb{I}_+$, $k \leq m_i$ and all i.

Let $k = m_i$, and denote $B^i(m_i) := A^i(m_i, m_i)$, then the functional equation in (54) becomes

$$\frac{B^i(rm_i)}{A^i(r, rm_i)} = \frac{B^i(m_i)}{A^i(1, m_i)}$$

that is if $A^i(r, rm_i) > 0$, and $B^i(rm_i) > 0$, follows $\frac{A^i(r,rm_i)}{A^i(1,m_i)} = \frac{B^i(rm_i)}{B^i(m_i)}$, which substituted into (54) leads to

$$\frac{A^i(rk, rm_i)}{B^i(rm_i)} = \frac{A^i(k, m_i)}{B^i(m_i)}.$$

It should be highlighted that $B^i(m_i) > 0$ for all m_i, is a necessary conditions for avoiding $a^i(k, m_i) = 0$ for all $k \leq m_i$. Indeed if $B^i(m_i) = 0$, then by the definition of $a^i(k, m_i)$, since $A^i(0, m_i) = 0$, it follows that $\sum_{k=1}^{m_i} a^i(k, m_i) = 0$ which, since we consider $a^i(k, m_i) \geq 0$, requires $a^i(k, m_i) = 0$ for all $k \leq m_i$ in order to be satisfied. Moreover, given $B^i(m_i) > 0$ also $A^i(r, rm_i) > 0$ is necessary for avoiding $a^i(k, m_i) = 0$ for all $k \leq m_i$, from (53) the condition $A^i(r, rm_i) = 0$ gives $A^i(rk, rm_i) = A^i(r(k-1), rm_i) = 0$ for all k which leads to $a^i(k, m_i) = 0$ for all k.

Denote with $Z^i(k, m_i) := \frac{A^i(k,m_i)}{B^i(m_i)}$, the previous condition is therefore

$$Z^i(rk, rm_i) = Z^i(k, m_i).$$

We now follow Donaldson and Weymark (1980) in order to show that the solution of the functional equation implies that $Z^i(k, m_i)$ could be defined as a function over nonnegative rational numbers.

Define $\eta = E(k,m_i)$ as the largest common divisor of k and m_i, for all $k, m_i \in \mathbb{I}_+$, $k \leq m_i$. It follows

$$\begin{aligned} Z^i(k,m_i) &= Z^i\left(\eta\frac{k}{\eta}, \eta\frac{m_i}{\eta}\right) = Z^i\left(\frac{k}{\eta}, \frac{m_i}{\eta}\right) \\ &= Z^i\left(\frac{k}{E(k,m_i)}, \frac{m_i}{E(k,m_i)}\right) \qquad \forall m_i > k \in \mathbb{I}_+. \end{aligned} \tag{55}$$

Notice that $E(k,m_i)$ is homogeneous of degree 1, therefore $Z^i\left(\frac{k}{E(k,m_i)},\frac{m_i}{E(k,m_i)}\right)$ is homogeneous of degree 0 in k and m_i. We could therefore define a function $Q^i : \mathbb{Z}_+ \to \mathbb{R}_+$ (where $\mathbb{Z}_+$ is the set of positive rational numbers including zero), such that

$$\begin{aligned} Q^i(\frac{k}{m_i}) &: = Z^i(k,m_i) \quad \text{if } 1 < k \leq m_i \\ Q^i(0) &: = 0 \quad \text{if } k = 0. \end{aligned} \tag{56}$$

From which $A^i(k,m_i) = Q^i(\frac{k}{m_i})B^i(m_i)$, thus, given the definition of $B^i(m_i)$, follows $Q^i(1) = 1$.

Part B) After substituting for (56) into (52) we get

$$\begin{aligned} &\mathcal{F}^i(r\mathbf{m})B^i(rm_i)\left[Q^i(\frac{rk}{rm_i}) - Q^i(\frac{r(k-1)}{rm_i}))\right] \\ = \ &\phi(r)\mathcal{F}^i(\mathbf{m})B^i(m_i)\left[Q^i(\frac{k}{m_i}) - Q^i(\frac{(k-1)}{m_i}))\right], \end{aligned}$$

that is, either $Q^i(\frac{k}{m_i}) = Q^i(\frac{(k-1)}{m_i})$ for all $k \leq m_i$, which is ruled out because in this case $a^i(k,m_i) = 0$ for all $k \leq m_i$, or

$$\mathcal{F}^i(r\mathbf{m})B^i(rm_i) = \phi(r)\mathcal{F}^i(\mathbf{m})B^i(m_i). \tag{57}$$

Consider $\mathcal{F}_0^i(m_i)$, we have that

$$\mathcal{F}_0^i(rm_i)B^i(rm_i) = \phi(r)\mathcal{F}_0^i(m_i)B^i(m_i), \tag{58}$$

that is, if $\mathcal{F}_0^i(m_i) > 0$, and $B^i(m_i) > 0$ for all $m_i \in \mathbb{I}_+$, then substituting in the previous functional equation we get

$$\frac{\mathcal{F}^i(\mathbf{m})}{\mathcal{F}_0^i(m_i)} = \frac{\mathcal{F}^i(r\mathbf{m})}{\mathcal{F}_0^i(rm_i)}.$$

Let $\frac{\mathcal{F}^i(\mathbf{m})}{\mathcal{F}_0^i(m_i)} := \mathcal{K}^i(\mathbf{m})$, then the previous condition is

$$\mathcal{K}^i(\mathbf{m}) = \mathcal{K}^i(r\mathbf{m})$$

for all $\mathbf{m} \in \mathcal{M}^n$, $r \in \mathbb{I}_+$.

Let $0 < n_+ \leq n$ the number of all the groups i such that $m_i \in \mathbb{I}_+$. Define $\eta^{n_+} = E^{n_+}(\mathbf{m})$ the largest common divisor of all the group sizes $m_1, m_2, ..m_i, .$

and m_n belonging to $\mathbb{I}_+$, for all $\mathbf{m} \in \mathcal{M}^n$. That is in $E^{n+}(\mathbf{m})$ we consider only the groups with at least one individual. It follows

$$\begin{aligned}\mathcal{K}^i(\mathbf{m}) &= \mathcal{K}^i\left(\eta^{n+}\frac{m_1}{\eta^{n+}}, \eta^{n+}\frac{m_2}{\eta^{n+}}, ..\eta^{n+}\frac{m_n}{\eta^{n+}}\right) \\ &= \mathcal{K}^i\left(\frac{m_1}{\eta^{n+}}, \frac{m_2}{\eta^{n+}}, ..\frac{m_n}{\eta^{n+}}\right) = \mathcal{K}^i\left(\frac{\mathbf{m}}{E^{n+}(\mathbf{m})}\right) \qquad \forall i,\ \mathbf{m} \in \mathcal{M}^n.\end{aligned}$$

It follows that $\mathcal{K}^i$ is homogeneous of degree 0. From which we could define $\mathcal{H}^i : \mathcal{Z}_1^n \to \mathbb{R}_+$ (where $\mathcal{Z}_1^n$ is the nth-dimensional Cartesian product of the set of positive rational numbers including zero such that they sum to 1), a function of the population shares of each group where $q_i = m_i / \sum_j m_j = m_i/m$, and $\mathbf{q}$ is the nth-dimensional vector of q_i, such that:

$$\begin{aligned}\mathcal{K}^i(\mathbf{m}) &: = \mathcal{H}^i(\mathbf{q}) && \text{for all } \mathbf{m} \in \mathcal{M}^n \text{ such that } m_i \neq 0 \\ \mathcal{K}^i(\mathbf{m}) &: = 0 && \text{for all } \mathbf{m} \in \mathcal{M}^n \text{ such that } m_i = 0.\end{aligned}$$

It follows

$$\mathcal{F}^i(\mathbf{m}) = \mathcal{H}^i(\mathbf{q})\mathcal{F}_0^i(m_i).$$

Part C) Substituting into the previous conditions we get

$$\mathcal{F}_0^i(rm_i)B^i(rm_i) = \phi(r)\mathcal{F}_0^i(m_i)B^i(m_i) \tag{59}$$

which is the first part of statement C in the lemma. The second part of the final statement is independent form the previous statements in parts A & B. Therefore we recover it without imposing the restrictions obtained in the previous parts. Let $\mathcal{F}^i(\mathbf{m})\left[A^i(k, m_i) - A^i(k-1, m_i)\right] = G^i(\mathbf{m}_{-i}, k, m_i)$ from (52) extended to the case in which also $k = m_i$, follows

$$G^i(r\mathbf{m}, rk, rm_i) = \phi(r)G^i(\mathbf{m}, k, m_i)$$

for all $k \leq m_i$, for all i, and for all $r \in \mathbb{I}_+$. Consider

$$\begin{aligned}\phi(rt)G^i(\mathbf{m}, k, m_i) &= G^i(rt\mathbf{m}, rtk, rtm_i) \\ &= \phi(r)G^i(t\mathbf{m}, tk, tm_i) \\ &= \phi(r)\phi(t)G^i(\mathbf{m}, k, m_i)\end{aligned}$$

from which if $G^i(\mathbf{m}, k, m_i) > 0$, we get the condition

$$\phi(rt) = \phi(r)\phi(t) \tag{60}$$

for all $r, t \in \mathbb{I}_+$. ■

Proof of Proposition 5 (continued)

We now find the solution for (60) :

Lemma 5 *A weakly monotonic function* $\phi : \mathbb{I}_+ \to]0, +\infty[$ *satisfies*

$$\phi(rt) = \phi(r)\phi(t)$$

for all $r, t \in \mathbb{I}_+$ *iff* $\phi(r) = r^\alpha$ *where* $\alpha \in R$, *for all* $r \in \mathbb{I}_+$.

Proof:

Notice that

$$\phi(rt) = \phi(r)\phi(t)$$

is a Cauchy functional equation, since $\phi(.)$ is defined over $\mathbb{I}_+$, we cannot make use of the solution valid for functions defined over dense sets (see Aczél 1966) but we will follow the results in Donaldson and Weymark (1980) in order to prove our claim.

The next result follows from Lemma 1 in Donaldson and Weymark (1980):

Lemma 6 (Donaldson and Weymark (1980)) *The function* $\phi : \mathbb{I}_+ \to]0, +\infty[$ *satisfies* $\phi(rv) = \phi(r)\phi(v)$ *for all* $r, v \in \mathbb{I}_+$ *if and only if*

$$\phi(t) = \prod_{i \in \mathbb{I}_+} c_i^{n_i},$$

where $c_i \in \mathbb{R}$ *is an arbitrary constant and* n_i *is the number of times the* i^{th} *prime number* ϱ_i $(\varrho_1 = 2,\ \varrho_2 = 3,\ \varrho_3 = 5,\ \varrho_4 = 7, ..)$ *occurs in the unique factorization of* t *into primes.*

Since $\phi(1v) = \phi(1)\phi(v)$, it is evident that $\phi(r)$ requires also $\phi(1) = 1$. We follow now the proof of Theorem 2 in Donaldson and Weymark (1980) in order to prove that given $\phi(r)$ is *positive* and *weakly monotonic*, the solution is $\phi(r) = r^\alpha, \alpha \in \mathbb{R}$. Theorem 2 in Donaldson and Weymark (1980) shows that if $\phi(r)$ is monotonically non-decreasing then $\phi(r) = r^\beta$, $\beta \geq 0$. Since we restrict $\phi(r)$ being *weakly monotonic*, we are left with the case in which $\phi(r)$ is monotonically non-increasing. Following exactly the same logical arguments as in the proof of Theorem 2 in Donaldson and Weymark (1980), we prove the second part of our statement.

Since $\phi(r) > 0$ then $c_i > 0$ for all $i \in \mathbb{I}_+$ it follows that we can write $c_i = \varrho_i^{\beta_i}$, $\beta_i \in \mathbb{R}$. We need to prove that if $\phi(r)$ is non increasing then $\beta_i = \beta$ for all $i \in \mathbb{I}_+$. Suppose the contrary. Since $\mathbb{I}_+$ has a countable number of elements, then there exist two real numbers $\beta_L < \beta_H$, and a disjoint partition of $\mathbb{I}_+ = (\mathbb{I}_+^L, \mathbb{I}_+^H)$ such that: $i \in \mathbb{I}_+^L \Leftrightarrow \beta_i \leq \beta_L$, and $i \in \mathbb{I}_+^H \Leftrightarrow \beta_i \geq \beta_H$.

Consider now a number $\bar{r} \in \mathbb{I}_+$, $\bar{r} > 1$, such that $n_i = 0$ for all $i \in \mathbb{I}_+^H$, that is $\bar{r}$ is factorized by all the prime numbers in $\mathbb{I}_+^L$. Therefore $\phi(\bar{r}) \leq \bar{r}^{\beta_L}$, as well as $\phi(\bar{r}^T) \leq \bar{r}^{T\beta_L}$ for all $T \in \mathbb{I}_+$. Consider now $\bar{r}+1$, or more generally $\bar{r}^T + 1$, no number ϱ_i with index $i \in \mathbb{I}_+^L$ is a prime factor of $\left(\bar{r}^T + 1\right)$, so $\left(\bar{r}^T + 1\right)$ could be decomposed into the product of prime factors ϱ_j where $j \in \mathbb{I}_+^H$. It follows $\phi\left(\bar{r}^T + 1\right) \geq \left(\bar{r}^T + 1\right)^{\beta_H}$. Putting this together with the inequality obtained before we get

$$\frac{\phi\left(\bar{r}^T + 1\right)}{\phi(\bar{r}^T)} \geq \frac{\left(\bar{r}^T + 1\right)^{\beta_H}}{\bar{r}^{T\beta_L}},$$

which could be rearranged as

$$\frac{\phi\left(\bar{r}^T + 1\right)}{\phi(\bar{r}^T)} \geq \left(\frac{\bar{r}^T + 1}{\bar{r}^T}\right)^{\beta_H} \left(\bar{r}^T\right)^{(\beta_H - \beta_L)}.$$

From the previous inequality follows

$$\lim_{T\to\infty} \frac{\phi\left(\bar{r}^T + 1\right)}{\phi(\bar{r}^T)} \geq \lim_{T\to\infty} \left(\frac{\bar{r}^T + 1}{\bar{r}^T}\right)^{\beta_H} \left(\bar{r}^T\right)^{(\beta_H - \beta_L)} = \infty,$$

which contradicts the fact that $\phi(.)$ is non-increasing. It follows that necessarily $\beta_i = \beta$ for all $i \in \mathbb{I}_+$, in which case $\phi(r) = r^{\beta}$. In order to satisfy non-increasingness of $\phi(r)$ we need to consider $\beta \leq 0$. It follows that weak monotonicity of $\phi(r)$ requires $\phi(r) = r^{\beta}$, $\beta \in \mathbb{R}$. ■

In order to complete the proof of the proposition we need to make use of the solution $\phi(r) = r^{\alpha}, \alpha \in \mathbb{R}$ and solve for part C in the previous Lemma, that is $\mathcal{F}_0^i(rm_i)B^i(rm_i) = r^{\alpha}\mathcal{F}_0^i(m_i)B^i(m_i)$ for all $r, m_i \in \mathbb{I}_+$ and for any group i.

Let $\mathcal{F}_0^i(m_i)B^i(m_i) := \mathcal{B}^i(m_i)$, it follows that

$$\mathcal{B}^i(rm_i) = r^{\alpha}\mathcal{B}^i(m_i)$$

for all $r, m_i \in \mathbb{I}_+$. Let $m_i = 1$, it derives $\mathcal{B}^i(r) = r^{\alpha}\mathcal{B}^i(1)$ for all $r \in \mathbb{I}_+$, from which

$$\mathcal{B}^i(m_i) = m_i^{\alpha}c_i \qquad \text{where } \; c_i = \mathcal{B}^i(1). \tag{61}$$

Substituting we get $\mathcal{F}_0^i(m_i)B^i(m_i) = m_i^\alpha \mathcal{F}_0^i(1)B^i(1) = m_i^\alpha c_i$. That is $\mathcal{F}_0^i(m_i)$ and $B^i(m_i)$ are not uniquely defined.

Substitute now the results from previous lemmata into the definition of $\mathcal{D}(\mathbf{y})$ and $\mathcal{S}(\mathbf{y})$. Consider first $\mathcal{D}(\mathbf{y})$, we have from Lemma 3 $\mathcal{D}(\mathbf{y}) = \sum_{i=1}^n \mathcal{F}^i(\mathbf{m})\mathcal{D}^i(\mathbf{y})$, that is

$$\mathcal{D}(\mathbf{y}) = \sum_{i=1}^n \mathcal{F}^i(\mathbf{m}) \sum\nolimits_{k=1}^{m_i} \left[A^i(k, m_i) - A^i(k-1, m_i)\right] \frac{1}{m_i} \left(\sum_{j \geq k} y_{(j)} - y_{(k)} \right),$$

From Lemma 4 we have: $A^i(k, m_i) = Q^i(\frac{k}{m_i})B^i(m_i)$, and $\mathcal{F}^i(\mathbf{m}) = \mathcal{H}^i(\mathbf{q})\mathcal{F}_0^i(m_i)$. According to (61) $\mathcal{F}_0^i(m_i)B^i(m_i) = m_i^\alpha c_i$. Therefore

$$\mathcal{F}^i(\mathbf{m})B^i(m_i) = \mathcal{H}^i(\mathbf{q})m_i^\alpha c_i.$$

Let $c_i Q^i(\frac{k}{m_i}) := V^i(\frac{k}{m_i})$, it follows that $V^i(1) = c_i$ is independent between groups, and given that $Q^i(.)$ and c_i could be chosen arbitrarily different between the groups, it follows that $V^i(\frac{k}{m_i})$ is independent between groups for all k/m_i.

Substituting we get:

$$\mathcal{D}(\mathbf{y}) = \sum_{i=1}^n \mathcal{H}^i(\mathbf{q})m_i^{\alpha-1} \sum\nolimits_{k=1}^{m_i} a^i(k, m_i) \left(\sum_{j \geq k} y_{(j)} - y_{(k)} \right),$$

where

$$a^i(k, m_i) = V^i(k/m_i) - V^i((k-1)/m_i)$$

and $\mathcal{S}(\mathbf{y}) = \sum_{i=1}^n \mathcal{H}^i(\mathbf{q})m_i^{\alpha-1} \sum_{k=1}^{m_i} a^i(k, m_i) \left[\sum_k y_k - \left(\sum_{j \geq k} y_{(j)} - y_{(k)} \right) \right]$.

The sufficiency part of the proof is easily established checking that the solutions satisfy LH and PR. ■

6.4 Proof Proposition 6:

Consider (23) , substituting for $\delta_F^i(p) = \int_p^1 \{F_i^{-1}(t) - F_i^{-1}(p)\}dt$, letting $\mu(F_i) = \int_0^1 F_i^{-1}(p)dp$ the average income of group i, it follows

$$\delta_F^i(p) = \mu(F_i) - \int_0^p F_i^{-1}(t)dt - (1-p)F_i^{-1}(p). \tag{62}$$

Denote with $D_i(F) = \int_0^1 v_i(p)\delta_F^i(p)dp$, the average deprivation within group i, therefore

$$D_i(F) = \mu(F_i)\int_0^1 v_i(p)dp - \int_0^1 [v_i(p)(1-p)] F_i^{-1}(p)dp - \qquad (63)$$
$$- \int_0^1 v_i(p)\int_0^p F_i^{-1}(t)dtdp.$$

Let $\phi^i(p) = \int_0^p v_i(t)dt$ (therefore $\phi^i(1) = \int_0^1 v_i(t)dt$), integrating by parts the last term in (63) we obtain

$$\int_0^1 v_i(p)\int_0^p F_i^{-1}(t)dtdp = \left[\int_0^p v_i(t)dt \int_0^p F_i^{-1}(t)dt\right]_0^1 - \int_0^1 \left(\int_0^p v_i(t)dt\right) F_i^{-1}(p)dp$$
$$= \phi^i(1)\mu(F_i) - \int_0^1 \phi^i(p)F_i^{-1}(p)dp.$$

Substituting into (63) and simplifying we obtain

$$\begin{aligned} D_i(F) &= \int_0^1 \phi^i(p)F_i^{-1}(p)dp - \int_0^1 [v_i(p)(1-p)] F_i^{-1}(p)dp \\ &= \int_0^1 \left[\phi^i(p) - v_i(p)(1-p)\right] F_i^{-1}(p)dp \\ &= \int_0^1 w_i(p)F_i^{-1}(p)dp, \end{aligned}$$

where $\phi^i(p) - v_i(p)(1-p) = w_i(p)$. Substituting into (23) we get (24) . ■

References

[1] Aczél J. (1966): *Lectures on functional Equations and Their Applications.* New York, London: Academic Press.

[2] Atkinson, A. B. (1970): On the measurement of inequality. *Journal of Economic Theory*, **2**, 244-263.

[3] Atkinson, A. B. (1987): On the measurement of poverty. *Econometrica* **55**, 749-764.

[4] Atkinson, A. B. (1992): Measuring poverty and differences in family composition. *Economica,* **59**, 1-16.

[5] Atkinson, A. B. and Bourguignon, F. (1982): The comparison of multidimensioned distribution of economic status. *Review of Economic Studies*, **49**, 183-201.

[6] Atkinson, A. B. and Bourguignon, F. (1987): Income distribution and differences in needs. In Feiwel (ed.): *Arrow and the Foundations of the Theory of Economic Policy,* 350-370. London: Macmillan.

[7] Atkinson, A. B. and Bourguignon, F. (1989): The design of direct taxation and family benefits. *Journal of Public Economics*, 3-29.

[8] Ben Porath, E. and Gilboa, I. (1994): Linear measures, the Gini index, and the income-equality trade-off. *Journal of Economic Theory*, **18**, 59-80.

[9] Bishop. J. A., Chakraborti, S. and Thistle, P.D. (1991): Relative deprivation and economic welfare: a statistical investigation with Gini-based welfare indices. *Scandinavian Journal of Economics*, **93**, 421-437.

[10] Blackorby, C. and Donaldson, D. (1978): Measures of relative inequality and their meaning in terms of social welfare. *Journal of Economic Theory*, **64**, 443-467.

[11] Blackorby, C. and Donaldson, D. (1980b): Ethical indices for the measurement of poverty. *Econometrica*, 48, 1053-1060.

[12] Blackorby, C. and Donaldson, D. (1984): Social criteria for evaluating population change. *Journal of Public Economics*, **25**, 13-33.

[13] Blackorby, C., Donaldson, D and Auersperg, M. (1981): A new procedure for the measuremet of inequality within and among population subgroups.Canadian Journal of Economics, 14, 665-85.

[14] Blackorby, C., Bossert, W., and Donaldson, D (1995): Intertemporal population ethics: critical level utilitarian principles. *Econometrica*, **63**, 1303-1320.

[15] Bossert, W. (1990): An axiomatization of the single-series Ginis. *Journal of Economic Theory*, **50**, 82-92.

[16] Bourguignon, F. (1989): Family size and social utility: income distribution dominance criteria. *Journal of Econometrics*, **42**, 67-80.

[17] Chakravarty, S. R. (1983): Ethically flexible measures of poverty. *Canadian Journal of Economics,* **16**, 74-85.

[18] Chakravarty, S. R. (1983a): A new index of poverty. *Mathematical Social Sciences,* **6**, 307-313.

[19] Chakravarty, S. R. (1988): Extended Gini indices of inequality. *European Economic Review,* **29**, 147-156.

[20] Chakravarty, S. R. (1990): Ethical Social Index Numbers. Berlin: Springer-Verlag.

[21] Chakravarty, S. R. (1997): Relative deprivation and satisfaction orderings. *Keio Economic Studies*, **34**, 17-31.

[22] Chakravarty, S. R., Chakraborti, A. B. (1984): On indices of relative deprivation. *Economics Letters*, **14**, 283-287.

[23] Chakravarty, S. R., Chattopadhyay, N. (1994): An ethical index of relative deprivation. In *Research on Economic Inequality*, **5**, 231-240.

[24] Chakravarty, S. R., Chattopadhyay, N. and Majumder, A. (1995): Income inequality and relative deprivation. *Keio Economic Studies*, **32**, 1-15

[25] Chakravarty, S. R. and Mukherjee, D. (1999): Measures of deprivation and their meaning in terms of social welfare. *Theory and Decision*, **47**, 89-100.

[26] Chambaz, C. and Maurin, E. (1998): Atkinson and Bourguignon's dominance criteria: extended and applied to the measurement of poverty in France. *Review of Income & Wealth,* **44**, 497-515.

[27] Chew, S. H. and Epstein, L. G. (1989): Axiomatic rank-dependent means. *Annals of Operations Research, **19**, 299-309.*

[28] Clark, S.R.; Hemming, R. and Ulph, D. (1981): On indices for the measurement of poverty. *Economic Journal*, **3**, 215-251.

[29] Dalton, H. (1920): The measurement of the inequality of incomes. *Economic Journal*, **20**, 348-361.

[30] Dardanoni, V. and Lambert, P. J. (1988): Welfare rankings of income distributions: a role for the variance and some insights for tax reform. *Social Choice and Welfare*, **5**, 1-17.

[31] Dasgupta, P., Sen, A. K. and Starrett, D. (1973): Notes on the measurement of inequality. *Journal of Economic Theory*, **6**, 180-187.

[32] Donaldson, D. and Weymark, J. A. (1980): A single-parameter generalization of the Gini indices of inequality. *Journal of Economic Theory*, **22**, 67-86.

[33] Donaldson, D. and Weymark, J. A. (1983): Ethically flexible Gini indices for income distributions in the continuum. *Journal of Economic Theory*, **29**, 353-358.

[34] Donaldson, D. and Weymark, J. A. (1986): Properties of fixed population poverty indices. *International Economic Review*, **27**, 667-688.

[35] Duclos, J. Y. (2000): Gini indices and the redistribution of income. *International Tax and Public Finance*, **7**, 141-162.

[36] Duclos, J. Y. and Grégoire, P. (1999): Absolute and relative deprivation and the measurement of poverty. LIS working paper n. 213/99.

[37] Ebert, U. (1988): Measurement of inequality: an attempt at unification and generalization. *Social Choice and Welfare*, **5**, 147-169.

[38] Ebert, U. (1997): Social welfare when needs differ: an axiomatic approach. *Economica*, **64**, 233-244.

[39] Ebert, U. (2000): Sequential generalized Lorenz dominance and transfer principles. *Bulletin of Economic Research*, **52:2**, 113-123.

[40] Ebert, U. (2000a): Social welfare, inequality, and poverty when needs differ: rank-dependent evaluation functions. University of Oldenburg, Discussion Paper V 200/00.

[41] Ebert, U. and Moyes, P. (1998): An axiomatic characterization of Yitzhaki's index of individual deprivation. University of Oldenburg, Discussion Paper V 193/98 Forthcoming *Economics Letters*.

[42] Fishburn, P. C. and Willig, R. D. (1984): Transfer principles in income redistribution. Journal of Public Economics, **25**, 323-328.

[43] Foster, J. (1984): On economic poverty: a survey of aggregate measures. In *Advances in Econometrics,* Vol. 3, (Basman, R. L. e Rhodes, G. F. ed.), 215-251. London: JAI Press Inc.

[44] Foster, J.; Greer, J. and Thorbecke, D. (1984): A class of decomposable poverty measures. *Econometrica*, **52**, 761-766.

[45] Foster, J. and Shorrocks, A. F. (1988a): Poverty orderings. *Econometrica*, **56**, 173-177.

[46] Foster, J. and Shorrocks, A. F. (1988b): Poverty orderings and welfare dominance. *Social Choice and Welfare*, **5**, 179-198.

[47] Foster, J. and Shorrocks, A. F. (1988c): Inequality and poverty orderings. *European Economic Review,* **32**, 654-662.

[48] Foster, J. and Shorrocks, A. F. (1991): Subgroup consistent poverty indices. *Econometrica*, **59**, 687-709.

[49] Gastwirth, J. L. (1971): A general definition of the Lorenz curve. *Econometrica*, **39**, 1037-1039..

[50] Hagenaars, A. J. M. (1986): *The Perception of Poverty.* Amsterdam: North-Holland.

[51] Hey, J. D. e Lambert, P. J. (1980): Relative deprivation and the Gini coefficient: comment.*Quarterly Journal of Economics,* 95, 567-73.

[52] Jenkins, S. and Lambert, P. J. (1993): Ranking income distributions when needs differ. *Review of Income & Wealth* **39**, 337-356.

[53] Jenkins, S. and Lambert, P. J. (1997): Three I's of poverty curves, with an analysis of UK poverty trends. *Oxford Economic Papers* **49**, 317-327.

[54] Jenkins, S. and Lambert, P. J. (1998a): Three I's of poverty curves and poverty dominance: TIPs for poverty analysis. In *Research on Economic Equality*, Vol **8.**

[55] Jenkins, S. and Lambert, P. J. (1998b): Ranking poverty gap distributions: further TIPs for poverty analysis. In *Research on Economic Equality*, Vol **8.**

[56] Kakwani, N. C. (1980): On a class of poverty measures. *Econometrica*, **48**, 437-446.

[57] Kakwani, N. C. (1984): The realtive deprivation curve and its applications. *Journal of Business and Economic Statistics*, **2**, 384-405.

[58] Karcher, T., Moyes, P. and Trannoy, A. (1995): *The stochastic dominance ordering of income distributions over time: the discounted sum of expected utility of incomes.* In (Barnett, W.A. et al. eds.) *Social Choice Welfare and Ethics*, 375-408. Cambridge: Cambridge University Press.

[59] Kolm, S. C. (1969): The optimal production of social justice. In *Public Economics* (Margolis, J. e Gutton, H. ed.), pp. 145-200. London: Mcmillan.

[60] Kolm, S. C. (1976a): Unequal inequalities. I. *Journal of Economic Theory*, **12**, 416-442.

[61] Kolm, S. C. (1976b): Unequal inequalities. II. *Journal of Economic Theory*, **13**, 82-111.

[62] Lambert, P. J. (1985): Social welfare and the Gini coefficient revisited. *Mathematical Social Science,* **9**, 19-26.

[63] Lambert, P. J. (1993): *The Distribution and Redistribution of Income: a Mathematical Analysis.* Manchester: Manchester University Press.

[64] Lambert, P. J. and Ramos, X. (2000): Welfare comparisons: sequential procedures for heterogeneous populations. Mimeo.

[65] Mehran, F. (1976): Linear measures of income inequality. *Econometrica*, **44**, 805-809.

[66] Mosler, K. and Muliere, P. (1998): Welfare means and equalizing transfers. *Metron*, **56**, 10-52.

[67] Muliere, P. and Scarsini, M. (1989): A note on stochastic dominance and inequality measures. *Journal of Economic Theory*, **49**, 314-323.

[68] Ok, E. and Lambert, P. J. (1999): On evaluating social welfare by sequential generalized Lorenz dominance. *Economics Letters,* **63**, 45-53.

[69] Paul, S. (1991): An index of relative deprivation. *Economics Letters.* **36**, 337-341.

[70] Peragine, V. (1998): Responsibility, compensation and income distribution. University of York, Discussion Paper 98/13. In *Logic, Game Theory and Social Choice* (De Swart ed.). Tilburg University Press: Tilburg.

[71] Quiggin, J. (1993): *Generalized Expected Utility Theory. The Rank-Dependent Model.* Dordrecht: Kluwer.

[72] Roemer, J. E. (1998): *Equality of Opportunity.* Cambridge MA: Harvard University Press.

[73] Runciman, W. G. (1966): *Relative Deprivation and Social Justice.* London: Routledge.

[74] Saposnik, R. (1981): Rank-dominance in income distributions. *Public Choice*, **36**, 147-51.

[75] Safra, Z. and Segal, U. (1998): Constant risk aversion. *Journal of Economic Theory*, **83**, 19-42.

[76] Seidl, C. (1988): Poverty measurement: a survey. In Welfare and Efficiency in the Public Sector (Bös D., Rose M. and Seidl C. eds.) 71-147. Heidelsberg: Springer-Verlag.

[77] Sen, A. K. (1973): *On Economic Inequality.* Oxford: Claredon Press. (1997) expanded edition with the annexe "*On Economic Inequality After a Quarter Century*" by Foster, J. And Sen, A.K..

[78] Sen, A. K. (1976): Poverty: an ordinal approach to measurement. *Econometrica*, **44**, 437-446.

[79] Shorrocks, A. F. (1983): Ranking income distributions. *Economica*, **50**, 3-17.

[80] Shorrocks, A. F. (1995): Revisiting the Sen poverty index. *Econometrica*, **63**, 1225-1230.

[81] Shorrocks, A. F. (1998): Deprivation profiles and deprivation indices. In *The Distribution of Welfare and Household Production: International Perspectives,* Jenkins, S. A., Kaptein, S. A. and van Praag, B. eds. 250-267. London: Cambridge University Press.

[82] Shorrocks, A. F. and Foster, J. E. (1987): Transfer sensitive inequality measures. *Review of Economic Studies*, **14**, 485-497.

[83] Spencer, B. D. and Fisher, S. (1992): On comparing distributions of poverty gaps. *Sankya,* **54B**, 114-126.

[84] Thon, D. (1979): On measuring poverty. *Review of Income and Wealth*, **25**, 429-440.

[85] Wang, S. S. and Young, V. R. (1998): Ordering risk: expected utility theory versus Yaari's dual theory of risk. *Insurance: Mathematics and Economics*, **22**, 145-161.

[86] Weymark, J. A. (1981): Generalized Gini inequality indices. *Mathematical Social Sciences*, **1**, 409-430.

[87] Weymark, J. A. (1995): An alternative proof of Ben Porath and Gilboa's generalized Ginis characterization theorem. *Journal of Economic Theory*, **66**, 573-580.

[88] Yaari, M. E. (1987): The dual theory of choice under risk. *Econometrica*, **55**, 99-115.

[89] Yaari, M. E. (1988): A controversial proposal concerning inequality measurement. *Journal of Economic Theory*, **44**, 381-397.

[90] Yitzhaki, S. (1979): Relative deprivation and the Gini coefficient. *Quarterly Journal of Economics*, 93, 321-324.

[91] Yitzhaki, S. (1982): Relative deprivation and economic welfare. *European Economic Review*, 17, 99-113.

[92] Yitzhaki, S. (1983): On an extension of the Gini inequality index. *International Economic Review,* **24**, 617-628.

[93] Zheng, B. (1997): Aggregate poverty measures. *Journal of Economic Surveys,* **11**, 123-162.

[94] Zheng, B. (1999): On the power of poverty orderings. *Social Choice and Welfare,* ***17*, .**

[95] Zoli, C. (1997): Intersecting generalized Lorenz curves and the Gini index. *Quaderni del Dipartimento di Economia Pubblica e Territoriale Università di Pavia* n. 2/1997.

[96] Zoli, C. (1999): Intersecting generalized Lorenz curves and the Gini index. *Social Choice and Welfare,* ***16*,** 183-196.