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Richard Cornes is Professor, School of Economics, University of Nottingham, and Roger Hartley is Professor, Department of Economics, Keele University

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Richard Cornes

School of Economics, University of Nottingham

and

Roger Hartley

Department of Economics, Keele University

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1. INTRODUCTION

The rent-seeking model of Tullock (1980) has stimulated a large literature on rent-seeking contests, of which Hillman (1989) and Nitzan (1994) provide useful surveys. Although Tullock's 'winner take all' model has been adapted and extended in numerous ways, there remain fundamental modeling issues, particularly concerning the limiting proportion of rent dissipation as the number of players becomes large, that have not been fully resolved. This is especially true if we want to permit players not only to be strictly risk averse but also to differ from one another with respect to their attitudes to risk. Here, as in other applications of noncooperative game theory, the intricate ways in which the decisions of heterogeneous players interact to generate an equilibrium have been a substantial obstacle to analysis.

The standard method of analysis defines each player's best response function, $\hat{q}_i\left(\mathbf{q}_{-i}\right)$, and characterizes a Nash equilibrium as a fixed point of a mapping provided by the *n* best responses, **f**: $\mathbb{R}^n \to \mathbb{R}^n$, where *n* is the [possibly large] number of players. Analysis then proceeds by investigating the properties of a mapping of potentially very high dimension. Common responses to this dimensionality problem are to restrict attention to 2-player examples, or to suppose that each of the *n* contestants is identical and then focus attention on the symmetric equilibrium. The first of these responses to the problem is adopted in the rent-seeking literature by Skaperdas and Gan (1995), and the second by Hillman and Katz (1984) and Konrad and Schlesinger (1997). Although both approaches afford useful insights, neither is wholly satisfactory.

However, there is an alternative way of thinking about the model that overcomes the 'curse of dimensionality' and enables us to explore further the implications of heterogeneity. This paper introduces and applies what we have elsewhere called the 'share function' to simplify the analysis of players' interactions[1]. Player *i*'s best response function may be written alternatively as $q_i = \hat{q}_i\left(Q - q_i\right)$ where *Q* is the sum of all players' choices, including that of player *i*. Under assumptions that are satisfied in the present paper, this can be rearranged to provide an explicit function of the form $q_i = \phi_i\left(Q\right)$. We call $\phi_i(Q)$ the replacement function - given a particular value of the sum of all players' choices, it identifies the quantity q_i such that, if this is removed from Q, player i's best response to the remainder precisely replaces the quantity removed. The share function, $s_i(Q)$, is then defined to be the ratio $q_i/Q = \phi_i(Q)/Q = s_i(Q)$. The share function allows us to characterize Nash equilibria by requiring that the sum of the individual share functions, each having as its domain an interval of the real line, should equal unity. This exploits the special structure of the game and thereby avoids the need to consider spaces of high dimension - indeed, throughout the analysis we need only consider mappings on intervals of the real line. This approach handles games involving large numbers of heterogeneous players with ease.

The simplicity of the share function approach enables us to extend the treatment of existence and uniqueness beyond the risk neutral setting considered, for example, by Szidarovszky and Okuguchi (1997). More significant, however, are its implications for the analysis of rent dissipation. Existing literature has established a presumption that if all players are risk averse there will be less than complete rent dissipation in the limit as the number of players becomes large. We are able to show that matters are more subtle than this. The limiting proportion of rent dissipated depends not only on the distribution of attitudes to risk among players, but also on whether or not an Inada-type condition is satisfied by the technology whereby individuals convert effort into increments of probability of winning the rent. It is also affected by interactions between risk aversion and heterogeneity. For example, we describe a nested sequence of games in which all players are strictly risk averse and yet the rent is fully dissipated in the limit.

The rent-seeking model, which we describe in Section 2, is essentially Tullock's, extended to accommodate strictly risk averse players and more general technologies. Section 3 introduces the share function and applies it to the simplest situation, in which each player is risk neutral and has access to a particularly simple linear technology for converting effort into incremental probability of winning the rent. Section 4 extends the basic model and establishes the existence of a unique equilibrium in a rent-seeking contest when players are risk averse with constant coefficient of risk aversion, but may differ from one another in terms of their attitude towards risk, and have access to nonlinear technologies. Section 5 presents our central conclusions on the comparative static properties of the model, particularly the behavior of rent dissipation as the game becomes large.

2. THE BASIC MODEL OF RENTSEEKING

Consider an n-player 'winner take all' rent-seeking contest with $n \geq 2$. Player i has an initial wealth level I_i and chooses the level of resources or 'effort', x_i, to devote to enhancing her probability of being the sole winner of an exogenous rent R. Introducing notation that will later prove useful, that probability may be variously written as

$$p_i = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)} = \frac{y_i}{Y} = \frac{y_i}{y_i + \tilde{Y}_i}, \qquad (1)$$

where $f_i(x_i)$ is an increasing concave function satisfying $f_i(0) = 0$ and $y_i = f_i(x_i)$, $Y = \sum_{j=1}^{n} y_j$ and $\tilde{Y}_i = Y - y_i$. The function $f_i(x_i)$ may be thought of as a production function exhibiting non-increasing returns to scale in the production of an intermediate input y_i. Rent-seeking contests of this kind are often called 'imperfectly discriminating' contests - see Hillman and Riley (1989). Some authors - for example, Szidarovszky and Okuguchi (1997) - call $f_i(x_i)$ player i's 'production function for lotteries'. Since $f_i(x_i)$ is everywhere increasing, its inverse function $g_i(\cdot) = f_i^{-1}(\cdot)$ is defined. The function $g_i(y_i)$, which measures the total cost of generating the level y_i of the intermediate input, is an increasing convex function and satisfies $g_i(0) = 0$. The intermediate input y_i has no natural interpretation, but the ratio y_i/Y has. It is simply player i's probability of winning the rent. This ratio, or 'share', plays a key role in subsequent analysis, and we will denote it by the variable σ_i. The specification of the production function $f_i(\cdot)$ determines the rate at which an increase in effort applied by player i enhances her probability of winning the rent.

The preferences over wealth of a risk averse player are described by a concave utility function $u_i(W_i)$, where W_i is the player's wealth. Each player is an expected utility maximizer, and player i's payoff function is

$$\text{Expected utility} = p_i u_i(I_i + R - x_i) + (1 - p_i) u_i(I_i - x_i).$$

We find it convenient to treat the intermediate good y_i, rather than x_i, as the player's choice variable. Accordingly, we rewrite the payoff function for $y_i + \tilde{Y}_i > 0$ as

$$\pi_i\left(y_i,\tilde{Y}_i\right) = \frac{y_i}{y_i+\tilde{Y}_i}u_i\left[I_i+R-g_i\left(y_i\right)\right]+\frac{\tilde{Y}}{y_i+\tilde{Y}_i}u_i\left[I_i-g_i\left(y_i\right)\right]. \qquad (2)$$

We also set $\pi_i(0, 0) = u_i(I_i)$. Player i chooses $y_i \geq 0$ to maximize her payoff, and a Nash equilibrium is a vector of consistent best responses, $(y_1^*, y_2^*,\dots, y_n^*)$.

The literature on rent-seeking has been concerned to establish the existence and uniqueness of Nash equilibrium and to consider the extent to which the resources used up in competing for the rent partially, or wholly, dissipate that rent. To tackle these questions analytically, it has proved necessary to impose further structure on the problem. This has been done by specifying more precisely the players' attitudes to risk as represented by the form of the utility functions $u_i(\cdot)$ in (2) and by assuming particular functional forms for the production functions $f_i(\cdot)$ that appear in (1). We will assume that each player is either risk neutral or strictly risk averse. Furthermore, we will assume that the coefficient of risk aversion is everywhere a constant for each player. This implies that each player's utility function takes the form

$$u_i\left(W_i\right) = -e^{-\alpha_i (W_i)}$$

where α_i (> 0) represents player i's constant degree of absolute risk aversion:

$$R_A = -u''(\cdot)/u'(\cdot) = \alpha_i\,.$$

The possibility that player i is risk neutral can be accommodated by the utility function $u_i(W_i) = W_i$. We refer to this situation as one in which $\alpha = 0$. We should emphasize that our general analysis allows the coefficient of risk aversion to vary across players. Thus, it accommodates heterogeneous players.

The focus of our analysis is on how attitudes to risk, particularly when they are heterogeneous, interact to determine the proportion of the rent dissipated in the limit. By studying CARA, we abstract from anomalies found by other scholars such as risk-averse players spending more on rent-seeking than risk-neutral players [Konrad and Schlesinger (1997)] or an increase in the number of identical players leading to a reduced total outlay on rent-seeking [Skaperdas and Gan (1995)]. It can be shown that CARA players' outlays are diminishing in the coefficient of risk aversion. In particular, risk-neutral participants always spend more than risk-averse ones. Furthermore, adding extra players increases total expenditure if all players are identical or is there are constant returns to scale, and equilibrium Y is always increasing in the number of players. Nevertheless, we will demonstrate that even in the well-behaved model of CARA, results can be surprising. Most notably, we show that it is possible to have all players strictly risk averse and yet for the whole rent to be dissipated in the limit. Such results cast doubts over claims that risk aversion reduces rent-seeking even where there is no anomalous behaviour and the number of players is large. Of course, for 'small' risks, CARA may also provide a good approximation even though preferences depart from CARA for larger risks. In any event, we include the risk neutral model, for which Szidarovszky and Okuguchi (1997) have recently provided a proof of equilibrium existence and uniqueness, as a special case.

Two specific technologies are commonly assumed in the existing literature. One is the linear technology, under which each individual expends effort x_i to generate y_i with constant returns to scale, so that a given proportion change in x_i implies the same proportion change in y_i. The alternative formulation, suggested by Tullock (1980), assumes a technology represented by a concave production function of the form $y_i = f_i\left(x_i\right) = x_i^{r_i}$, where $0 < r_i < 1$. Although this is a very convenient form for analytical work, we shall argue that it is potentially misleading insofar as it confounds two independent aspects of the technology. In the first place, it represents decreasing returns, or increasing unit costs - and does so in a very simple and tractable form. However, it also implies that the marginal product of effort x_i in generating the intermediate input y_i is unbounded, since if $f_i\left(x_i\right) = x_i^{r_i}$, then $\lim_{x_i \to 0} f_i'\left(x_i\right) = \lim_{x_i \to 0} r_i x_i^{r_i - 1} = \infty$. This characteristic is familiar as one of the Inada conditions frequently encountered in growth theory. This is not a necessary implication of decreasing returns to scale. Consider, for example, $y = (x + 1)^{1/2} - 1$ where $x \geq 0$, $y \geq 0$. This satisfies the requirement that $f(0) = 0$ and exhibits decreasing returns to scale, but the marginal product in the neighborhood of the origin is finite: $f'(0) = ½$. Moreover, we shall show that whether the marginal product is bounded or unbounded above has important implications for the answers to questions about rent dissipation. As a consequence, when we introduce decreasing returns to scale, we will distinguish explicitly between two possibilities, according to whether or not there exists a finite upper bound to the marginal product of x_i.

Section 3 analyses the model in which all players are risk neutral and all have access to linear technologies. Our intention here is expository, to introduce the share function approach by applying it to the simplest model. Consequently, the treatment is fairly informal. Formal propositions, together with their proofs, are presented in Sections 4 and 5, where we allow for more general technologies and strict risk aversion.

3. A SIMPLE EXAMPLE: LINEAR TECHNOLOGY AND RISK NEUTRALITY

Suppose that each of n risk neutral players has access to a linear technology for converting the input x into the output y. We allow the technology to vary across players. For player i, we assume that $x_i = a_i y_i$, where a_i is an exogenously fixed unit cost coefficient. If $a_i > a_j$, player i has to apply more effort than does player j in order to increase her probability of winning by a given increment. Player i's payoff function is

$$\pi_i\left(y_i, \tilde{Y}_i\right) = \frac{y_i}{y_i + \tilde{Y}_i}\left[I + R - a_i y_i\right] + \frac{\tilde{Y}_i}{y_i + \tilde{Y}_i}\left[I - a_i y_i\right]. \qquad (3)$$

Standard manipulation of first and second order conditions confirm that a necessary and sufficient condition for a value $y_i > 0$ to maximize $\pi_i(y_i, \tilde{Y}_i)$ is

$$a_i y_i^2 + 2a_i \tilde{Y} y_i + \left(a_i \tilde{Y}_i^2 - R\tilde{Y}\right) = 0 \qquad (4)$$

This is a quadratic. Of its two solutions, only one can be positive and therefore economically interesting. Taking this solution, and also taking account of the fact that the non-negativity constraint on y_i may bind, the necessary and sufficient condition for $y_i \geq 0$ to be the solution is

$$y_i = \max\left\{\sqrt{\frac{R\tilde{Y}_i}{a_i}} - \tilde{Y}_i, 0\right\} \qquad (5)$$

This is the player's best response function, $\hat{y}_i\left(\tilde{Y}_i\right)$. Alternatively, if the term $\tilde{Y}_i$ is everywhere replaced in (4) by $(Y - y_i)$, simplification of the resulting expression yields

$$y_i = \max\left\{Y - \frac{a_i Y^2}{R}, 0\right\}. \qquad (6)$$

Equation (6), which expresses y_i as a function of the aggregate, Y, is the **replacement function**, $\phi_i(Y)$.

Finally, if both sides of (6) are divided by $Y > 0$, the resulting function is the **share function**, $s_i(Y)$. Again, it incorporates the requirement expressed by the first order condition (4), but this time in a functional form which has as its dependent variable player i's share, y_i/Y. Since this is the function we intend to exploit, let us summarize our finding in the following fact:

Proposition 3.1: If player i is risk neutral, with access to a linear technology such that $_i$ = $_i y$, then a share function exists for that player. Moreover, it has the following form[2]:

$$_i(Y) = \max\left[1 \quad \frac{a_i}{R}, 0\right] \qquad \text{for all } Y > 0. \qquad (7)$$

Figure 1 shows the best response, replacement and share functions of a risk neutral player with linear technology. Panel (a) shows the graph of equation (5), the best response function. Observe that it is not monotonic. Therefore, proofs of results that depend on the assumption of monotonicity of best response functions are not available to us in this model. The graph of the replacement function, shown in Panel (b), can be constructed very simply from the one in Panel (a). At each point - say $\left[\tilde{Y}', \hat{y}(\tilde{Y}')\right]$ - on the graph of the best response function, construct a square with that point as its top left corner. Then its top right hand corner represents the corresponding point on the graph of the replacement function. This construction simply adds the quantity $\hat{y}(\tilde{Y}')$ horizontally to the quantity $\tilde{Y}'$, so that the graph in panel (b) maps $\hat{y}(\tilde{Y})$ against $Y = \tilde{Y} + \hat{y}(\tilde{Y})$. A further purely geometric device enables us to construct the graph of the implied share function, shown in Panel (c), from that of the replacement function. Draw the ray from the origin that passes through a given point, say (Y', y_i'), on the graph of the replacement function. At the point on this ray where $Y = 1$, its height measures the share value implied by (Y', y_i'). The associate pointed on the graph of the share function is therefore (Y', σ_i'). In the present example, $s_i(Y)$ is piece-wise linear and strictly decreasing in Y for any $\sigma_i > 0$. Its value approaches one as $Y \to 0$, falls linearly to zero at the point where $Y = R/a_i$, and remains zero thereafter. Although this purely geometric argument does not contribute any fresh economic insights, it does usefully stress the observation that the three functions are no more than alternative ways of presenting the same information. The choice between them should be determined by convenience.

Figure 2 shows the share functions for four individuals with different unit costs. The same figure also shows the graph of the aggregate share function, obtained in this and in other cases by adding the graphs of the individual players' share functions vertically. A Nash equilibrium is an allocation such that, given the prevailing value of Y - say Y^* - the sum of all players' share values is unity. The value of Y^* is determined by the condition that

$$\sum_i s_i\left(Y^*\right) = \sum_i \left\{\max\left[1 - \frac{a_i Y^*}{R}, 0\right]\right\} = 1. \tag{8}$$

The thick line in Figure 2 is the graph of the vertical sum of the individual share functions, and the Nash equilibrium value, Y^*, is the unique value at which this sum is unity.

The properties of the individual share functions imply the following three properties of the aggregate share function: (i) it is piece-wise linear and continuous, (ii) for sufficiently small values of Y its value exceeds one, while for sufficiently large values of Y its value is zero, (iii) it is strictly decreasing in Y whenever it is positive. From these observations we can infer our second fact:

Proposition 3.2: In a rent-seeking contest involving n risk neutral players, each with a [possibly different] linear technology, there exists a unique Nash equilibrium.

Students of rent-seeking contests have been particularly concerned with the question of whether, and under what conditions, the full value of the rent is dissipated by the resources expended in the contest. This is an easy question to answer in the current context. Recall, however, that the value of resources expended is not measured by Y.

Rather, it is the value of $X = \Sigma_i x_i$ that we need to examine. Doing so, we have our third fact:

Proposition 3.3: Suppose that, in a rent-seeking contest with n risk-neutral players, m ($\leq n$) are active. The proportion of the rent dissipated will not exceed $(m - 1)/m$, and will equal this if all active players are identical.

Proof: The amount of rent dissipated at a Nash equilibrium is

$$X = \sum_{i \in S_C} x_i = \sum_{i \in S_C} a_i y_i = \sum_{i \in S_C} (1 - \sigma_i) \frac{R}{Y} y_i = \sum_{i \in S_C} \sigma_i (1 - \sigma_i) R$$

where S_C is the set of strictly positive contestants. Let $\varepsilon_i \equiv \sigma_i - 1/m$. Then $\Sigma_j \varepsilon_i = 0$ and

$$\begin{aligned} \frac{X}{R} &= \sum_{i \in S_C} \left(\frac{1}{m} + \varepsilon_i\right)\left(1 - \frac{1}{m} - \varepsilon_i\right) \\ &= \sum_{i \in S_C} \left(\frac{1}{m}\right)\left(1 - \frac{1}{m}\right) + \sum_{i \in S_C} \left[\varepsilon_i \left(1 - \frac{2}{m}\right) - \varepsilon_i^2\right] \\ &= \frac{m-1}{m} - \sum_{i \in S_C} \varepsilon_i^2 \\ &\leq \frac{m-1}{m} \end{aligned}$$

with equality holding if $\sigma_i = \sigma_j = 1/m$ for all $i, j \in S_C$.

Note particularly that, if the n players are identical - that is, if each player has the same technology - then the proportion of rent dissipated is $X/R = (n - 1)/n$.

It is instructive to consider one further simple exercise in the context of the present model. Suppose we are interested in the outcome of this game in the limit when it is played by a large number of players with differing unit cost coefficients. One simple way of generating such a game is as follows. Imagine a population consisting of 2 types of individual with unit input requirement coefficients $\underline{a}$ and $\overline{a}$ [$> \underline{a}$]. We will later consider a generalization of this discrete distribution. Use of share functions shows in a very simple and direct manner the implications of increasing the number of contestants by repeated draws from such a population. Figure 3 shows the share functions of the two types. Let the number of $\underline{a}$ types drawn from the population be $\underline{n}$. Consider for the moment the sum of the share functions of $\underline{a}$ types. As their number grows, so too does the value of Y that would represent a Nash equilibrium if only low cost individuals were contesting the rent. There is a value of $\underline{n}$ at which this value exceeds the value of Y at which high cost types would choose not to compete for the rent. In Figure 3, for example, the Nash equilibrium is Y^* and only low cost types contest the rent. This situation arises if

$$\underline{n}\left(1-\frac{1}{\underline{n}}\right)\frac{R}{\underline{a}} > \frac{R}{\bar{a}},$$

or

$$\underline{n} > \frac{\bar{a}-\underline{a}}{\bar{a}}.$$

Thus, if the number of low cost types is sufficiently large, the Nash equilibrium will not involve positive expenditure of effort by high cost types. They simply do not find it worthwhile to contest the rent. Their presence in the population makes no difference to the way the game is played if the number of low cost types is sufficiently high.

This observation may be extended in a natural and straightforward way to accommodate many types, distinguished by attitude to risk. Let a_τ denote the unit cost of a type τ individual, and adopt the convention that if $i < j$, then $a_i < a_j$. A type 1 individual, therefore, is the type with lowest unit cost. Finally, let the number of players of type τ be denoted by n_τ. Then the following is true:

Proposition 3.4: Let there be risk neutral players of T types, with unit cost coefficients $a_1 < a_2 < \ldots < a_T$. Then if $n_1 > \frac{a_2 - a_1}{a_2}$, only type 1 players will devote a positive level of input into contesting the rent. As n_1 tends to infinity, the rent is wholly dissipated.

The participation of the low unit cost types drives Y up to a level at which none of the other types finds it worthwhile to enter the contest. The final part of proposition III.4 is an immediate consequence of the previous proposition and holds even if the numbers of all other types tends to infinity.

In the two sections that follow we allow for strict risk aversion and a more general convex technology for converting individual effort into probability of winning. Our objective is to test the robustness of these propositions with respect to such extensions and to establish analogous propositions in the more general setting.

4. EQUILIBRIUM EXISTENCE AND UNIQUENESS WITH STRICT RISK AVERSION AND NONLINEAR TECHNOLOGIES

Throughout this section, we maintain the following two assumptions:

A.1 For all i, preferences over wealth are represented by

either (i) $u_i(W_i) = -e^{-\alpha_i W_i}$ where $\alpha_i > 0$ [player i is strictly risk averse]

or (ii) $u_i(W_i) = W_i$ [player i is risk neutral].

A.2 The production function, $f_i(x_i)$, satisfies the following conditions:

$f_i(0) = 0$, $f_i'(\cdot) > 0$, $f_i''(\cdot) \le 0$.

Since we find it useful to use the cost function, $g_i(\cdot) = f_i^{-1}(\cdot)$, it is worth observing that assumption **A.2** has the following implications for $g_i(\cdot)$: $g_i(0) = 0$, $g_i'(\cdot) > 0$, and $g_i''(\cdot) \ge 0$. Suppose that player i is strictly risk averse, with a constant coefficient of absolute risk aversion and access to a nonlinear production function for lotteries. Her payoff function is:

$$\pi_i(y_i, \tilde{Y}_i) = \frac{y_i}{y_i + \tilde{Y}_i}\left\{-e^{-\alpha_i[I_i + R - g_i(y_i)]}\right\} + \frac{\tilde{Y}}{y_i + \tilde{Y}_i}\left\{-e^{-\alpha_i[I_i - g_i(y_i)]}\right\} \quad (10)$$

The appendix shows that, if player i's payoff reaches a maximum at a positive value of y_i, then the first order condition requires the value of the implied share, $\sigma_i = y_i/Y$, to satisfy the following condition:

$$g_i'(\sigma_i Y)Y = \frac{(1-\sigma_i)}{\beta(\alpha_i) - \alpha_i \sigma_i}, \quad (11)$$

where $\beta(\alpha_i) \equiv \dfrac{\alpha_i}{\left(1-e^{-\alpha_i R}\right)}$ for $\alpha_i > 0$ and $\beta(0) = 1/R$. Moreover, this condition is not only necessary but also sufficient for *i*'s payoff to be maximized by a positive value of y_i.

It is important to allow for the possibility that, for a given level of *Y*, there may be no positive share value that satisfies (11). In this case, the player's share is zero. Summarizing, we have the following proposition:

Proposition 4.1: If **A.1** and **A.2** hold, a share function for player *i* exists. The player's share, σ_i, takes the value 0 if and only if $f_i'(0) < \infty$ and $Y \geq [\beta(\alpha_i)g'(0)]^{-1}$. Otherwise, it is determined by the following condition:

$$\frac{\left(1-\sigma_i\right)}{\beta\left(\alpha_i\right)-\alpha_i\sigma_i} = g_i'\left(\sigma_i Y\right)Y \qquad (12)$$

In general, this expression cannot be rearranged to express the share as an explicit function of *Y*. However, it is still possible to infer the important properties of the implied function. First, suppose that there is a pair of values (σ_i, *Y*), with $\sigma_i > 0$, that satisfy (12). Then an increase in *Y* implies an increase in the right hand side of (12). Given that $\beta(\alpha_i) > \alpha_i$, σ_i must fall in order to increase the value of the left hand side and reduce that of the right hand side, thereby maintaining the required equality. Hence, for all *Y* such that $0 < Y < [\beta(\alpha_i)g'(0)]^{-1}$, σ_i is a decreasing function of *Y*. Second, as *Y* tends towards zero, so too does the right hand side of (12), implying that $\lim_{Y\to 0} \sigma_i = 1$. It is important to identify more precisely how σ_i behaves as *Y* varies. Two special cases help to reveal significant features.

Special case 1 - Linear technology: If the technology is linear, player *i*'s share can be written as an explicit function of *Y*. Let $g_i'(y) = 1$ everywhere. Rearrangement of (12) yields

$$s_i(Y) = \max\left\{\frac{1-\beta(\alpha_i)Y}{1-\alpha_i Y},\ \ 0\right\}.$$

Figure 4 shows the graphs of share functions corresponding to various assumed values of α_i. A higher value of α_i implies a lower threshold value of Y at which the player's share falls to zero, thereafter remaining zero as Y increases further.

Special case 2 - risk neutrality: Another revealing special arises when we allow for a nonlinear technology but suppose that the individual is risk neutral. In this case, interior solutions are characterized by the equality

$$1-\sigma = \frac{Yg_i'(\sigma Y)}{R}$$

Clearly, the precise shape of the graph of the share function depends upon the form of the total cost function. More specifically, we can show that if $g_i'(0) = 0$, then the player's share approaches zero as Y grows without ever actually reaching zero. This is a consequence of the observation that the left hand side of (12) is bounded above [by $\beta(\alpha_i)^{-1}$]. Hence, as $Y \to \infty$, for (12) to be satisfied we must have $\sigma_i Y \to 0$ and, a fortiori, $\sigma_i \to 0$. However, if $g_i'(0)$ is positive, then there is a finite value Y^o such that $\sigma_i = 0$ for all $Y \geq Y^o$. Figure 4 shows the share functions corresponding to two technologies, one of which implies a zero marginal cost, the other a strictly positive marginal cost, at $y = 0$.

It may help to summarize the salient properties of the share function:

Proposition 4.2: Player i's share function, $s_i(Y)$, has the following properties:

(i) $s_i(Y) \to 1$ as $Y \to 0$,

(ii) $s_i(Y)$ is continuous,

(iii) For all values of $Y < [\beta(\alpha_i)g_i'(0)]^{-1}$, $s_i(Y) > 0$ and $s_i(Y)$ is strictly decreasing,

(iv) For all values of $Y \geq [\beta(\alpha_i)g_i'(0)]^{-1}$, $s_i(Y) = 0$,

(v) If $g_i'(0) = 0$, $s_i(Y) \to 0$ as $Y \to \infty$,

Recall that a Nash equilibrium is an allocation at which the sum of share values over all players equals one. The properties of the aggregate share function that are implied by those of the players' share functions imply that, whether $g_i'(0)$ is zero or strictly positive, a unique Nash equilibrium exists:

Proposition 4.3: If **A.1** and **A.2** hold, the *n*-person rent-seeking game has a unique Nash equilibrium.

5. COMPARATIVE STATICS AND RENT DISSIPATION

The properties established in the last section allow us to address a number of comparative static issues. A particular concern of students of rent-seeking contests has been the issue of rent dissipation - what proportion of the exogenous rent, R, is used up in the noncooperative process of competition for that rent? This, too, is our ultimate concern. However, we begin by considering the consequences for an individual of an exogenous change in the aggregate value of Y. One may interpret this as the consequence of a change in the technology or preferences of other players, or of a change in the number of players who are contesting the rent.

Proposition 1 tells us that $s_i(Y)$ is everywhere nonincreasing, and is strictly decreasing wherever $s_i(Y) > 0$. Therefore, we can infer immediately that, if $Y^1 > Y^0$, then either $s_i(Y^0) > s_i(Y^1) \geq 0$ or $s_i(Y^0) = s_i(Y^1) = 0$. The fact that the share σ_i [$= y_i/Y$] is nonincreasing in Y, of course, provides no decisive information about the change in y_i. Indeed, this may move in either direction, even in the very simplest situation involving linear technology and a risk-neutral player. Therefore, the response of the effort, x_i, cannot generally be signed. However, in spite of this, we can unambiguously sign the response of i's payoff to an exogenous change in Y. First, note that the value of player i's payoff can be thought of as a function of $s_i(Y)$ and Y. Denote this function by $\nu_i(s_i(Y), Y)$. Then the effect on player i's payoff of a change in Y can be signed:

Proposition 5.1: Assume **A.1** and **A.2**. Then, if $Y^1 > Y^0$,

$\nu_i[s_i(Y^0), Y^0] \geq \nu_i[s_i(Y^1), Y^1]$, with strict inequality if $s_i(Y^0) > 0$.

Proof: If $s_i(Y^0) = 0$, then $s_i(Y^1) = 0$ by Proposition 4.2.(iii) and, trivially, the result holds with both sides being equal to zero.

If $s_i(Y^0) > 0$, then by Proposition 4.2.(iii),

$1 - s_i(Y^1) > 1 - s_i(Y^0)$

$$\therefore Y^1[1 - s_i(Y^1)] > Y^0[1 - s_i(Y^0)]$$

$$\nu_i\left[s_i\left(Y^0\right), Y^0\right] = \max_{y_i}\left\{\frac{y_i}{Y^0 - Y^0 s_i\left(Y^0\right) + y_i}\left[u\left(I_i + R - g_i\left(y_i\right)\right) - u\left(I_i - g_i\left(y_i\right)\right)\right] + u\left(I_i - g_i\left(y_i\right)\right)\right\}$$

$$> \max_{y_i}\left\{\frac{y_i}{Y^1 - Y^1 s_i\left(Y^1\right) + y_i}\left[u\left(I_i + R - g_i\left(y_i\right)\right) - u\left(I_i - g_i\left(y_i\right)\right)\right] + u\left(I_i - g_i\left(y_i\right)\right)\right\}$$

$$= \nu_i\left[s_i\left(Y^1\right), Y^1\right].$$

[Note that, by assumption, the first maximizer is never zero].

This result can be exploited to explore the consequences of admitting more players to a rent-seeking contest, since we know from the properties of share functions that adding more players to such a contest either increases or leaves unchanged the equilibrium level of Y. Moreover, a change in Y is the only mechanism that can affect the payoff of an individual player with given technology and attitude towards risk.

Proposition 5.1, together with the properties that we have established of the individual share functions, lead directly to the following comparative static properties:

Proposition 5.2: Assume **A.1** and **A.2**. If additional players are admitted to a rent-seeking game and at least one new player actively participates, then

(i) None of the original players switches from nonparticipation to active participation,

(ii) The probability of an original active participant winning the rent falls,

(iii) All original active participants are made worse off.

If, in addition, all the players are identical, or else they differ in their attitudes towards risk but have access to the same linear technology, then total expenditure on rent-seeking increases.

We now consider the issue of rent dissipation. In particular, we are interested in what happens to the proportion of rent dissipated when the number of players becomes large. Because of the simplicity with which the aggregate share function can be obtained, and the Nash equilibrium thereby identified, this is a relatively simple question to address once the individual share functions are known.

We have already pointed out that whether the share value merely approaches zero as Y grows or actually becomes zero at a finite value of Y depends critically on the precise nature of the technology. This turns out to be crucial for determining the extent of rent dissipation as the number of players grows large. Accordingly, we distinguish between two cases, and introduce the following definition:

Definition: The technology is **linearizable** if $f'(x_i)$ is bounded above for all $x_i > 0$. Otherwise, it is nonlinearizable.

When assumption A.2 is in force, $f_i'(x_i)$ is non-decreasing as x_i falls to zero. Linearizability imposes a finite upper bound on the marginal product, so it has a limit, which we write $f_i'(0)$ as x_i falls to zero. Nonlinearizability means that the marginal product becomes arbitrarily large as we approach the origin: we write $f_i'(0) = \infty$. The idea behind our terminology is simple. It would not be sensible to approximate a technology by a linearization that embodies an infinitely large marginal product. Such an approximation would effectively deny the economic fact of scarcity. It may seem natural to assume that technologies are linearizable. However, nonlinearizable technologies are commonly encountered in economic modeling. In growth theory, for example, fulfillment of the Inada conditions implies nonlinearizability at the origin - see Burmeister and Dobell (1970, p. 35) or Barro and Sala-i-Martin (1995, p. 16). Consider the following family of production functions: $y_i = \left(x_i + k^{1/r}\right)^r - k$, $k \geq 0$, $0 < r < 1$. In Tullock (1980), $k = 0$ by assumption. This implies that $f(x_i)$ is nonlinearizable, since $f(x_i) = x_i^r \rightarrow f'(0) = \infty$ for $0 < r < 1$. However, if k is strictly positive, then the marginal product is finite for all values of x_i, including $x_i = 0$.

We denote by $G^{(n)}$ the game played by n players. We will consider a sequence of rent-seeking contests $\left\{G^{(n)}\right\}_{n=2}^{\infty}$ constructed as follows. There is a sequence of non-negative numbers $\{\alpha_i\}_{i=1}^{\infty}$. The n players in $G^{(n)}$ exhibit constant absolute risk aversion and have coefficients of risk aversion α_1, α_2, …, α_n. Each has access to the same technology $f(\cdot)$, which satisfies **A.2**. We have already established that $G^{(n)}$ has a unique equilibrium. We denote the equilibrium vectors of the effort variables in $G^{(n)}$ by $\mathbf{x}^{(n)}$ and those of the intermediate inputs by $\mathbf{y}^{(n)}$. The aggregates are denoted by $X^{(n)} = \sum_j x_j^{(n)}$ and $Y^{(n)} = \sum_j y_j^{(n)}$. We now consider the limits of these sequences of equilibria as n becomes very large. In particular, we examine what happens to the proportion of rent dissipated, X/R, as n becomes large.

(a) Rent-seeking with a linearizable technology and identical players

Throughout this subsection we assume that $f'(0) < \infty$; or, equivalently, $g'(0) > 0$. For the moment, suppose that all players are identical: $\alpha_i = \alpha$ for all i. We know that $\mathbf{y}^{(n)}$ satisfies $\sum_{j=1}^{n} s_j\left(Y^{(n)}\right) = 1$, and symmetry implies that $s_j\left(Y^{(n)}\right) = 1/n$ for every player. Therefore,

$$\frac{n-1}{n\beta(\alpha)-\alpha} = Y^{(n)} g'\left(\frac{Y^{(n)}}{n}\right) \tag{13}$$

Our earlier results imply that $\left\{Y^{(n)}\right\}_{n=2}^{\infty}$ is a non-decreasing sequence, and that $s_i(Y)$ vanishes for $Y \geq [\beta(\alpha)g'(0)]^{-1}$. Hence the sequence is bounded and has a limit $\hat{Y}$. Taking the limit in (13) shows that $\hat{Y} = [\beta(\alpha)g'(0)]^{-1}$. Now recall that $x_i^{(n)} = g\left(y_i^{(n)}\right)$. Then

$$X^{(n)} = \sum_{i=1}^{n} x_i^{(n)} = ng\left(Y^{(n)}/n\right) \to \hat{Y}g'(0) = \left[\beta(\alpha)\right]^{-1} = \rho(\alpha)R \text{ as } n \to \infty,$$

where

$$\rho(\alpha) \equiv \frac{1-e^{-\alpha R}}{\alpha R} < 1 \text{ for } \alpha > 0, \text{ and } \rho(0) \equiv 1.$$

Clearly, if players are risk neutral the rent is fully dissipated. However, if they are all strictly risk averse, then even in a very large game the factor ρ sets an upper limit to the proportion of rent dissipated. Furthermore, since ρ decreases with α and approaches 0 for large α, the proportion of rent that is dissipated in equilibrium falls (to zero) as the identical players become increasingly risk averse. This is summarized in the following proposition:

Proposition 5.3: If each of *n* players is risk-neutral or strictly risk averse with a common non-negative value of α, and if each has access to the same linearizable technology, then as *n* tends to infinity the proportion of rent dissipated tends to $\rho(\alpha)$, where $\rho(0) = 1$, ρ decreases with α, and $\lim_{\alpha \to \infty} \rho(\alpha) = 0$.

(b) Linearizable technology and heterogeneous players

Heterogeneity of attitudes to risk significantly modifies these conclusions. Consider first an example with two types of player: $\alpha_i = \underline{\alpha}$ or $\overline{\alpha}$ (> $\underline{\alpha}$). If there are enough players with lower risk aversion, then the other type will not participate in the contest. All players will therefore be of type $\underline{\alpha}$. This observation can clearly be generalized to many types. The intuition remains: there is a finite integer, $\underline{n}$, such that if the number of players of the lowest risk averse type exceeds $\underline{n}$, all other types will not wish to contest the rent. The contest then involves only the $\underline{n}$ players with low risk aversion. This is essentially the same result as Proposition 3.3.

This observation prompts consideration of the generalization to the case where all α_i may differ. We might expect that the effective "population" value of the coefficient of risk aversion is the greatest lower bound of the individual values. With a qualification, this expectation proves justified.

Proposition 5.4: If $\underline{\alpha} = \inf_{i=1,\ldots,\infty} \alpha_i$ and $\alpha_i > \underline{\alpha}$ for all *i*, then the proportion of rent dissipated tends to $\rho(\underline{\alpha})$ as $n \to \infty$.

Proof: in the Appendix.

Both requirements in the proposition are necessary. Consider, for example, the case where $\alpha_1 = \alpha_2 = 0$ and, for $i \geq 3$, α_i is chosen so that its share function is zero at the value of Y for which the share function of a risk neutral player is ½. Then only the two risk neutral players participate and (by Proposition 3.3) $X^{(n)}/R \to$ ½. Yet inf $\alpha_1 = 0$, so we might expect full dissipation of the rent. However in this example it is not the case that $\alpha_1 >$ inf α_i, so the proposition is not violated. The problem is that, not only do we require less risk averse players to exclude their more risk averse competitors, but we also need enough players left in the game for the limiting results to hold. The second requirement in the proposition ensures this.

The following corollary shows that we will have full risk dissipation in the limit even if all players are risk averse, provided that the coefficient of risk aversion has no positive lower bound.

Corollary 1: If all players are strictly risk averse and $\inf_{i=1,\ldots,\infty} \alpha_i = 0$, then rent is fully dissipated in the limit.

A natural way to generate the α_i is randomly from some distribution. In this case we can use Proposition 5.4 to prove the following corollary:

Corollary 2: Let F be a distribution function on the non-negative real numbers and let $\underline{\alpha} \geq 0$ be the greatest lower bound of its support. Let $\alpha_1, \alpha_2, \ldots$ be independent draws from this distribution. Then

$$\lim_{n\to\infty} X^{(n)}/R = \rho(\underline{\alpha}) \text{ with probability one.}$$

(b) Rent-seeking with a nonlinearizable technology and identical players

We now consider games in which, as an individual's level of input tends to zero, her marginal productivity becomes infinitely large: $f'(0) = \infty$. To facilitate the presentation, we will focus on the form $f(x) = x^r$, where $0 < r < 1$. Recall that this is precisely the Tullock technology.

First, consider a situation in which all players have the same coefficient of risk aversion, $\alpha > 0$. We know that the typical player's share function is given implicitly by the condition

$$\frac{1-\sigma}{\beta(\alpha)-\alpha\sigma} = Yg'(\sigma Y)$$

Suppose that $f(x) = x^r$, or $g(y) = y^{1/r}$.

Then $g'(y) = [1/r]y^{(1-r)/r}$ and $\frac{1-\sigma}{\beta(\alpha)-\alpha\sigma} = \frac{Y\sigma^{(1-r)/r}Y^{(1-r)/r}}{r}$

Since players are identical, $\sigma = 1/n$, so that

$$\frac{1-\frac{1}{n}}{\frac{1}{\rho(\alpha)R}-\alpha\sigma} = \frac{n^{r/(1-r)}Y^{1/r}}{r} \qquad (14)$$

$$\therefore\ X = nx = ny^{1/r} = (ny)^{1/r}.n^{(r-1)/r} = Y^{1/r}.n^{(r-1)/r}$$

Therefore, using (14) ,

$$\frac{X}{R} = \frac{r\rho(\alpha)\left(1-\frac{1}{n}\right)}{1-\frac{\alpha\rho(\alpha)R}{n}}.$$

Thus we have the following proposition:

Proposition 5.5: Let all players have the same coefficient of risk aversion α, and also access to the same nonlinearizable technology: $y = x^r$, where $0 < r < 1$. Then, in the limit as n grows large, the proportion of rent that is dissipated is $X/R = r\rho(\alpha)$.

A similar, but somewhat simpler, analysis leads to the following proposition in the case where the identical players are all risk neutral:

Proposition 5.6: If all players are risk neutral and have access to the same nonlinearizable technology, $y = x^r$, where $0 < r < 1$, then in the limit as n grows large, the proportion of rent that is dissipated is $X/R = r$.

Even though all players are risk averse, **Proposition 5.6** shows that $r < 0$ sets an upper limit to the proportion of rent that is dissipated, even in the limit. **Proposition 5.5** shows that if all players are identical and strictly risk averse, then the limiting equilibrium aggregate expenditure is reduced still further. Note that the limiting aggregate expenditure on rent seeking is reduced by exactly the same factor as when $f'(0) < \infty$. This might suggest that **Proposition 5.4** changes in a similar way. However the next section shows that this is not the case.

(d) Nonlinearizable technology and heterogeneous players.

We now consider the implications of heterogeneity. For simplicity, assume that there are just two types of player: $\alpha_i = \overline{\alpha}_1$ if i is odd, and $\alpha_i = \overline{\alpha}_2$ if i is even. The game $G^{(2n)}$ has $2n$ players, n of each type. Then the following proposition can be established:

Proposition 5.7: Let there be two types of player: type k players have risk aversion coefficients $\overline{\alpha}_k$ for k = 1, 2, and a common nonlinearizable technology described by $y = x^r$. Suppose odd (even) numbered are of type 1 (2). Then $\lim_{n\to\infty} X^{(2n)} / R = \rho(\tilde{\alpha})r$, where

$$\rho(\tilde{\alpha}) = \frac{\left(1 - e^{-\tilde{\alpha}R}\right)}{\tilde{\alpha}R}$$

and $$\tilde{\alpha} = \beta^{-1}\left\{\frac{\left[\beta(\overline{\alpha}_1)\right]^{r/(r-1)} + \left[\beta(\overline{\alpha}_2)\right]^{r/(r-1)}}{\left[\beta(\overline{\alpha}_1)\right]^{1/(r-1)} + \left[\beta(\overline{\alpha}_2)\right]^{1/(r-1)}}\right\}, \qquad (15)$$

Proof: In the appendix.

The limiting aggregate expenditure is identical to that in a sequence of games with identical players, having risk aversion coefficient given by (15) and production function x^r. The proposition shows that, when players differ with respect to their attitudes towards risk, results differ qualitatively from the case in which the marginal product is bounded.

Recall that, when $f'(0) < \infty$, the reduction corresponds to the risk aversion coefficient of the less risk averse type of player. Here, even if one of the types is risk neutral and the other strictly risk averse, the limiting equilibrium expenditure is less than when all players are risk neutral. This is a consequence of the fact that all individuals remain positive contestants, however high the equilibrium value of aggregate *Y*.

A natural extension suggests itself. Even when the coefficient of risk aversion is distributed in the population according to the distribution function *F* satisfying $F(\alpha) > 0$ for all $\alpha > 0$, less of the limiting equilibrium rent is dissipated than when all players are risk neutral. The following proposition summarizes this observation:

Proposition 5.8: Let the distribution of attitudes to risk be described by the function $F(\alpha)$, where $F(\cdot) > 0$ for all $\alpha > 0$, and let there be a common nonlinearizable technology described by $y = x^r$. Suppose there are *n* players of each type. Then $\lim_{n\to\infty} X^{(n)}/R = \rho(\tilde{\alpha})r$,

where

$$\rho(\tilde{\alpha}) = \frac{\left(1 - e^{-\tilde{\alpha}R}\right)}{\tilde{\alpha}R}$$

and

$$\tilde{\alpha} = \beta^{-1}\left(\frac{\int[\beta(\alpha)]^{r/(r-1)}dF(\alpha)}{\int[\beta(\alpha)]^{1/(r-1)}dF(\alpha)}\right).$$

Clearly $\tilde{\alpha} > 0$ unless *F* is entirely concentrated on $\alpha = 0$. In contrast to the linearizable case, in which the more risk averse types make zero effort, the asymptotic property of the share functions implies that all players apply positive effort, and their presence in the game influences the outcome and reduces the proportion of rent that is dissipated by comparison with games in which only those with low risk aversion participate. However, this result is not completely disconnected from the linearizable case, for it can be shown that, as $r \to 1$, then $\beta(\tilde{\alpha}) \to \beta(\underline{\alpha})$ where $\underline{\alpha}$ is the greatest lower bound of the support of *F*. Hence $\tilde{\alpha} \to \underline{\alpha}$ as $r \to 1$.

6. CONCLUSION

Use of the share function to model an individual's behavior facilitates the analysis of rent-seeking games in which, by increasing the effort devoted to the contest, each player can increase her probability of winning an exogenous rent. We have exploited the simplicity of this approach to provide a thorough analysis of such games when all players are risk averse with constant coefficients of risk aversion. The approach handles heterogeneous players easily, and we are able to allow attitudes to risk to differ across players. In particular, we are able to reach strong conclusions concerning the proportion of rent that is dissipated in the limit as the number of heterogeneous players becomes large. Our analysis enables us to make the observation that the limiting proportion of rent that is dissipated depends not only on the risk attitudes of players but also on details of the technology available for converting effort into increments of probability of success - details whose significance has not hitherto been fully recognized in existing literature.

FOOTNOTES

1. Cornes and Hartley (2000) defines the share function approach and applies it to a model of open access resource exploitation.
2. Individual share functions are not well-defined for $Y = 0$. However, this is not a problem, since in this model the Nash equilibrium will necessarily involve a strictly positive value of Y. This observation remains true in our more general formulation in Section 4.

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APPENDIX

I. Proof of Proposition 4.1

The payoff function of a strictly risk averse player with general technology is

$$\pi\left(y,\tilde{Y}\right)=\frac{y}{y+\tilde{Y}}\left\{-e^{-\alpha\left[I+R-g(y)\right]}\right\}+\frac{\tilde{Y}}{y+\tilde{Y}}\left\{-e^{-\alpha\left[I-g(y)\right]}\right\}$$
$$=-\left\{\frac{ye^{\alpha\left[g(y)-I-R\right]}}{y+\tilde{Y}}+\frac{\tilde{Y}e^{\alpha\left[g(y)-I\right]}}{y+\tilde{Y}}\right\}. \qquad \text{(A1)}$$

A necessary condition for $\pi(\cdot)$ to be maximized by a positive value of y is that its partial derivative with respect to y be zero:

$$\frac{\partial\pi}{\partial y}=-\frac{Y\left\{y\alpha g'(y)e^{-\alpha R}+e^{-\alpha R}\right\}-ye^{-\alpha R}+Y\tilde{Y}\alpha g'(y)-\tilde{Y}}{Y^2}=0 \qquad \text{(A2)}$$

For $Y > 0$, a zero value of $\partial\pi(\cdot)/\partial y$ is equivalent to a zero value of the numerator. Writing σ for $y/\left(y+\tilde{Y}\right)$, this can be rearranged as

$$\frac{(1-\sigma)}{\dfrac{\alpha}{\left(1-e^{-\alpha R}\right)}-\alpha\sigma}-g'(\sigma Y)Y=0\cdot \qquad \text{(A3)}$$

Partial differentiation of (A2) again with respect to y holding $\tilde{Y}$ fixed, after some tedious manipulation, reveals that, at any point where $\frac{\partial\pi}{\partial y}=0$, it must be true that $\frac{\partial^2\pi}{\partial y^2}<0$. This implies that for any given $Y > 0$ there can only be at most one stationary point, and that such a point indeed characterizes a global maximum of $\pi(\cdot)$ with respect to y.

This implies that there can be at most one positive stationary point and that, if such a point exists, it is the global maximum of $\pi(\cdot)$ with respect to y. Otherwise, the global maximum is at $y = 0$ if and only if $\frac{\partial\pi}{\partial y}\leq 0$ at y = 0. This can be rewritten as $Yg'(0) \geq [\beta(\alpha)]^{-1}$. For this to hold, we need $g'(0) > 0$, which is equivalent to $f'(0) < \infty$, and as $Yg'(0) \geq [\beta(\alpha)]^{-1}$. Finally, observe that when $\sigma = 1$, the left hand side of (A3) is negative. If $[\beta(\alpha)]^{-1} > Yg'(0)$, the right hand side of (A3) is positive at $\sigma = 0$ and, by continuity, (A3) holds for some $\sigma \in (0, 1)$. If $[\beta(\alpha)]^{-1} \leq Yg'(0)$, then $y = 0$ is optimal as we have shown. This completes the proof of Proposition 4.1.

The argument to this point establishes that the equation (A3) uniquely characterizes an interior solution to the player's maximization problem for $Y > 0$. It is possible that, for a given positive value of Y, there is no non-negative value of y that satisfies (A3). Consider the left hand side of (A3). This takes the value - $Yg'(Y) < 0$ when $\sigma = 1$. Furthermore, since $\alpha\big/\left(1 - e^{-\alpha R}\right) > \alpha$ and $g'(\cdot) > 0$, it is a decreasing function of σ. Therefore (A3) can only be satisfied by a positive share if the left hand side is positive when $\sigma = 0$ - that is, if $1/\beta(\alpha) - Yg'(0) > 0$.

In summary, this shows that a share function exists. It takes the value 0 if and only if $f'(0) < \infty$ and $Y \geq [\beta(\alpha)g'(0)]^{-1}$. Otherwise, it is uniquely determined by (A3).

II. Proof of Proposition 5.4:

Denote the share function of a player with constant coefficient of risk aversion $\alpha \geq 0$ by $s(Y; \alpha)$. The first step is to prove that $s(\cdot)$ is continuous in α. (Continuity in Y was established in Proposition **4.2**). We have

$$\frac{1 - s(Y;\alpha)}{\beta(\alpha) - \alpha s(Y;\alpha)} = Yg'\left[s(Y;\alpha)Y\right]$$

when $Y < [\beta(\alpha)g'(0)]^{-1}$ and $s(Y; \alpha) = 0$ when $Y \geq [\beta(\alpha)g'(0)]^{-1}$. Suppose we had $\alpha^{(n)} \to \alpha^0$ as $n \to \infty$ but not $s(Y; \alpha^{(n)}) \to s(Y; \alpha^0)$. Then we could find $\delta > 0$ such that $s(Y; \alpha^{(n)}) \in S^0$ where $S^0 = \{\sigma : 0 \leq \sigma \leq 1,\ \sigma - s(Y; \alpha^0) \geq \delta\}$ for infinitely many n. Since S^0 is closed and bounded, this implies the existence of a subsequence of $\{s(Y; \alpha^{(n)})\}$ convergent to σ^0, say. Taking limits on this subsequence and using the continuity of β and g' we deduce that $\sigma^0 = 0$ if $Y \geq [\beta(\alpha^0)g'(0)]^{-1}$ and

$$\frac{1 - \sigma^0}{\beta\left(\alpha^0\right) - \alpha^0\sigma^0} = Yg'\left(\sigma^0 Y\right) \text{ otherwise.}$$

By our previous results, $\sigma^0 = s(Y; \alpha^0)$, which means that $s(Y; \alpha^{(n)}) \in S^0$ for some $s(Y; \alpha^{(n)})$ in the subsequence. This contradiction proves continuity.

We now establish that the limit of $\{Y^{(n)}\}$ is $\rho(\underline{\alpha})R/g'(0)$, by first proving that this is an upper bound on the sequence and then showing that the bound is approached arbitrarily closely.

Since $\beta(\alpha_i) > \beta(\underline{\alpha})$ for all i and $s(Y;\alpha) = 0$ if and only if $Y \geq [\beta(\alpha)g'(0)]^{-1}$, we deduce that $s\{[\beta(\underline{\alpha})g'(0)]^{-1}; \alpha_i\} = 0$ for all i, which means that $\sum_{i=1}^{n} s_i\left\{\left[\beta(\underline{\alpha})g'(0)\right]^{-1};\alpha_i\right\} = 0$ for all n. Hence, $Y^{(n)} < [\beta(\underline{\alpha})g'(0)]^{-1} = \rho(\underline{\alpha})R/g'(0)$. We next show that, if $Y < \rho(\underline{\alpha})R/g'(0)$, then there is an n for which $Y^{(n)} > Y$. We can prove this by exhibiting an n such that $\sum_{i=1}^{n} s_i(Y) > 1$. So, choose an $\overline{\alpha} > \underline{\alpha}$ such that $\beta(\overline{\alpha})g'(0)Y = 1$. This is always possible as $\beta(\alpha)$ is continuous, increasing and unbounded above. For $\underline{\alpha} \leq \alpha \leq (\underline{\alpha}+\overline{\alpha})/2$, we have $\beta(\alpha) \leq \beta[(\underline{\alpha}+\overline{\alpha})/2] < \beta(\overline{\alpha})$ so that $[\beta(\alpha)g'(0)]^{-1} > [\beta(\overline{\alpha})g'(0)]^{-1} = Y$. Hence, $s(Y;\alpha) > 0$ for $\underline{\alpha} \leq \alpha \leq (\underline{\alpha}+\overline{\alpha})/2$ and the continuity of $s(Y;\cdot)$ means that there is an $\underline{s} > 0$ such that $s(Y;\alpha) \geq \underline{s}$ for all $\underline{\alpha} \leq \alpha \leq (\underline{\alpha}+\overline{\alpha})/2$. The assumption in the proposition allows us to choose an n for which $\underline{\alpha} \leq \alpha_i \leq (\underline{\alpha}+\overline{\alpha})/2$ for more than $\underline{s}^{-1}$ values of i where $\underline{s}^{-1} \leq n$. Hence, $\sum_{i=1}^{n} s_i(Y) > 1$ as required and we may conclude that $\lim_{n\to\infty} Y^{(n)} = \rho(\underline{\alpha})R/g'(0)$.

Next we show that limiting shares fall uniformly to zero with n. For $Y > 0$, define $s^M(Y) = \sup_{i=1,\ldots,\infty} s(Y;\alpha_i)$. Note that, for $n = 1, 2, \ldots$, there is an $\alpha^{(n)}$ such that $\beta(\alpha^{(n)}) Y^{(n)}g'(0) = 1$. Since the inverse of β is continuous and $Y^{(n)} \to [\beta(\underline{\alpha})g'(0)]^{-1}$ as $n \to \infty$, we deduce that $\alpha^{(n)} \to \underline{\alpha}$ as $n \to \infty$. The definition of $\alpha^{(n)}$ means that $s(Y^{(n)};\alpha) = 0$ for $\alpha > \alpha^{(n)}$, so that $s^M(Y^{(n)}) \leq \sup_{\underline{\alpha}\leq\alpha\leq\alpha^{(n)}} s\left(Y^{(n)};\alpha\right)$. Now A.2 implies that $g'[Ys(Y;\alpha)] \geq g'(0)$ and we can rearrange the expression displayed above to give

$$s(Y;\alpha) \leq \frac{1-\beta(\alpha)Yg'(0)}{1-\alpha Yg'(0)}$$

whence $\alpha Yg'(0) \leq 1$. Hence

Hence $s^M\left(Y^{(n)}\right) \leq \sup_{\underline{\alpha} \leq \alpha \leq \alpha^{(n)}} \frac{1-\beta(\alpha)Y^{(n)}g'(0)}{1-\alpha Y^{(n)}g'(0)}$

$$\leq \frac{1-\beta(\underline{\alpha})Y^{(n)}g'(0)}{1-\alpha^{(n)}Y^{(n)}g'(0)}$$

$$\to 0 \text{ as } n \to \infty.$$

We have used the fact that $\alpha^{(n)}Y^{(n)}g'(0) < 1$ and $\underline{\alpha}/\beta(\underline{\alpha}) < 1$.

The proof is concluded by translating these results into the untransformed variables. Since $g(\cdot)$ is convex and $g(0) = 0$,

$$y_i^n g'(0) \leq g\left(y_i^{(n)}\right) = g\left[s_i\left(Y^{(n)}\right)Y^{(n)}\right] \leq s_i\left(Y^{(n)}\right)\frac{g\left[s^M\left(Y^{(n)}\right)Y^{(n)}\right]}{s^M\left(Y^{(n)}\right)}$$

for $n = 1, \ldots, \infty$ and $i = 1, \ldots, n$. Summing over i and using $x_i^{(n)} = g\left(y_i^{(n)}\right)$ gives

$$g'(0)Y^{(n)} \leq X^{(n)} \leq Y^{(n)}\frac{g\left[s^M\left(Y^{(n)}\right)Y^{(n)}\right]}{s^M\left(Y^{(n)}\right)Y^{(n)}}.$$

Taking the limit and using the fact that $g(y)/y \to g'(0)$ as $y \to 0$, we obtain the result that $\lim_{n\to\infty}X^{(n)} = g' \lim_{n\to\infty}Y^{(n)} = R$ as required.

IV: Proof of **Proposition 5.7**:

The share function of type k satisfies

$$\frac{1-\tilde{s}\left(Y;\bar{\alpha}_k\right)}{\beta\left(\bar{\alpha}_k\right)-\bar{\alpha}_k\tilde{s}\left(Y;\bar{\alpha}_k\right)} = \frac{1}{\alpha}[Y]^{1/\alpha}\left[\tilde{s}\left(Y;\bar{\alpha}_k\right)\right]^{(1-\alpha)/\alpha} \text{ for } k = 1, 2. \tag{A4}$$

For a Nash equilibrium value $Y^{(2n)}$, the sum of all players' share values equals unity:

$$ns\left(Y^{(2n)};\bar{\alpha}_1\right) + ns\left(Y^{(2n)};\bar{\alpha}_2\right) = 1 \tag{A5}$$

Since $0 \leq s\left(Y^{(2n)};\bar{\alpha}_k\right) \leq 1$ for $k = 1, 2$, we can deduce that $s\left(Y^{(2n)};\bar{\alpha}_k\right) \to 0$ as $n \to \infty$.

Taking limits in (A4) establishes the limit

$$\left[Y^{(2n)}\right]^{1/1-r} s\left(Y^{(2n)};\bar{\alpha}_k\right) \to \left[\frac{r}{\beta(\bar{\alpha}_k)}\right]^{r/(1-r)} \quad \text{as } n \to \infty \text{ for } k = 1, 2. \qquad \text{(A6)}$$

Multiplying both sides of (A5) by $[Y^{(2n)}]^{1/1-r}$ and letting $n \to \infty$ gives

$$n^{-1}\left[Y^{(2n)}\right]^{1/(1-r)} \to \left[\frac{r}{\beta(\bar{\alpha}_1)}\right]^{r/(1-r)} + \left[\frac{r}{\beta(\bar{\alpha}_2)}\right]^{r/(1-r)},$$

and so

$$ns\left(Y^{(2n)};\bar{\alpha}_k\right) \to \frac{\left[\dfrac{r}{\beta(\bar{\alpha}_k)}\right]^{r/(1-r)}}{\left[\dfrac{r}{\beta(\bar{\alpha}_1)}\right]^{r/(1-r)} + \left[\dfrac{r}{\beta(\bar{\alpha}_2)}\right]^{r/(1-r)}} \quad \text{for } k = 1, 2.$$

Hence,

$$X^{(2n)} = n\left[s\left(Y^{(2n)};\bar{\alpha}_1\right)Y^{(2n)}\right]^{1/r} + n\left[s\left(Y^{(2n)};\bar{\alpha}_2\right)Y^{(2n)}\right]^{1/r} \to \frac{r}{\beta(\tilde{\alpha})} = r\rho(\tilde{\alpha})R$$

where

$$\tilde{\alpha} = \beta^{-1}\left\{\frac{\left[\beta(\bar{\alpha}_1)\right]^{r/(r-1)} + \left[\beta(\bar{\alpha}_2)\right]^{r/(r-1)}}{\left[\beta(\bar{\alpha}_1)\right]^{1/(r-1)} + \left[\beta(\bar{\alpha}_2)\right]^{1/(r-1)}}\right\}. \qquad \text{(A7)}$$

With this relationship established, the rest of the proposition follows in a straightforward manner.

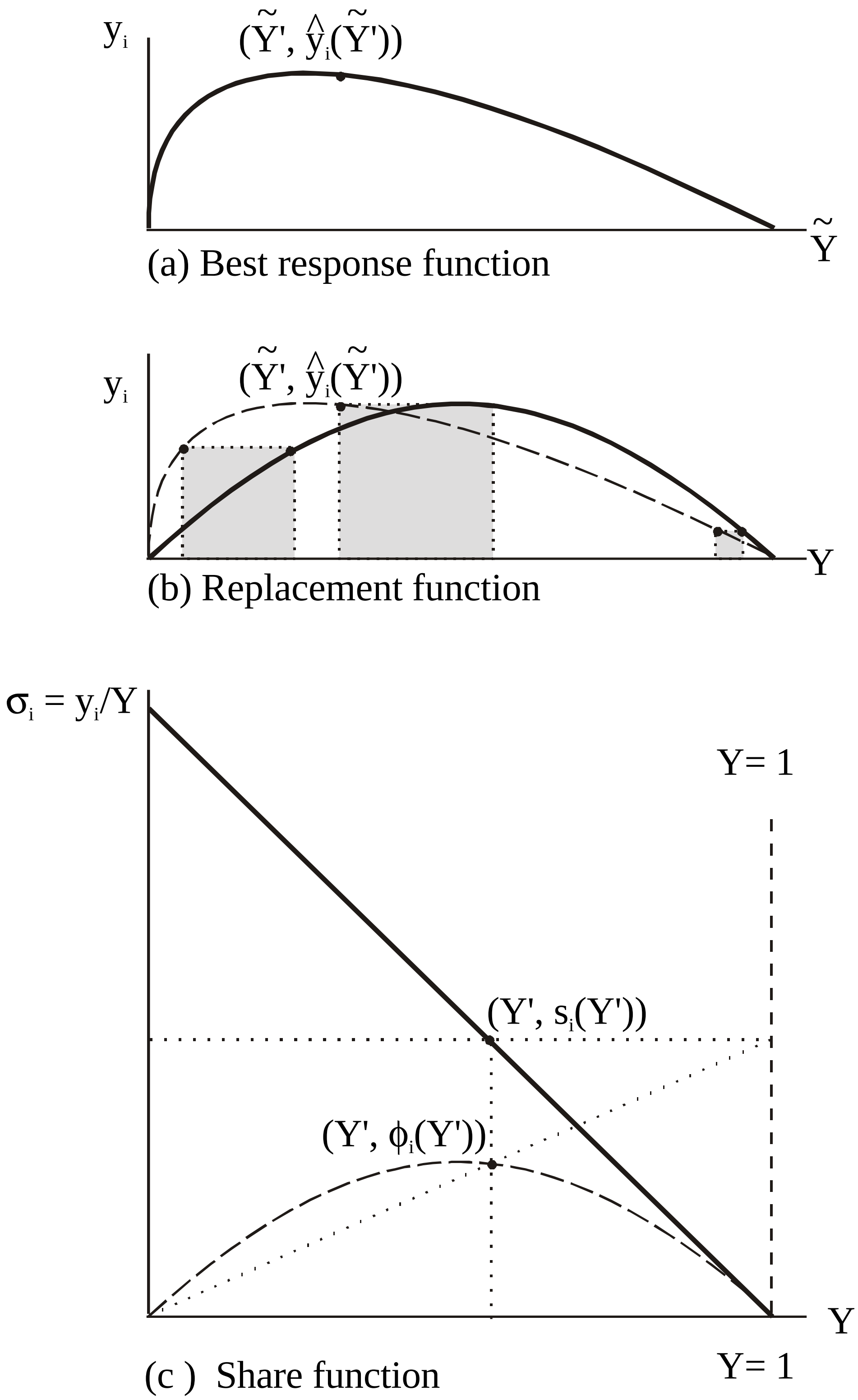

Figure 1 : Best response, replacement and share functions
in a simple rent-seeking model

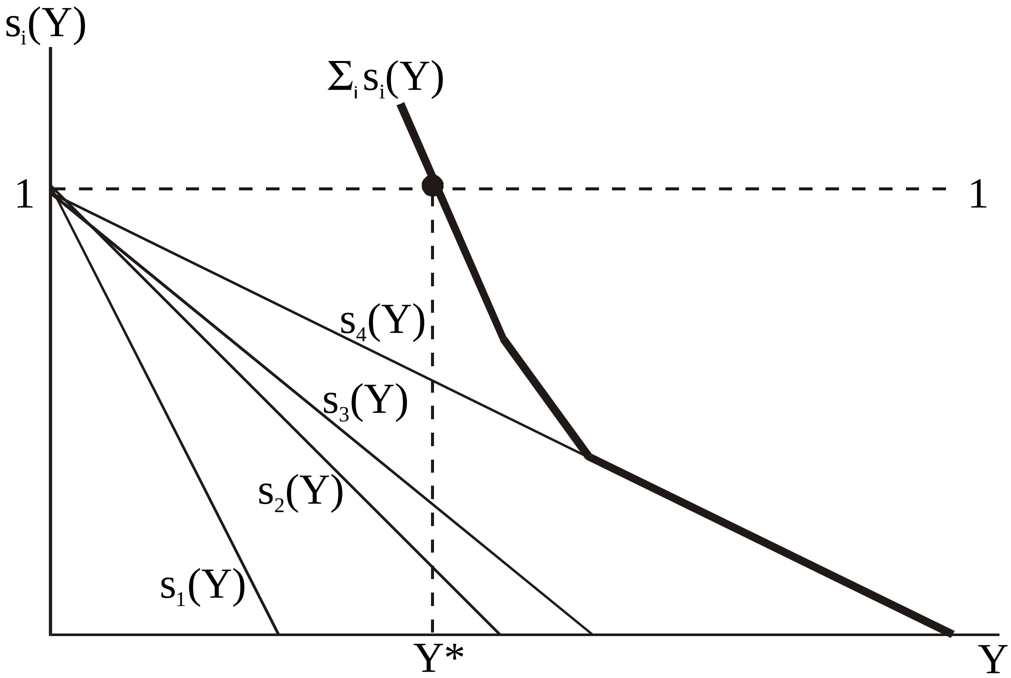

Figure 2: Rentseeking with four risk-neutral players and linear technologies

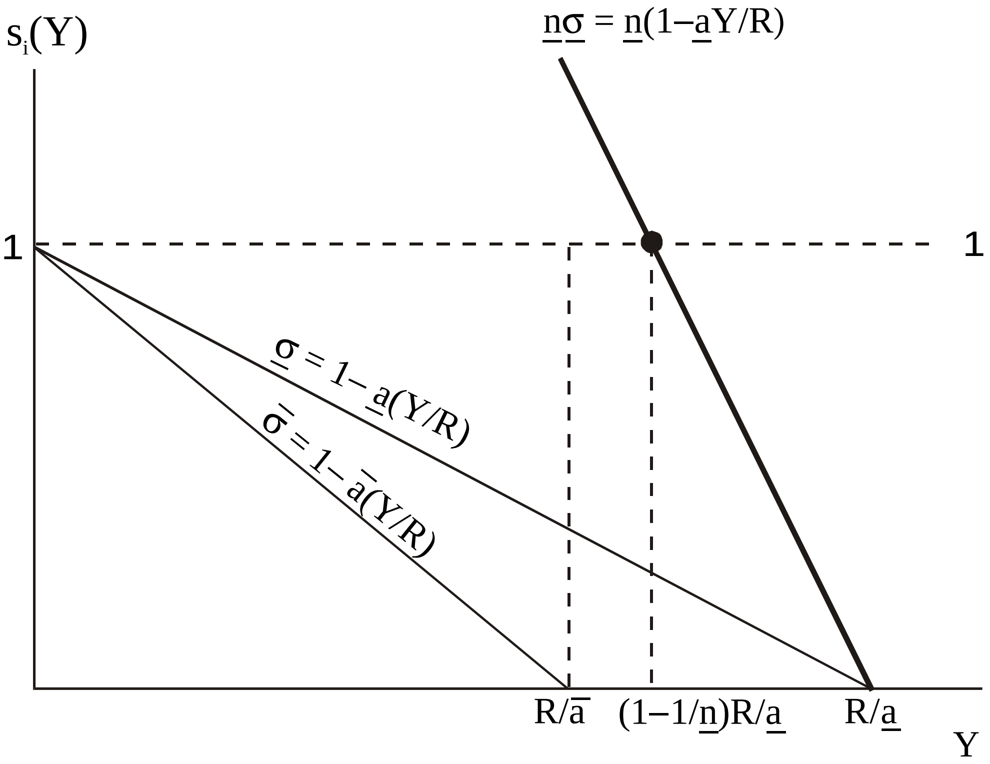

Figure 3: Rent-seeking equilibrium with two types of risk-neutral players and linear technologies

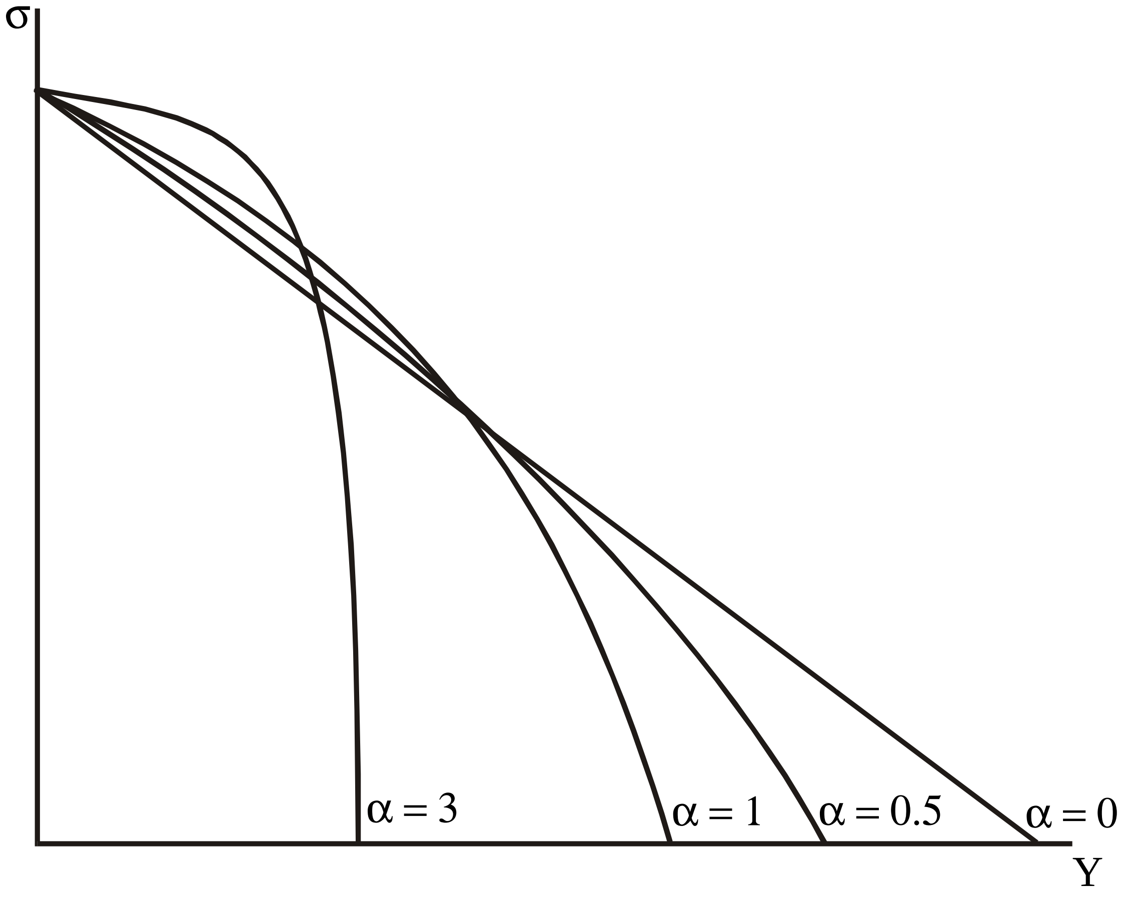

Figure 4: Share functions for various degrees of risk aversion and linear technology.

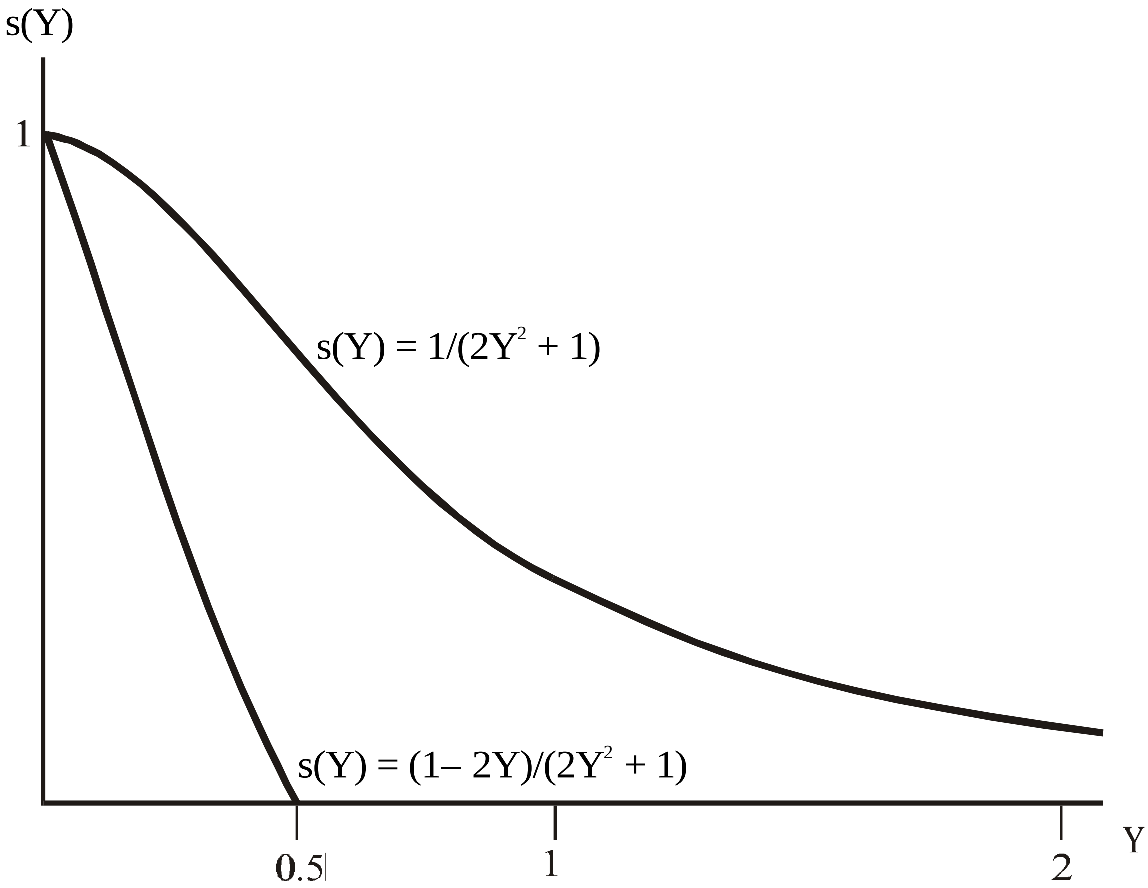

Figure 5: Two share functions for a risk neutral player:
If $g(y) = y^2$, $s(Y) = 1/(2Y^2 + 1)$
If $g(y) = y^2 + 2y$, $s(Y) = (1- 2Y)/(2Y^2 + 1)$