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Abstract

Player *i*'s payoff in a noncooperative game is generally expressed as a function of the vector of strategies of all players. However, in some games - 'simply reducible games' - the payoff of player *i* is a function of two arguments - the strategy chosen by *i*, and the sum of the strategies of all players in the game. Cournot oligopoly, public good provision, cost- and surplus-sharing, and open access resource exploitation are all simply reducible games.

We define the 'share function' of a simply reducible game. We indicate its role in the analysis of equilibrium existence, uniqueness and comparative static properties of simply reducible games, and apply it to a model of open access resource exploitation. Finally, we suggest further applications and extensions of our approach.

JOINT PRODUCTION GAMES AND SHARE FUNCTIONS

I. INTRODUCTION

At the analytical heart of many economic models is a noncooperative game on which a good deal of special structure is typically imposed. Often, each player's payoff depends on two arguments - one is that player's own chosen strategy, and the other is the unweighted sum of every player's strategy choice. Shubik comments that such games "clearly have much more structure than a game selected at random. How this structure influences the equilibrium points has not yet been explored in depth" [Shubik (1982), p.325]. We refer to such games as 'simply reducible games'. Some authors, such as Corchon (1996), have called them aggregative games.

Several authors have indeed exploited the aggregative structure of games that arise in specific contexts. Friedman (1982) refers to a proof - first suggested by Selten (1970) and later used by Szidarovszky and Yakowitz (1977) - of the existence of a unique Nash equilibrium in the Cournot oligopoly model. This exploits the observation that, under certain assumptions, a player's best response function implies the existence of a function of the form $q_i^* = \phi_i(Q)$, where q_i^* is player i's best response and Q is the total of all players' choices, including that of player i. We define $\phi_i(Q)$ formally in Section II, where we also suggest labelling it a 'replacement function'. It permits a simple characterisation of Nash equilibrium as an allocation at which $\Sigma_i\phi_i(Q) = Q$. Novshek (1984, 1985) uses a somewhat more general replacement correspondence, which he calls a 'backward reaction mapping', to establish existence of equilibrium under less restrictive assumptions. More recently, Okuguchi (1993) and Cornes, Hartley and Sandler (1999) have exploited the replacement function to analyse existence and uniqueness of equilibrium in the simply reducible games that arise in models of oligopoly and public good provision[1]. Corchon (1994, 1996) exploits the aggregative structure of such games using a slightly different approach.

In these papers, individual players' reaction functions - and, as a consequence, their replacement functions - are assumed to be everywhere monotonic non-increasing. In Cournot oligopoly models, this property is implied by the Hahn stability conditions which are often imposed - see Vives (1999, Ch.4) for a recent survey of this literature. In models of pure public good provision, the assumption that both goods are normal also has this implication. However, it is important to allow for non-monotonic reaction and replacement functions. In the first place, even a slight relaxation of the Hahn conditions in the Cournot model raises the possibility of locally increasing reaction functions. More significantly, proportional cost and surplus sharing models, of which the open access resource exploitation model is an important example, do not typically imply monotonic reaction functions even when the normality assumption is adopted. Watts (1996) provides a useful summary of this literature. In other significant applications, too - for example in the literature on rentseeking surveyed by Nitzan (1994) - reaction functions typically fail to be monotonic.

This paper extends existing analyses by introducing and exploiting the idea of a 'share function'. Player *i*'s share function expresses the share q_i^*/Q as a function of the aggregate Q: $\beta_i(Q) = \phi_i(Q)/Q$. We believe that the share function provides an analytical tool with at least four notable virtues. First, it is simple: it permits a straightforward characterisation of Nash equilibrium. If each player's behavior is described by a well-defined share function, then a Nash equilibrium is simply an allocation at which $\sum_i \beta_i(Q) = 1$. In common with the $\phi_i(Q)$ function, this characterisation avoids the need to consider the mappings in potentially high dimensions that are generated by best response functions. Instead, analysis proceeds by considering a simple function f: $A \rightarrow B$ in which A and B are real numbers. This contrasts with the reaction function approach, which involves mappings in a space of dimension equal to the number of players in the game. Second, it is powerful: because of the simple way in which it models the interaction of players, it can be used to analyse not only the existence and uniqueness of Nash equilibrium, but also its comparative static properties. Furthermore, it can handle models involving many heterogeneous players whose reaction functions are not necessarily monotonic. By contrast, the reaction function approach is not well suited to handling heterogeneous players, and typically has to resort either to 2-player models or else to identical players in order to avoid what Bellman (1957, p. ix) called the 'curse of dimensionality'. Third, it is versatile: it is applicable to any model which has the aggregative structure to which we have alluded - indeed, our concluding section suggests extensions of the approach which further extend its potential applicability. Finally, its simplicity permits a straightforward geometric representation which aids intuition and makes it an attractive expository approach as well as a powerful research tool.

Section II provides a formal definition of a simply reducible game and of replacement and share functions. Our aim is not simply to provide an alternative proof of existence and uniqueness, but to suggest a simple and, we hope, helpful modelling tool that provides an elementary and systematic method of analysing a wide variety of applications. To achieve this aim, we also provide an informal discussion of the general nature and strategy of the share function approach. Section III demonstrates the existence of a unique Nash equilibrium, and explores comparative static issues, in a simple and well-known open access resource model. Section IV shows how the approach may be used to look at other surplus and cost sharing rules. Section V draws attention to the wide range of additional applications of the approach, and suggests two extensions that further increase its scope. Section VI concludes.

II. SIMPLY REDUCIBLE GAMES AND SHARE FUNCTIONS

(i) A definition

Consider a finite game. Let I be the set of players, of whom there are n. Player i's strategy choice is denoted by the non-negative scalar q_i, $i = 1, \ldots, n$, and the vector of strategy choices by all the players is $\mathbf{q} \equiv (q_1, q_2, \ldots, q_n)$. Player i's payoff, or utility, is u_i. The payoff of player i is generally a function of the vector of every player's chosen strategy:

$$u_i = \eta_i(\mathbf{q}) = \eta_i(q_i, \mathbf{q}_{-i})$$

where $\mathbf{q}_{-i} \equiv (q_1, \ldots, q_{i-1}, q_{i+1} \ldots, q_n)$ is the vector of strategy choices of others.

In a simply reducible game player i's payoff function $\nu_i(\cdot)$ can be expressed as a function of two scalars. One is the player's own chosen strategy, q_i, and the other is the simple unweighted sum of the choices made by all players in the game, including player i. We denote this sum by $Q = \Sigma_j\, q_j$. Here is a formal definition:

Definition: The game $\mathcal{G}$ is a **simply reducible game** if and only if the payoff function for each player i may be written as

$$u_i = \pi_i\left(q_i, \sum_{j=1}^{n} q_j\right) = \pi_i\left(q_i, Q\right). \qquad (1)$$

(i) The replacement function of player *i*

In a simply reducible game, each player has preferences over her own choice variable, q_i, and some aggregate, Q. We will later provide a more detailed discussion and interpretation of this aggregate. In the Cournot model, it is the sum of all firms' chosen output levels. In a public good provision model, it is the total provision of the public good, and may include not only the contributions of all explicitly labelled players, but also government provision. For the moment, it is sufficient to note that each player's behavior is typically described by the best response function:

$$q_i^{BR} = \hat{q}_i\left(\tilde{Q}_i\right), \qquad (2)$$

where $\tilde{Q}_i \equiv \sum_{\substack{j=1 \\ j\neq i}}^{n} q_j$. For a given Q, let $\Psi_i(Q)$ denote the set of q_i satisfying $0 \le q_i \le Q$ and

$$q_i = \hat{q}_i\left(Q - q_i\right). \qquad (3)$$

For a given value of Q, this set may be empty or may have many elements. In the sequel we shall concentrate on situations in which, for any value of Q, $\Psi_i(Q)$ contains the single element $\phi_i(Q)$. We call $\phi_i(Q)$ the replacement function of player i. More formally,

Definition: If, for any $Q \ge 0$, either $\Psi_i(Q)$ is empty or $\Psi_i(Q) = \{\phi_i(Q)\}$, we will call $\phi_i(Q)$ the **replacement function of player *i***.

Note that the domain of $\phi_i(Q)$ is the set of Q for which $\Psi_i(Q)$ is non-empty. Our reason for using the label "replacement function" is simple. Consider a given value of the total, Q. Now ask the question: "Given Q, is there an amount Z, $0 \le Z \le Q$, such that, if the quantity Z were taken away from Q, player i's best response to the remaining quantity, $(Q - Z)$, would precisely replace Z?" If, for any given quantity Q, there is a unique Z with this property, then it is described by the replacement function.

Figure 1 illustrates these points. The thinner of the two continuous curves is the graph of player *i*'s best response function. Consider any point on this graph - say the point C'. The corresponding point on the graph of the replacement function is the point D', where C'D' = C'F'. The points D' and E' are obtained by completing the square with side C'F'. The point D' represents the allocation $\left(q_i{}', \tilde{Q}_i + q_i{}'\right) = \left(q_i{}', Q'\right)$. The thick line in the figure - the graph of the replacement function - is obtained by applying this construction to every point on the graph of the best response function. Note that, for $Q < q_i^0$, where q_i^0 = $\hat{q}_i(0)$, the set $\Psi_i(Q)$ is empty. Here, q_i^0 is the value that maximises the payoff of player *i* in a 1-player "game". In the model of Cournot oligopoly it is the monopoly output.

It is easy to see from this construction that, if two points on the graph of $\hat{q}_i(\cdot)$ lie on the same line of slope –45°, the "graph" of Ψ_i will contain two points for some Q. Excluding such a possibility ensures that $\phi_i(\cdot)$ is a function with domain $Q \geq q_i^0$.

(ii) The aggregate replacement function

Suppose that a replacement function can be defined for every player in a game. Then we can define the aggregate replacement function of the game:

Definition: Let every player in the *n*-player game G have a replacement function, $q_i = \phi_i(Q)$. Then the **aggregate replacement function of G** is the function $\Phi(Q)$ where

$$\Phi(Q) \equiv \sum_{j=1}^{n} \phi_j(Q). \qquad (4)$$

The domain of $\Phi(Q)$ is the [possibly empty] intersection of the domains of the individual replacement functions.

It follows readily from the definitions that $(q_1^*,\ldots, q_n^*)$ is a Nash equilibrium if and only if $q_i^* = \phi_i(Q^*)$ for all *i* where $Q^* = \sum_{j=1}^{n} q_j^*$. Thus Q^* is an equilibrium aggregate action if and only if $Q^* = \Phi(Q^*)$. So examining Nash equilibria reduces to the study of fixed points of a one-dimensional aggregate replacement function.

Our motivation should now be clear. The investigation of existence and uniqueness of Nash equilibrium, together with its comparative static properties, can be undertaken by examining the properties of a function - $\Phi(Q)$ - defined on the real line. This is much easier than the reaction function approach, which involves mappings defined in a Euclidean space, the dimension of which is as large as the number of players.

Cornes, Hartley and Sandler (1999) show that, in the standard pure public good model in which both the private and the public goods are normal, the replacement function for the game is continuous and monotonic nonincreasing [and strictly decreasing wherever positive]. It follows from these properties that a Nash equilibrium exists and is unique.

(iii) The share function

Under various different labels, the replacement function has been introduced and exploited by a number of writers, mainly in the context of oligopoly theory. The analyses with which we are familiar - for example, Okuguchi (1993) - typically make assumptions that ensure that the aggregate replacement function is monotonic nonincreasing [and strictly decreasing wherever positive]. However, to deduce this useful property can require rather strong assumptions on demand and/or cost functions. In addition, there are important examples of simply reducible games drawn from fields outside oligopoly theory that do not necessarily imply a monotonic aggregate replacement function, but do imply that the ratio $\phi_i(Q)/Q$ is strictly decreasing when $\phi_i(Q)/Q > 0$. For this reason, for any $Q > 0$, we find it convenient to work, not directly with the replacement function as we have defined it above, but rather with the 'share' function of player i.

Definition: Let every player in the n-player game G have a replacement function, $q_i = \phi_i(Q)$. Then, for $Q > 0$, the function $\beta_i(Q) = \phi_i(Q)/Q$ is the **share function of player *i***.

The share function of player i answers the question "Given Q, is there a proportion, b, $0 \leq b \leq 1$, such that, if the proportion b were taken away from Q, player i's best response to the remaining quantity, $(1 - b)Q$, would precisely replace the proportion b?" Given our definition of the individuals' share functions, we can define an aggregate share function:

Definition: Let every player in the game G have a share function, $\hat{b}_i = \beta_i(Q)$ with domain $Q \geq q_i^0$. Then the **aggregate share function of G** is the function $B(Q)$ with domain $Q \geq \max_{j=1,\ldots,n} q_j^0$, where $B(Q) \equiv \sum_{j=1}^{n} \beta_j(Q)$.

Using the aggregate share function, a Nash equilibrium can be characterised as follows: $Q^* > 0$ is an equilibrium value of Q if and only if $B(Q^*) = 1$. In the application in Section III, we first establish the existence of a well-defined share function $\beta_i(\cdot)$ for each player. We then prove that, for each player i, $\beta_i(\cdot)$ is continuous and strictly decreasing for all $\beta_i(\cdot) > 0$, and that $\sup \beta_i(\cdot) \geq 1$ and $\inf \beta_i(\cdot) = 0$. Since $B(\cdot)$ inherits these properties, we can immediately conclude that there exists only one positive Nash equilibrium value Q^*.

III. THE OPEN ACCESS RESOURCE PROBLEM

Imagine an open access resource, to which we will refer as a fishing ground. The total catch of fish obtained from the ground, X, depends upon the aggregate level L of an input that is applied to the fishing ground. We will think of this as labor. Exploitation of the open access resource is described by a production function $F(L)$ that exhibits diminishing returns to labor. Player i chooses her level of variable input, ℓ_i, taking the input levels of all other players as given. The proportion of total output that is consumed by i, x_i, equals the proportion of total variable input that she supplies: $x_i / X = \ell_i / L$. Player i's preferences are represented by a utility function $u_i(x_i, \ell_i)$. Player i's payoff to strategy profile $(\ell_1, \ldots, \ell_n)$ is $u_i(x_i, \ell_i)$ where $x_i = \frac{\ell_i}{L} F(L)$ if $L > 0$ and is $u_i(0, 0)$ otherwise.

This is clearly a reducible game. Furthermore, its formal structure is precisely that of the proportional surplus sharing game. We make the following assumptions:

A.1 $u_i(x_i, \ell_i)$, is quasi-concave, locally non-satiable, non-decreasing in x_i, non-increasing in ℓ_i, continuous and continuously differentiable[2] for x_i, $\ell_i > 0$. Both x_i and ℓ_i are normal.

A.2 $F(L)$ is increasing, strictly concave, continuous and continuously differentiable for $L > 0$, and $F(0) = 0$[3].

A.3 Either (i) There exists a value of $L > 0$ such that $u_i(F(L), L) = u_i(0, 0)$

or (ii) For all $L \geq 0$, $u_i(F(L), L) \leq u_i(0, 0)$[4].

These assumptions are, for the most part, standard. Our characterisation of normality in **A.1** follows Watts (1996). Suppose that the allocation (x_i', ℓ_i') is in player i's demand set when the budget set is $px_i - w\ell_i \leq m_i'$. Both goods are normal if, for any $m_i'' > m_i'$, the demand set associated with the budget set $px_i - w\ell_i \leq m_i''$ contains at least one point (x_i'', ℓ_i'') such that $x_i'' \geq x_i'$ and $\ell_i'' \leq \ell_i'$. Bearing in mind the fact that ℓ_i is a 'bad' this implies that, if preferences are strictly convex, income expansion paths in (x_i', ℓ_i') space are downward-sloping. **A.3** says that the indifference curve through the origin either crosses the graph of the production function twice (the origin is common to both) or lies above it for all positive values of L. Sufficient conditions for **A.3** are either (a) $F'(L) \to 0$ as $L \to \infty$ or (b) the MRS of the indifference curve through the origin is unbounded. In particular, an upper bound imposed on the input of player i corresponds to vertical indifference curves at $\ell_i = \bar{\ell}_i$, so that (b) is satisfied. We allow for the possibility that some individuals may choose to supply a zero level of labor input in equilibrium.

Consistent with our discussion in the previous section, we denote player i's share of total input at any allocation by b_i: $b_i = \ell_i/L$. Because of the dependence of x_i on ℓ_i and L, both player i's payoff and also her marginal rate of substitution between x_i and ℓ_i can be written as functions of b_i and L:

$$u_i\left(x_i, \ell_i\right) = u_i\left(b_i F(L), b_i L\right) \equiv \omega_i\left(b_i, L\right)$$

$$-\frac{\partial u^i\left(x_i, \ell_i\right) / \partial \ell_i}{\partial u^i\left(x_i, \ell_i\right) / \partial x_i} = s_i\left(x_i, \ell_i\right) = s_i\left(b_i F(L), b_i L\right) \equiv \sigma_i\left(b_i, L\right). \qquad (6)$$

Now consider the response of x_i to a change in ℓ_i when the input levels of all other players are taken as given. This response, which is *i*'s marginal rate of transformation of input into consumption, can be expressed as a function of b_i and L. By differentiating the share formula for *i*, holding all other players' input levels fixed, we obtain the following expression for *i*'s marginal rate of transformation:

$$\left[\frac{\ell_i}{L}\right]F'(L)+\left[\frac{(L-\ell_i)}{L}\right]\frac{F(L)}{L}=b_i F'(L)+\left[1-b_i\right]\frac{F(L)}{L}\equiv\tau_i\left(b_i,L\right). \tag{7}$$

Note that $\tau_i(\cdot)$ is a convex combination of the marginal and average products evaluated at L. Assumption **A.2** implies that $F'(L) < F(L)/L$ for all L. Consequently, for any value of L, $F'(L) \leq \tau_i(b_i, L) \leq F(L)/L$. This simple observation plays a key role in the subsequent argument.

Denoting player *i*'s most preferred quantities as $\hat{x}_i$ and $\hat{\ell}_i[=\hat{b}_i L]$, and allowing for the possibility that at equilibrium $\hat{x}_i = \hat{\ell}_i = 0$, the first order conditions can be written as

$$\sigma_i\left(\hat{b}_i,L\right)\geq\tau_i\left(\hat{b}_i,L\right) \tag{8}$$

$$\left[\sigma_i\left(\hat{b}_i,L\right)-\tau_i\left(\hat{b}_i,L\right)\right]\hat{b}_i=0 \tag{9}$$

and $\hat{x}_i/\hat{\ell}_i = F(L)/L$, or $\hat{x}_i=\hat{b}_i F(L)$. (10)

(8) and (9) state that either $\hat{b}_i$ is positive, in which case $\sigma_i\left(\hat{b}_i,L\right)=\tau_i\left(\hat{b}_i,L\right)$, or else $\hat{b}_i$ = 0 and $\sigma_i(0, L) \geq \tau_i(0, L)$. It is often assumed that the players are competitive producers whose objective is profit-maximisation. In this case, the marginal rate of substitution is simply *w/p*, where *w* and *p* are, respectively, the input and output prices. It is readily confirmed that, when the input levels of all other players are held constant, the sharing rule implies that x_i is a concave increasing function of ℓ_i. Since $u_i(\cdot)$ is quasi-concave by assumption, conditions (8) - (10) are not only necessary but also sufficient for $\hat{b}_i$ to solve player *i*'s problem.

Our demonstration of the existence of a unique Nash equilibrium in the open access resource game follows directly from a number of straightforward facts. The first concerns the existence of the share function of player *i*:

<u>Fact 1</u>: For any $L > 0$, there is at most one $\hat{b}_i$ satisfying (8) - (10).

<u>Demonstration.</u> Assumption **A.1** implies that $\sigma_i(b_i, L)$ is non-decreasing in b_i. Consider the expression for $\tau_i(b_i, L)$. Since L is constant, so too are $F'(\cdot)$ and $F(\cdot)/L$. Furthermore, assumption **A.2** implies that $F'(\cdot) < F(\cdot)/L$. Therefore, as b_i increases, relatively more weight falls on the strictly smaller of the two constant terms in the convex combination in (7). Therefore, $\tau_i(b_i, L)$ is strictly decreasing in b_i.

Since $\sigma_i(\cdot)$ is everywhere non-decreasing in b_i and $\tau_i(\cdot)$ is everywhere strictly decreasing in b_i, there is at most one value of $\hat{b}_i$ consistent with the first-order conditions. ■

Figure 2 summarises the reasoning that establishes Fact 1. Consider a particular value of L, say L^0 in the figure. Suppose that there is a share, b_i^0, such that $b_i^0L^0$ is a best response to $(1 - b_i^0)L^0$. If b_i^0 is strictly positive, it is characterised by equality of $\sigma_i(\cdot)$ and $\tau_i(\cdot)$. By demonstrating that, for any such given value of L, $\sigma_i(\cdot)$ is nondecreasing in b_i and $\tau_i(\cdot)$ is everywhere increasing in b_i, we have established the uniqueness of such a share. For any $L > 0$ for which conditions (8) - (10) have a unique solution, we will write $\beta_i(L)$ for that solution. Note that Fact 1 does not constrain the domain of β_i. However, as well as drawing attention to a useful property of the share function, the next result shows that the domain is an unbounded interval (if non-empty).

Fact 2: Suppose $L^1 > L^0 > 0$ and $\beta_i(L^0)$ exists. Then $\beta_i(L^1)$ exists and $\beta_i(L^1) \leq \beta_i(L^0)$. The latter inequality is strict if $\beta_i(L^0) > 0$.

Demonstration: Inspection of (6) makes it clear that, under our normality assumption, $\sigma_i(\cdot)$ is a nondecreasing function of each of its arguments. Furthermore, inspection of (7) reveals that, given the concavity of the production function, $\tau_i(\cdot)$ is a decreasing function of each of its arguments. Therefore

$$\sigma_i(\beta_i(L^0), L^1) \geq \sigma_i(\beta_i(L^0), L^0) \geq \tau_i(\beta_i(L^0), L^0) > \tau_i(\beta_i(L^0), L^1).$$

Hence either (a) $\sigma_i(b_i, L^1) \geq \tau_i(b_i, L^1)$ for $0 \leq b_i \leq 1$ or (b) $\sigma_i(\hat{b}_i, L^1) = \tau_i(\hat{b}_i, L^1)$ for some $\hat{b}_i$. In case (a), $\beta_i(L^1) = 0 \leq \beta_i(L^0)$ and the inequality is strict if $\beta_i(L^0) > 0$. In case (b), since $\sigma_i(\times, L^1)$ is non-decreasing and $\tau_i(\times, L^1)$ is strictly increasing, we have $\beta_i(L^1) = \hat{b}_i < \beta_i(L^0)$. Note that (b) can only occur if $\beta_i(L^0) > 0$. ■

Again, Figure 2 summarises the argument. Briefly stated, an increase in the value of L from L^0 to L^1 shifts the graphs of both $\sigma_i(\cdot, L)$ and $\tau_i(\cdot, L)$ to the left. Therefore, if player i's most preferred share is initially positive, it must fall.

Nothing in our argument to this point excludes the possibilities that β_i has empty domain, exhibits downward jumps or has a strictly positive limit as $L \rightarrow \infty$, any of which would pose difficulties for the existence of a Nash equilibrium. Fortunately, Fact 3 rules out such problematic behavior.

Fact 3: If assumption **A3**(i) holds and $0 < b_i \leq 1$, there is a value of L such that $b_i = \beta_i(L)$. If **A3**(ii) holds, $\beta_i(L) = 0$ for all $L > 0$.

Demonstration: Figure 3 provides an intuitive justification of this fact. The graph marked $X = F(L)$ represents the aggregate technology. Choose any given value of b_i such that $0 < b_i \leq 1$. Take any point on the graph of $X = F(L)$, such as A. The point a is the point on the ray OA with the property that $Oa/OA = b_i$. As A moves along the graph of $F(L)$, the point a traces out the graph of the function $x_i = b_i F(\ell_i / b_i)$ for the chosen value of b_i. Note that the slope of the graph through a equals that of the graph through A: $\partial x_i(\ell_i, b_i)/\partial \ell_i = F'(L)$ evaluated at $\ell_i/b_i = L$. It follows that both graphs share a common tangent at the origin.

If Assumption **A.3**(ii) holds, $F'(0) \leq s_i(0, 0)$ and we can deduce that $\beta_i(L) = 0$ for all $L > 0$. If **A.3**(i) holds [the case drawn in the figure] a point of intersection I exists, at some positive value of L, between the graph of $F(\cdot)$ and player i's indifference curve through the origin. Hence, the graph of $x_i = b_i F(\ell_i/b_i)$ intersects this indifference curve at some point S in the positive orthant. Conditions (8) - (10) are satisfied with $\hat{b}_i = b_i$ at some point on the graph between 0 and S. This follows from the observation that at S, player i's marginal rate of substitution exceeds both the marginal product and also the average product. It therefore exceeds their convex combination. At the origin, the opposite is true. Continuity ensures that equality holds for some intermediate value of L. ◎

These facts ensure that β_i is (a) continuous, (b) strictly decreasing where positive, (c) approaches or equals 0 and under **A.3**(i) $\beta_i(L) = 1$ for some L, whilst under **A.3**(ii) $\beta_i(L) = 0$ for all $L > 0$. If **A.3**(ii) holds for all i, then $L = 0$ is the only Nash equilibrium. Otherwise, $B(L) = \sum_{j=1}^{n} \beta_j(L) \geq 1$ for some L and satisfies (a), (b) and (c). Consequently there is a unique $L^* > 0$ satisfying $B(L^*) = 1$. This establishes the next theorem.

Theorem 1 [Existence and Uniqueness]: Given assumptions **A.1 - A.3**, the open access resource game has a unique Nash equilibrium.

Figure 4 summarizes our argument to this point. It shows the graphs of the individual share functions associated with a 3-player game. Each is continuous and strictly decreasing for positive values of the share, and the figure shows a situation in which each reaches zero at some finite value of L. The aggregate share function, $B(L)$, inherits these properties of the individual share functions. Consequently, there exists a unique value, L^*, at which $B(L^*) = 1$.

The present approach also allows us to derive some simple comparative static results, all of which flow directly from the following theorem concerning an individual's behavioral and welfare response to a change in the aggregate L.

Theorem 2 [Comparative statics for the open access resource model]: Consider the open access resource game in which assumptions **A.1 - A.3** hold. Then, if $L^2 > L^1$ and $\beta_i(L^1)$ exists,

(i) $L^1\,[1-\beta_i(L^1)] < L^2\,[1-\beta_i(L^2)\,]$,

and (ii) $\omega_i(\beta_i(L^1), L^1) \geq \omega_i(\beta_i(L^2), L^2)$ for all $i \in I$, with strict inequality if $\beta_i(L^1) > 0$.

Proof: (i) If $\beta_i(L^1) = 0$, then $\beta_i(L^2) = 0$ by Fact 2, so that (i) follows immediately.

If $\beta_i(L^1) > 0$, Fact 2 implies that $\beta_i(L^2) < \beta_i(L^1)$. Hence,

$1 - \beta_i(L^2) > 1 - \beta_i(L^1) \Rightarrow L^2[1 - \beta_i(L^2)] > L^1[1 - \beta_i(L^1)]$ for all $i \in I$,

which establishes claim (i).

Now write $AP(L)$ for $F(L)/L$ and note that $AP(L)$ is strictly decreasing in L. Then (i) implies that

$$\omega_i\left(\beta_i\left(L^1\right), L^1\right) = \max_{\ell_i}\ u_i\left(\ell_i\, AP\left[L^1 - \beta_i\left(L^1\right)L^1 + \ell_i\right], \ell_i\right)$$
$$\geq \max_{\ell_i}\ u_i\left(\ell_i\, AP\left[L^2 - \beta_i\left(L^2\right)L^2 + \ell_i\right], \ell_i\right) = \omega_i\left(\beta_i\left(L^2\right), L^2\right)$$

with strict inequality if $\beta_i(L^1) > 0$. This proves (ii). ■

Any shock that leads to a higher equilibrium value of L will leave non-participants unaffected and reduce the welfare of participants. Its effect on preferred levels of input, by contrast, is ambiguous. Theorem 2 can be exploited to generate comparative static propositions covering situations in which L itself is treated as endogenous. Imagine a situation in which initially there are n_1 players exploiting an open access resource. Call this game G^1. Now imagine an alternative situation in which those n_1 players have been joined by others. Call this the game G^2. In short G^2 is obtained from G^1 by simply adding extra players so that if I^k denotes the set of players in game k, then $I^1 \subset I^2$. Let L^{k*} denote the aggregate labor input at the Nash equilibrium of game G^k.

The comparative static results that interest us can be readily obtained as corollaries of theorem 2.

Corollary 1: Let the open access game G^2 be formed by taking the game G^1 and adding extra players, so that $I^1 \subset I^2$. Then, given assumptions **A1**, **A2** and **A3**,

(i) $L^{1*} \leq L^{2*}$ with strict inequality if some player not in I^1 makes a positive contribution in G^2,

(ii) $L^{1*}\,[1-\beta_i(L^{1*})] \leq L^{2*}\,[1-\beta_i(L^{2*})]$ for all $i \in I^1$ with strict inequality if $L^{1*} > L^{2*}$,

(iii) The Nash equilibrium payoffs to all players in I^1 are no larger in G^2 than in G^1. Moreover, if $L^{2*} > L^{1*}$, then the payoffs are strictly less for those who make positive contributions in G^1.

Proof: Let $B^k(L)$ be the aggregate share function of G^k, $k = 1, 2$. Then

$$B^2(L^1) = B^1(L^1) + \sum_{j \in I_2 - I_1} \beta_j\left(L^1\right).$$

Since $\beta_j(L) \geq 0$ for all j, $B^2(L^{1*}) \geq B^1(L^{1*}) = 1$.

Since $B^2(L)$ is everywhere non-increasing, it must be true that, at the equilibrium of G^2,

$$1 = B^2(L^{2*}) \leq B^2(L^{1*}) \text{ and } L^{2*} \geq L^{1*}.$$

If, in addition, $\beta_j(L^{2*}) > 0$ for some $j \notin I_1$, then $\beta_j(L^{1*}) > 0$ and all preceding inequalities are strict. Having established that the equilibrium level of L cannot fall, and will generally rise, we can infer (ii) and (iii) immediately from Theorem 2.

Corollary 1 tells us that the addition of more players cannot improve, and will generally lower, the equilibrium utility levels of the existing players. Note that this has been established without assuming that the players have identical preferences. Furthermore, existing players' contributions may increase or decrease when extra players join the game.

Now suppose that each player is subjected to the same upper limit on the individual input level. This simply involves adding to the open access resource game a set of constraints of the form $\ell_i \leq \ell^{\max}$, where we restrict attention to the case where the limit is the same for all individuals. If preferences differ across individuals, such a quota may bind for some players, but not others. Then Theorem 2 allows us to infer the consequences of such regulations for those exploiters of the resource for whom the regulation does not bind. If the aggregate output of a subset of regulated players is constrained to be less than their unconstrained Nash equilibrium levels, then such a regulation will increase the payoffs to the remaining unregulated players:

Corollary 2: Let the open access resource game be modified by the introduction of a uniform quota on the level of individual inputs, of the form $\ell_i \leq \ell^{\max}$ for all $i \in I$. Then, denoting the resulting 'regulated equilibrium' by L^{R*},

(i) $L^{R*} \leq L^*$,

(ii) For any player i who does not face a binding constraint in the regulated equilibrium, $L^{R*}\,[1-\beta_i(L^{R*})] < L^*\,[1-\beta_i(L^*)]$,

(iii) The Nash equilibrium payoffs to all players who do not face a binding constraint at the regulated equilibrium are at least as high as in the unregulated equilibrium. Moreover, if the constraint binds for some player, then any player making a positive contribution in the regulated equilibrium who is not bound is strictly better off.

Again, to demonstrate this we merely need to observe that the regulation truncates individual share functions, so that i's share at any allocation is given by $b_i = \min\{\beta_i(L), \ell^{\max}/L\}$. Each point on the share function of the regulated game is either unchanged or moved to the left - $B^R(L) = \sum_{i \in I} \min\left\{\beta_i(L), \ell^{\max}/L\right\}$ - from which the claims of corollary 2 follow immediately.

Finally, consider the consequences of two types of shock for the equilibrium of the open access resource game with a fixed number of players. First, consider the effects of an idiosyncratic technological shock which exogenously increases the inherent productivity, or ability, of player i, while leaving that of all other players unaffected. This may be modelled in the following way. Suppose that each unit of nominal input by player j generates e_j units of effective input, where e_j is an exogenous parameter, so that $\ell_j = e_j h_j$. The variable h_j may be interpreted, for example, as the actual number of hours applied by player j to the productive activity. All our analysis up to this point assumes that $e_j = 1$

throughout for all players. We now consider the implications for equilibrium of an increase in player i's ability when that of every other player remains at unity.

Player i's payoff and marginal rate of substitution functions may be written as

$$u_i(x_i, h_i) = u_i\left(x_i, \frac{\ell_i}{e_i}\right) = u_i\left(b_i F(L), \frac{b_i L}{e_i}\right) \equiv \omega_i(b_i, L, e_i)$$

and

$$s_i(x_i, \ell_i) = s_i\left(b_i F(L), \frac{b_i L}{e_i}\right) \equiv \sigma_i(b_i, L, e_i).$$

Player i's preferences over (x_i, h_i) obey assumption **A.1.** Assumptions **A.2** and **A.3**, also, continue to hold. Consider an increase in e_i from e_i^1 to e_i^2, holding b_i and L constant. Player i enjoys the same value of x_i, and applies a lower level of h_i. Consequently, $\omega_i(\cdot)$ increases and $\sigma_i(\cdot)$ falls. As a consequence of the latter, if $\sigma_i(\cdot)$ now falls short of $\tau_i(\cdot)$, player i can further enhance her payoff by increasing her share b_i. Consequently, $\beta_i(L, e_i^2) > \beta_i(L, e_i^1)$ - the increase in i's ability implies a rightward shift in the graph of her share function in Figure 4, and therefore also in that of the aggregate share function. The equilibrium level of L therefore increases. Since the share functions of all other players are unchanged, we can appeal to theorem 2 to infer the consequences of the change in player i's ability:

Corollary 3: In the open access game, let player i's inherent productivity increase, while that of every other player remains constant. Then, denoting the initial and final equilibrium values of L by L^{1*} and L^{2*} respectively,

(i) $L^{2*} \geq L^{1*}$,

(ii) Either $L^{2*} = L^{1*}$ and $\beta_i(L^{2*}) = \beta_i(L^{1*}) = 0$, or $L^{2*} > L^{1*}$ and $\beta_i(L^{2*}) > \beta_i(L^{1*}) \geq 0$,

(iii) Player i's payoff will not fall and, if $\beta_i(L^{2*}) > 0$, it will rise,

(iv) Payoffs of players other than i will not rise and, if $\beta_i(L^{2*}) > 0$, they will fall.

A generalised technological shock may be modelled by supposing that every player's inherent ability takes a common value, e. We compare the equilibria associated with two values of e: $e = e^1 = 1$, and $e = e^2 > 1$.

We have already seen that an increase in each player's inherent ability shifts that player's share function to the right. In the present exercise, therefore, the graph of every player's share function shifts to the right. Therefore, so too does the aggregate share function. Therefore the aggregate level of input rises. However, the payoff of any individual player may rise or fall. Other things held constant, her own increase in ability enhances her payoff, but the adjustments by others lead to an increase in the level of exploitation that hurts her. We cannot draw the strong conclusions that we inferred in the case of an idiosyncratic shock. The one unambiguous conclusion concerns the level of aggregate exploitation:

Corollary 4: Let every player *i*'s inherent productivity increase from $e^1 = 1$ to $e^2 > 1$. Then, denoting the initial and final equilibrium values of L by L^{1*} and L^{2*} respectively,
$L^{2*} > L^{1*}$.

IV OTHER COST AND SURPLUS SHARING RULES

We define a simple surplus sharing rule as one in which player *i*'s consumption is a function of *i*'s input and of the aggregate input level of all players. Clearly, the proportional sharing rule is only one of many such rules. Other well-known examples include the equal share rule, in which $x_i = (1/n)F(L)$, and the 'equal benefit rule', in which $x_i = (1/n)F(L) + (\ell_i - L/n)F'(\cdot)$ - see Corchon and Puy (1998) for a recent discussion of these rules. Furthermore, any sharing rule that is a convex combination of simple sharing rules, where the weights are themselves exogenous, also gives rise to a simply reducible game. To show the versatility of our approach, we sketch its application to the exogenous sharing rule, of which the equal sharing rule is a special case.

(a) The exogenous shares model

Suppose that player *i* receives an exogenous share θ_i of total output, regardless of her own individual contribution to production. The restriction $\Sigma_j\theta_j = 1$ ensures that total output is just exhausted by payments to the players. The equal shares rule is obtained by putting $\theta_i = 1/n$ for all *i*. Player *i*'s payoff under the exogenous shares rule is $u_i(\theta_i F(L),\ell_i)$.

For simplicity, concentrate on the case where player i is at an interior solution. Then the first-order condition is

$$-\frac{\partial u_i/\partial \ell_i}{\partial u_i/\partial x_i} = \theta_i F'(L).$$

The left-hand side of this expression is precisely player i's marginal rate of substitution. Again, we can express this as a function of player i's chosen share and of total input. Using the superscript 'e' where necessary to remind ourselves that the context is that of the exogenous sharing rule,

$$\sigma_i^e(b_i^e, L) = -\frac{\partial u_i/\partial \ell_i}{\partial u_i/\partial x_i} = s_i\left(x_i, \ell_i\right) = s_i\left(\theta_i F(L), b_i^e L\right) \quad (11)$$

Inspection of (11), combined with **A.1**, reveals that $\sigma_i^e(b_i^e, L)$ is strictly increasing in both arguments. Under the exogenous sharing rule, $x_i = \theta_i F(L)$. Therefore, holding other players' input levels fixed while perturbing that of player i, i's marginal rate of transformation is

$$\tau_i^e(b_i^e, L) = \theta_i F'(L).$$

$\tau_i^e(\cdot)$ is independent of ℓ_i and decreasing in L. Consequently, an exogenous sharing rule implies a horizontal graph for $\tau_i^e(\cdot)$ in Figure 2. This slight modification does not change the essential features of our analysis of existence and uniqueness. An increase in the value of L shifts the graph of $\tau_i^e(\cdot)$ against b_i down, while shifting that of $\sigma_i^e(\cdot)$ to the left. Consequently, the value of b_i^e satisfying the first order conditions must decline as L increases. The share function under the exogenous sharing rule, $\beta_i^e(L)$, is monotonic nonincreasing. Indeed, Facts 1 - 3 all hold for the exogenous sharing rule under our assumptions **A.1** to **A.3**.

The comparative static results with respect to i's payoff response to a change in L differ from those of the proportional sharing rule. This reflects the well-established presumption that the Nash equilibrium under exogenous shares involves underprovision of input. Theorem 2 is replaced by

Theorem 3 [Comparative statics for the exogenous shares model]: Consider the exogenous surplus sharing game in which assumptions **A.1** - **A.3** hold. Then, if $L^2 > L^1$,

(i) $L^1\,[1- \beta_i^e(L^1)] < L^2\,[1- \beta_i^e(L^2)\;]$

and (ii) $\omega_i(\beta_i^e(L^1), L^1) \leq \omega_i(\beta_i^e(L^2), L^2)$ for all $i \in I$, with strict inequality if $\beta_i^e(L^1) > 0$.

[Note that part (i) is identical to Theorem 2. It is in part (ii) that the inequality is reversed.]

Proof: (i) As for Theorem 2.

(ii) Since $F(\cdot)$ is strictly increasing, (i) implies that

$$\omega_i\left(\beta_i^e\left(L^2\right), L^2\right) = \max_{\ell_i \geq 0}\; u_i\left(\theta_i F\left[L^2 - L^2\beta_i^e\left(L^2\right) + \ell_i\right], \ell_i\right)$$

$$\geq \max_{\ell_i \geq 0}\; u_i\left(\theta_i F\left[L^1 - L^1\beta_i^e\left(L^1\right) + \ell_i\right], \ell_i\right) = \omega_i\left(\beta_i^e\left(L^1\right), L^1\right)$$

with strict inequality if $\beta_i(L^1) > 0$. ■

Consequently, Corollaries 1 - 4 continue to apply provided we replace statements that a player is better or worse off with their opposite.

(b) Cost sharing models

Models of cost sharing have essentially the same structure as those of surplus sharing. Consequently, our approach can be applied to the cost-sharing model with minor modifications. To justify this claim, consider the proportional cost sharing game. It has a similar structure to that of the proportional surplus sharing model. To analyse a cost sharing game, it is most convenient to represent the technology by the cost function, $C(X)$, which describes the minimum cost required to produce X units of output - player i's payoff is now a function of output x_i, given by $u_i\left(x_i, \frac{x_i}{X} C(X)\right)$ where $X = \sum_{i=1}^{n} x_i$. We retain **A.1** and **A.3** and replace **A.2** by

A.2' $C(X)$ is increasing, strictly convex, continuous and continuously differentiable for $X > 0$, and $C(0) = 0$.

We define player *i*'s marginal rates of substitution and of transformation exactly as in the surplus-sharing example. However, their arguments are (g_i, X), where g_i is player *i*'s share of total output: $g_i = x_i/X$. Moreover, player *i*'s marginal rate of transformation equals the reciprocal of the convex combination of the marginal and average costs implied by the total output level:

$$\tau_i\left(g_i, X\right) \equiv \frac{1}{g_i C'(X) + \left[1 - g_i\right]\frac{C(X)}{X}}.$$

Comparison with (7) shows that, where the surplus sharing model generates a convex combination of marginal and average products of a concave production function, the cost sharing model generates the inverse of a convex combination of marginal and average costs associated with a convex cost function. Figures 2 - 4, with suitable re-labelling of the axes, remain qualitatively valid for the cost-sharing model. Consequently, existence and uniqueness of equilibrium and comparative statics are established along exactly analogous lines.

V FURTHER APPLICATIONS AND EXTENSIONS

We asserted in the introduction that the class of simply reducible games is much larger and more interesting than it might appear at first sight. The most convincing way to validate this claim is to provide examples of commonly analysed models that are covered by our analysis. Here is a list of commonly studied models that imply simply reducible games:

(a) Schelling *n*-person binary choice games [Schelling (1978)]

(b) Pure public good provision [Cornes and Sandler (1985, 1996), Bergstrom, Blume and Varian (1986)]

(c) Impure public good provision [Cornes and Sandler (1984, 1996)]

(d) Cournot oligopoly with undifferentiated output [Friedman (1982)]

(e) Simple surplus sharing [Moulin (1995), Watts (1996)]

(f) Simple cost sharing [Moulin (1995), Watts (1996)]

(g) Open access resource exploitation [Dasgupta and Heal (1979), Cornes and Sandler (1983)]

(h) Technological complementarities [Bryant (1983), Cooper (1999)]

(i) Financial intermediation [Cooper (1999)]

(j) Models of transfrontier pollution [Tulkens (1979), Barrett (1997), Hoel (1997), Chander and Tulkens (1997)]

(k) Rentseeking contests with risk averse players [Nitzan (1994)]

These applications typically allow players to choose the value of a continuously variable quantity. The apparent exception is the Schelling game, which offers players a binary choice. However, if players in Schelling games are allowed to play mixed strategies, the resulting model is a simply reducible game with continuously variable strategic choices. When analysing these games, many authors assume that all players have identical payoffs, and focus on symmetric Nash equilibria. Our approach offers a simple and attractive approach to such games which permits heterogeneity and is not restricted to the analysis of symmetric equilibria.

We should emphasise that we do not claim that all the models in our list have the same structure and properties as the proportional surplus sharing model. Indeed, the models of technological complementarity and intermediation inherently involve the possibility of multiple equilibria and hence the need to solve a co-ordination problem. Our claim is that they all have a simply reducible structure, and that their analysis may be facilitated by exploiting this structure and examining share functions rather than by analysing conventional reaction functions. The resulting analysis is significantly simplified by the dramatic reduction of the dimension of the space in which it is undertaken.

Furthermore, it is possible to extend our analysis in at least two ways that greatly expand its scope. First, there are models of pure public good provision and of oligopoly with differentiated products in which each player cares about, not an unweighted sum, but a more general aggregate of all players' choices. It may be shown that, if the aggregator function is additively separable, such games may be transformed into simply reducible games and the share function approach may be applied. Second, there are interesting *n*-person sharing games in which it is not possible to reduce the domain of the function describing the individual player's economic environment to the real line, but where it is possible to reduce it to dimension K, where $K < n$. Such games, which may be called 'K-reducible' games, are a topic for another paper.

VI CONCLUSION

A simply reducible game is one in which the payoff of every player can be expressed as a function of that player's own strategy choice and the sum of the strategies chosen by all players in the game. The models of pure public good provision, Cournot oligopoly with undifferentiated products, proportional cost and surplus sharing, and the open access resource game are all simply reducible games. The replacement and share functions of a simply reducible game provide a simple analysis of existence, uniqueness and the comparative static properties of equilibrium. In contrast to the usual approach, which considers fixed points of mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$, where n is the number of players and may be large, our approach enables us to base the analysis on a function defined on the real line. By treating the total quantity Q as the parametric independent variable in describing each player's behavior, the present approach obviates the need to solve a high order simultaneous system of equations in order to find a Nash noncooperative equilibrium. This greatly simplifies the analysis, and seems particularly promising in models with heterogeneous individuals. Finally, we have suggested two ways in which the concept of simple reducibility may be usefully generalised and its scope thereby increased.

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FOOTNOTES

1. We are grateful to Wolfgang Buchholz for drawing our attention to Okuguchi's paper.
2. Watts (1996) does not assume differentiability of $u_i(\cdot)$. However, we should point out that we make this assumption purely for expository reasons. The proofs in the sequel go through in the nondifferentiable case if $s_i(b_i, L)$ is interpreted as the slope of a separating line to the upper preference set of player i at $(x_i, \ell_i) = (b_iF(L), b_iL)$.
3. These assumptions ensure that $\lim_{L \to 0} f'(L)$ exists provided we allow $+\infty$ as a limiting value. For convenience, we write $f'(0)$ for this limit in the sequel.
4. The indifference curve may be steeper than the graph of $X = F(L)$ in the neighborhood of the origin. This creates no problems for our analysis. It simply means that player i will never choose a positive share. The important content of **A.3** is that, if the indifference curve is less steep at O, it must also intersect the graph of the production function elsewhere at some finite value of L.

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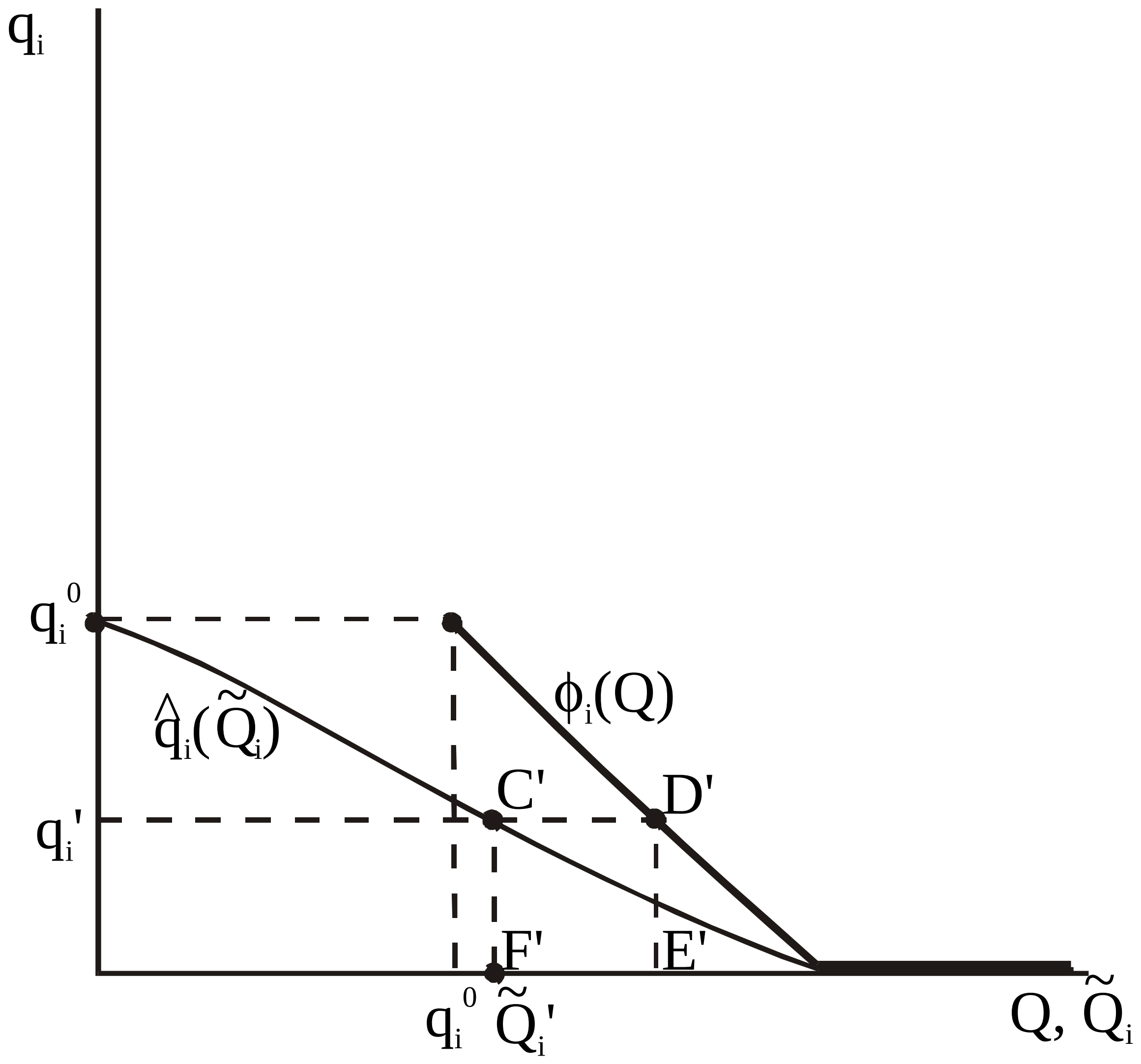
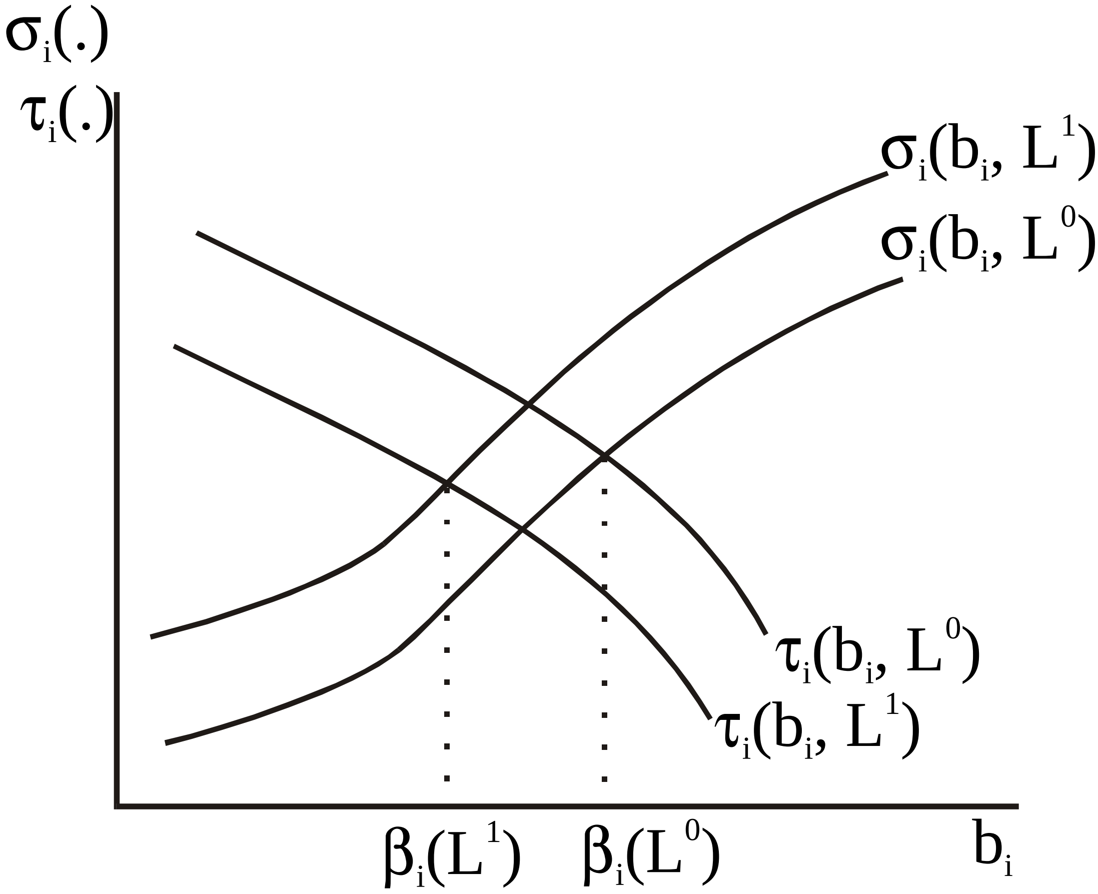


Figure 1



$\sigma_i(.)$
$\tau_i(.)$
$\sigma_i(b_i, L^1)$
$\sigma_i(b_i, L^0)$
$\tau_i(b_i, L^0)$
$\tau_i(b_i, L^1)$
$\beta_i(L^1)$
$\beta_i(L^0)$
b_i

Figure 2

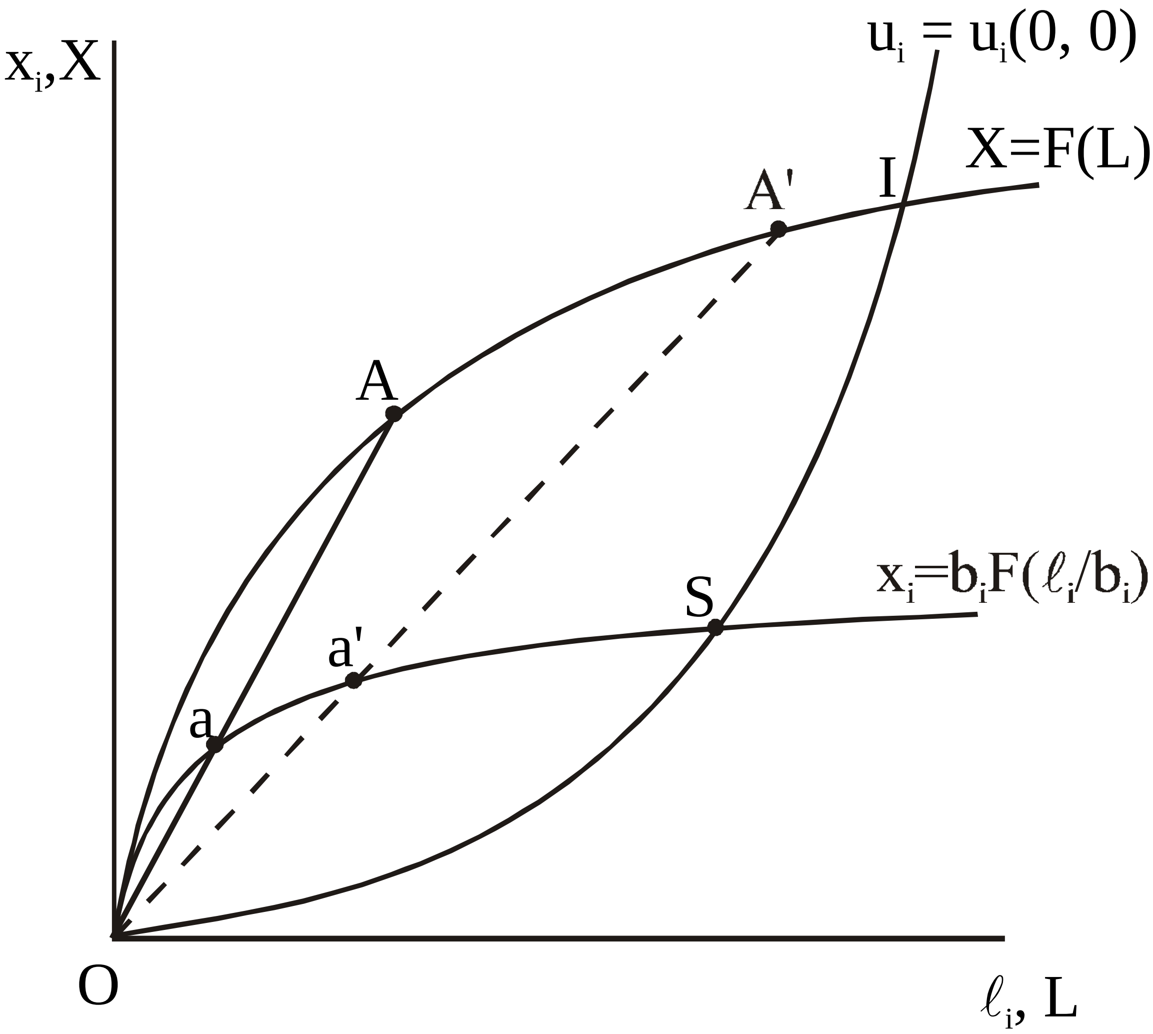


Figure 3

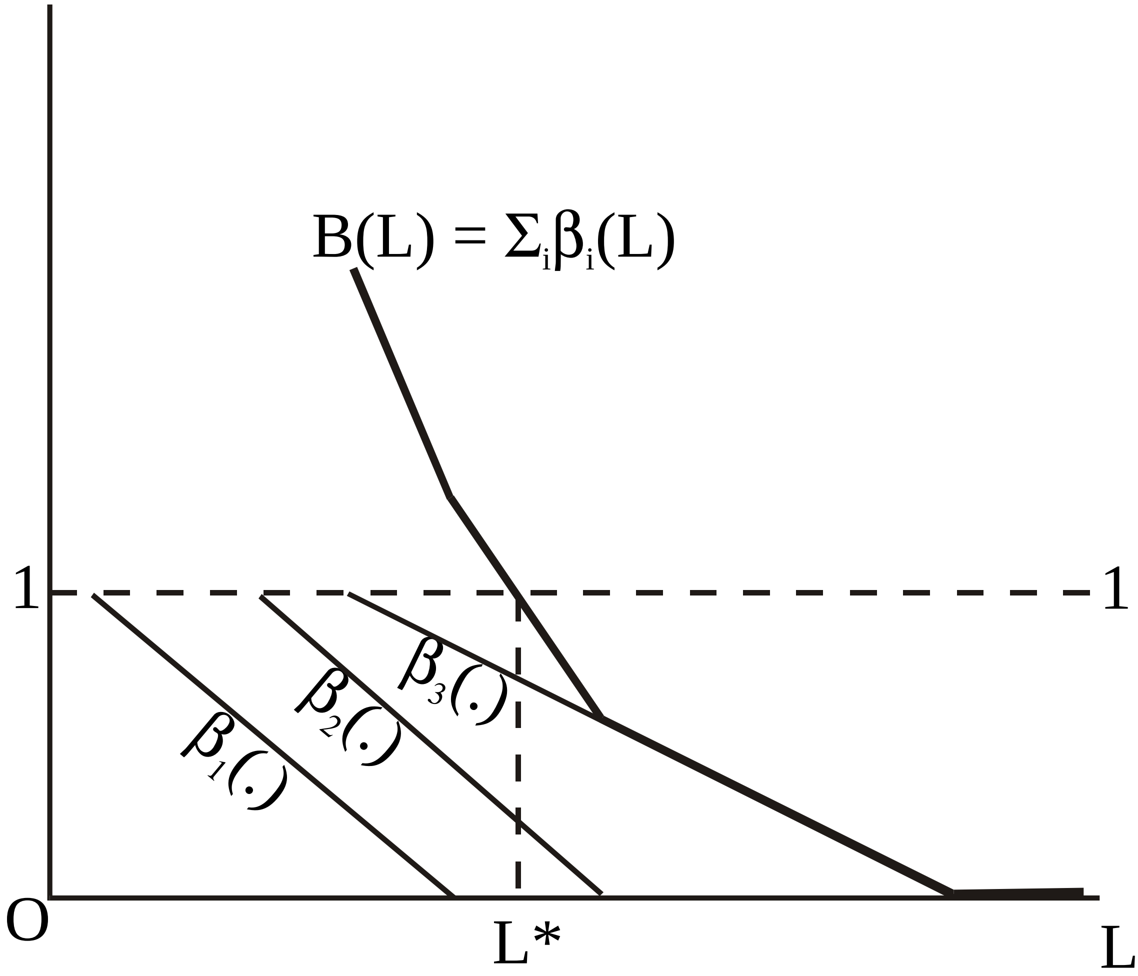


Figure 4