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INVERSE STOCHASTIC DOMINANCE, INEQUALITY MEASUREMENT AND GINI INDICES

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Inverse stochastic dominance, inequality measurement and Gini indices.

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December 14, 2000

Abstract

We investigate the relationship between the third degree inverse stochastic dominance criterion introduced in Muliere and Scarsini (1989) and inequality dominance when Lorenz curves intersect. We propose a new definition of transfer sensitivity aimed at strengthening the Pigou-Dalton Principle of Transfers. Our definition is dual to that suggested by Shorrocks and Foster (1987). It involves a regressive transfer and a progressive transfer both from the same donor and leaving the Gini index unchanged. We prove that finite sequences of these transfers and/or progressive transfers characterize the third degree inverse stochastic dominance criterion. This criterion allows to make unanimous inequality judgements even when Lorenz curves intersect. The Gini coefficient becomes relevant in these cases in order to conclusively rank the distributions.

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1 Introduction

Since Kolm (1969), Atkinson (1970) and Sen (1973) the relevance of the Lorenz dominance criterion for inequality comparisons of income distributions is widely recognized. An income distribution Lorenz dominates another if and only if it shows less dispersion in income shares, or equivalently if shows a reduction in inequality evaluated by all inequality averse and symmetric indices (S-Convex).

There are many cases, however, in applied analysis, where Lorenz curves intersect and therefore this result cannot be applied. Moreover, it is not necessarily the case that income shares alone should be relevant for inequality evaluations, income differentials (as argued in Kolm, 1976) could be relevant, and possibly also combined evaluations of both.

When two Lorenz curves intersect only once, Shorrocks and Foster (1987) show that one still obtains inequality dominance according to relative indices whose ranking is consistent with the third degree stochastic dominance criterion, provided the Lorenz curve of the distribution with the lower variance intersects the other one from the above[1]. This result holds if the inequality evaluation satisfies a Strong version of Kolm's (1976) Principle of Transfer Sensitivity[2]. This result provides an ethical rôle for the third degree stochastic dominance criterion and the use of the variance in inequality comparisons.

However it is possible to approach the problem from a different perspective, instead of looking at standard stochastic dominance conditions, consider the *inverse* stochastic dominance criterion introduced by Muliere and Scarsini (1989). Both dominance criteria are equivalent for the first two degrees of dominance but differ afterwards. As is well known the direct stochastic dominance approach is linked to the concept of welfare dominance according to utilitarian functions while the inverse approach is linked to welfare dominance in terms of the linear rank-dependent Yaari dual social welfare functions (YSWFs).[3]

[1] The result holds for distributions with the same mean and population size. Davies and Hoy (1995) provides a generalization of the result to the case of multiple intersections which considers continuous distributions. See also Atkinson (1973).

[2] The principle requires that inequality reduces as the effect of a variance-preserving combination of a transfer from a rich to a poor person at low income levels and a reverse transfer at higher incomes.

[3] As shown in Zoli (1999) when distributions with the same means are compared, inverse stochastic dominance conditions are equivalent to welfare dominance in terms of YSWFs, provided the weighting function applied to aggregate incomes satisfies appropriate restric-

In this paper we investigate the relationship between the third degree *inverse* stochastic dominance and inequality dominance when Lorenz curves intersect.

The respect of the Principle of Positional Transfer Sensitivity PPTS suggested by Mehran (1976) and Kakwani (1980)[4], has been shown in Zoli (1997, 1999) to be necessary in order to characterize inequality averse YSWFs consistent with the third degree inverse dominance criterion[5].

Following Zoli (1997) we show that the same restrictions on the Yaari weighting function implied by the PPTS could be obtained if we generalize the principle allowing for transfers of different amounts from the same donor to a poorer and a richer recipient, the only requirement being that these transfers have to leave unchanged the Gini coefficient. Exploiting the equivalence relation between welfare dominance for the obtained class of YSWFs and third degree inverse stochastic dominance it is therefore possible to characterize classes of inequality indices consistent with the latter dominance criterion. The main result of the paper is that the indices which are consistent with third degree inverse dominance are S-Convex and satisfy the above mentioned Strong version of PPTS, without being restricted in principle to belong to the linear inequality measures class in Mehran (1976). These results are consistent with those in Chateauneuf and Wilthien (2000) and Aaberge (2000). The intuitive argument providing the basis of the result was available in Zoli (1997), in this paper we provide a formal proof in order to establish it rigorously. Instead of approaching the problem trying to show directly the equivalence between third degree inverse dominance and sequences of transfers, we exploit the equivalence between inverse dominance and welfare dominance in terms of YSWFs and investigate which are the set of transfers which are consistent with welfare dominance. This different perspective allows us to provide an immediate and simple intuition of why the Strong version of PPTS requires to leave unchanged the Gini index. Remember that the Gini index attaches equal weights to the impact

tions. This result applies, under analogous conditions, also to dominance associated to the linear inequality measures in Mehran (1976) which could be considered the ethically based inequality measures associated to the YSWFs.

[4]The PPTS requires a combination of a progressive transfer at low income levels and a regressive transfer at higher income levels, both of the same amount, and occurring between individuals with a given proportion of the population in between them, to be welfare enhancing or inequality reducing.

[5]See also Chateauneuf, Gajdos and Wilthien (1999).

of a transfer, for a given distance in the income ranking between the donor and the recipient, no matter what is their position in the ranking. Therefore any composite transfer which is not affecting the index will be considered inequality reducing by all positional transfer sensitive indices attaching higher weights to transfers occurring at the bottom of the income distribution. It is exactly the dominance for all this measures which is equivalent to third degree inverse dominance.

The paper is organized as follows. In the first section we build up the set up, developing the analysis both for continuous and discrete income distributions. In the central part of the paper we will provide the main result of characterization of third degree inverse stochastic dominance in terms of sequences of transfers for discrete distributions[6]. We also generalize the class of inequality indices consistent with third degree inverse dominance in order to allow for intermediate inequality equivalence (Bossert and Pfingsten, 1990). Following Zoli (1997, 1999), we show that, because of the equivalence between inequality dominance according to all indices belonging to this class and third degree inverse stochastic dominance, when the intermediate Lorenz curves intersect once, dominance for all the indices belonging to this class is ensured simply checking that the distribution with the higher Lorenz curve at the lowest percentiles exhibits a lower "Intermediate Gini coefficient" (the Gini coefficient satisfying intermediate equivalence).

The results could be extended in order to take into account comparisons involving multiple intersections of the Lorenz curves. The subgroup decomposition of the intermediate Gini coefficient is still involved.

Finally in the last section, in order to highlight the link between inverse dominance and the Gini index, following Muliere and Scarsini (1989) and Yitzhaki (1982, 1983) we investigate the relationship between higher degrees of inverse dominance and welfare dominance according to the parameterized family of Extended Gini indices.

2 Preliminaries

It could be useful to consider the analogy between income distribution and random variables. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Ω could be seen as

[6]The proof will be provided in the appendix, and will be divided into two parts. The first part is of immediate interpretation, because it relies on the simple intuition discussed in the text. The second part is more technical and tedious.

a population of individuals, (either discrete or continuous) and Y is a non-negative random variable, with finite expectation. In our framework $Y(\omega)$ is the income level of agent $\omega \in \Omega$, therefore Y is called *income profile.* If $\Omega = \{\omega_1, \omega_2, .. \omega_i, ..\omega_n\}$, and P attaches equal mass $1/n$ to each ω_i, we have the *n-dimensional empirical case,* where $Y(\omega_i) = y_i$, and the income profile is represented by the n-dimensional vector $\mathbf{y} = (y_1, y_2, .y_i, .y_n)$.

Let $F(y)$ denote the (discrete or continuous) cumulative income distribution function of an income profile with support $(0, +\infty)$ and finite mean $\mu(F) = \int_0^{+\infty} y \; dF(y)$. Let $\mathcal{F}$ the set of all such cumulative distributions, and $\mathcal{X}^n$ and $\mathcal{F}^n$, the set of n-dimensional income profiles and the set of the associated distributions. Moreover, let

$$F^{-1}(p) = \inf \{y : F(y) \geq p\} \quad \text{with} \quad 0 \leq p \leq 1$$

be the left continuous inverse distribution function showing the income of an individual at the $100p^{th}$ percentile of the distribution. It follows that $\mu(F) = \int_0^1 F^{-1}(p)dp$. Next, making use of Gastwirth (1971), the Lorenz curve for F is defined as:

$$L_F(p) = \int_0^p \frac{F^{-1}(t)}{\mu(F)} dt.$$

An income profile shows less dispersion than another, in terms of income shares, if its Lorenz curve is nowhere below that of the other profile[7]. We will say that distribution $F \in \mathcal{F}$ *Lorenz dominates* distribution $G \in \mathcal{F}$, $F \succsim_L G$, if and only if

$$L_F(p) \geq L_G(p) \text{ for all } p \in [0, 1] .$$

If $\mu(F) = \mu(G)$ this condition is equivalent to second degree stochastic dominance, which in turn is equivalent to Generalized Lorenz dominance obtained comparing Generalized Lorenz curves $GL_F(p) = \mu(F)L_F(p)$.

The second degree stochastic dominance order is equivalent to welfare dominance once we consider inequality averse welfare functions which are not decreasing in income levels (see Kolm, 1969 and Shorrocks, 1983). When Generalized Lorenz curves intersect Shorrocks and Foster (1987), Dardanoni

[7]See Atkinson (1970), Kolm (1969), Sen (1973), Dasgupta, Sen and Starrett (1973), Rothschild and Stiglitz (1973) and Fields and Fei (1978).

and Lambert (1988), and Davies and Hoy (1994, 1995) provide a normative justification for using the third degree stochastic dominance order[8].

In this paper instead of looking at higher degree of stochastic dominance[9] we shift the attention to the concept of *inverse stochastic dominance* introduced by Muliere and Scarsini (1989). For $F \in \mathcal{F}$ we denote

$$F_1^{-1}(p) = F^{-1}(p), \qquad F_k^{-1}(p) = \int_0^p F_{k-1}^{-1}(t)dt, \qquad 0 \leq p \leq 1$$

and define the k-th degrees inverse stochastic dominance $\succcurlyeq_k^{-1}$as follows:

Definition 1 *Given two income distributions F and $G \in \mathcal{F}$, $F \succcurlyeq_k^{-1} G$ if and only if $F_k^{-1}(p) \geq G_k^{-1}(p)$ for all p, $0 \leq p \leq 1$.*

As shown by Muliere and Scarsini (1989) direct and inverse stochastic dominance are not equivalent for degrees higher than the second.

We will review results introduced in Zoli (1999), providing normative justification for the use of *third degree inverse stochastic dominance* (henceforth 3ISD) and we will investigate the implications of these results for inequality comparisons, as well as for providing a characterization of 3ISD in terms of transfers of incomes within an income profile.

We will denote by $I(F)$ and $W(F)$ respectively the *inequality* and *welfare* indices associated to distribution $F \in \mathcal{F}$. Since those indices are defined over distribution functions they are symmetric and population invariant, that is they depend only on the income distribution and are invariant w.r.t. replications. In the empirical case it means that the indices implicitly satisfy the following properties:[10]

Axiom 2 (Anonimity (An)) *The evaluation is invariant w.r.t. permutations of the income profile $(y_1, y_2, .y_i, .y_n)$.*

Let $\mathbf{y}^r$, r positive integer ($r \in \mathbb{I}$) be the distribution obtained from the income profile $\mathbf{y} \in \mathcal{X}^n$ replicating it r times, that is $\mathbf{y}^r = (\underbrace{\mathbf{y},,..\mathbf{y},..,\mathbf{y}}_{r \text{ times}}) \in \mathcal{X}^{rn}$.

[8] See also Withmore (1980) and Menezes et al. (1980) for a discussions concerning comparisons of risky prospects.

[9] See for instance Fishburn and Willig (1984).

[10] The anonymity property requires that the only factor relevant in inequality and welfare comparisons is the level of income. Therefore we can restrict our attention to ordered distributions.

Axiom 3 (Population Principle (PP)) *Income profiles* $\mathbf{y} \in \mathcal{X}^n$, *and* $\mathbf{y}^r \in \mathcal{X}^{rn}$ *receives the same evaluation for all* $r, n \in \mathbb{I}$.

This axiom suggested by Dalton (1920) (see also Dasgupta, Sen and Starrett, 1973) allows to compare distributions with different population size[11]. In this paper we will restrict attention to both inequality and welfare indices satisfying An and PP.

Following the ethical approach to inequality measurement, social welfare and inequality judgements are supposed to be *consistent*, i.e. they must produce the same rankings over distributions with equal means[12].

Definition 4 *An inequality index* $I(.)$ *and a social welfare function* $W(.)$ *are consistent if for any* F *and* $G \in \mathcal{F}$ *such that* $\mu(F) = \mu(G)$, $I(F) \leq I(G)$ *whenever* $W(F) \geq W(G)$.

The consistency property will allow us to interpret the results we will discuss for welfare dominance in terms of inequality dominance when distributions with equal means are compared.

The following properties of welfare and inequality indices allow to introduce inequality aversion in the evaluation. We consider n-dimensional empirical income profiles. We define a *progressive transfer* as a transfer of income from a rich to a poor individual which is not affecting their positions in the income ranking[13]; the transfer is called *regressive* if it goes in the opposite direction.

Axiom 5 (Principle of Transfers (PT)) *Given two distributions* F *and* $G \in \mathcal{F}^n$ *such that* F *is obtained from* G *through a progressive transfer then* $W(F) \geq W(G)$.

[11] According to PP the evaluation considers only relative frequencies or sample weights attached to each household receiving a given income. Following Yaari (1988) the terminology PP allows to restrict our attention to normalized income profiles with a population of unit mass.

[12] The consistency condition has been discussed in Blackorby and Donaldson (1984), Ebert (1987), Dutta and Esteban (1992), and Ben Porath and Gilboa (1994).

[13] The definition of progressive transfer and related Principle of Transfers is due to Fields and Fei (1978). The Principle of Transfers has been introduced in the literature by Pigou (1912) and Dalton (1920). For a discussion of different versions of the Principle of Transfers in inequality and welfare measurement see Mosler and Muliere (1998).

Welfare does not decrease (equivalently inequality is not increased $I(F) \leq I(G)$ for a consistent index $I(.)$) as the effect of a progressive transfer. More formally, assuming individuals are ranked in ascending order according to their incomes, let $\Delta W_{i+\rho,i}(\delta)$ denote the change in social welfare due to a transfer δ of income from the $(i+\rho)^{th}$ individual to the i^{th} individual, that leaves unchanged their rank in the distribution. According to PT $\Delta W_{i+\rho,i}(\delta) \geq 0$.

It is well known that the properties An and PT correspond to imposing S-Concavity on the equality ordering induced by the indices (therefore S-Convexity on the consistent inequality counterpart), and that over distributions of same total income and population dimension the inequality ordering is equivalent to the Lorenz ranking.

The Pigou-Dalton Principle of Transfers (PT) has been strengthened requiring to social welfare to be more sensitive to transfers that take place lower down in the distribution[14]. In this paper we concentrate on a version of the transfer sensitivity axiom, discussed in Mehran (1976) and Kakwani (1980) which place attention on the positions of the individuals in the income ranking.[15]

Definition 6 (Principle of Positional Transfer Sensitivity, PPTS) *For any pair of individuals i and j, if $j > i$ then $\Delta W_{i+\rho,i}(\delta) \geq \Delta W_{j+\rho,j}(\delta)$.*

This formulation of PPTS implies that a combination of a progressive transfer and a regressive transfer of the same amount and involving individuals with the same distance in the income ranking is not reducing welfare if the progressive transfer takes place at lower income levels than the regressive transfer.

For the purpose of investigating the relationship between welfare and inequality dominance and inverse stochastic dominance, we restrict our attention to rank-dependent evaluation functions.

[14]This is the so-called Principle of Transfer Sensitivity, which requires that a progressive transfer between persons with given income difference is valued more if these incomes are lower than if they are higher (Kolm, 1976, p. 87).

[15]Notice that while for inequality aversion considerations, putting enphasis on ranks of the individuals or on their income level is irrelevant, once we specify transfer sensitivity the two perspectives lead to different results. Chateauneuf, Gajdos and Wilthien (1999) discuss a combined approach reconciling PPTS and Kolm's PTS, obtaining characterizations valid for general rank-dependent utilitarian functions.

A *Yaari dual Social Welfare Function* (YSWF) $\tilde{W}(.)$ is an additive evaluation function according to which the social evaluation of an income profile is a weighted average of the incomes, which are linearly aggregated, and weighted according to their position in the income ranking, that is, following Yaari (1987, 1988):[16]

$$\tilde{W}(F) = \int_0^1 F^{-1}(p)v(p)dp, \tag{1}$$

where $v(p) \geq 0$ (which is independent from $F(y)$) is the weight attached to the income of the fraction of the population ranked p. Throughout the paper we will suppose that $v(p)$ is a continuous and twice differentiable function[17]. The family of Extended Gini based welfare functions[18] $\Xi_\theta(.)$ is obtained for $v(p) = \theta(1-p)^{\theta-1}$ where $\theta \geq 1$. For $\theta = 2$ we get the Gini based welfare function[19] $\Xi_2(F) = \mu(F)\left[1 - \Gamma(F)\right]$ where $\Gamma(F)$ is the Gini coefficient

$$\Gamma(F) = 1 - \frac{2}{\mu(F)} \int_0^1 (1-p)F^{-1}(p)dp. \tag{2}$$

Within Yaari's evaluation framework the effect of a progressive transfer could be checked as follows (see Mehran, 1976). Each income unit represents a fraction of the total population dp, since (1) is defined over distribution functions, it is invariant w.r.t. replications of the population, therefore dp could be arbitrarily small. We consider therefore a small transfer $\delta > 0$ from a tiny fraction $dp > 0$ of the population at the $100(i+\rho)^{th}$ percentile of the distribution to a fraction $dp > 0$ of poorer individuals at the $100i^{th}$ percentile, where $\delta > 0$ is such that the ranking of the income units considered is not

[16]See also Weymark (1981), Ebert (1988), Quiggin (1993), Ben Porath and Gilboa (1994) and Safra and Segal (1998).

[17]Within the appendix we will consider also cases in which $v(p)$ is not differentiable. Notice that for the empirical case $\tilde{W}(F)$ reduces to

$$W_Y(\mathbf{y}) = \sum_i y_{(i)} \left(V[i/n] - V[(i-1)/n]\right)$$

where $y_{(i)} \leq y_{(i+1)}$ and $V(t) = \int_0^t v(p)dp$.

[18]See Donaldson and Weymark (1983) and Yitzhaki (1983).

[19]The welfare index $\Xi_2(.)$ can be interpreted as the *equally-distributed-equivalent income* measure for the Gini coefficient if we apply the method of deriving social welfare functions from inequality indices suggested in Blackorby and Donaldson (1978).

affected as a consequence of the transfer. If welfare is evaluated according to (1) then:

$$\Delta\tilde{W}_{i+\rho,i}(\delta) = -\delta v(i+\rho)dp + \delta v(i)dp \tag{3}$$

It follows that a YSWF satisfies PT, if and only if $v'(p) \leq 0$ (See Mehran 1976, Yaari, 1987, 1988), and also PPTS if and only if $v''(p) \geq 0$ (See Mehran, 1976, Kakwani, 1980 and Zoli, 1999).[20]

As shown in Wang and Young (1998) and Zoli (1999) welfare dominance in terms of YSWFs corresponds to the criterion of *inverse* stochastic dominance.[21] Particularly, once we restrict attention to YSWFs where $v'(p) \leq 0$ and $v''(p) \geq 0$, welfare dominance is equivalent to 3ISD for income distributions with equal means, and generally:

Proposition 7 (Zoli (1999)) *For any YSWF satisfying PT and PPTS:* $\tilde{W}(F) \geq \tilde{W}(G)$ *if and only if* $F \succcurlyeq_3^{-1} G$ *and* $\mu(F) \geq \mu(G)$.

That is, if we consider inequality orderings consistent with the YSWF ordering then for comparisons of distributions with the same mean 3ISD replace Lorenz dominance.

The relevance of the Gini index $\Gamma(F)$ as an inequality summary statistic will become clear from the following result.

We specify the Rawlsian leximin criterion making use of the inverse distribution functions of F and G; $F \succ_{\min} G$ means that F dominates G according to the *leximin criterion,* that is, there exists an interval $(0, p^*)$ on which $F^{-1}(p) \neq G^{-1}(p)$ and $F^{-1}(p) \geq G^{-1}(p)$.[22]

Proposition 8 (Zoli (1999)) *If the Lorenz curves for* F *and* G *cross once,* $\mu(F) = \mu(G)$, *and* $F \succ_{\min} G$, *then* $\tilde{W}(F) \geq \tilde{W}(G)$ *for all YSWFs satisfying PT and PPTS if and only if* $\Gamma(F) \leq \Gamma(G)$.

This result, if combined with that in the previous proposition, shows that when it is impossible to compare distributions unanimously in terms of S-Concave evaluation functions, once we consider 3ISD as an ethically acceptable criterion, then in the most common case of single crossing, only the leximin and Gini index information is sufficient for providing a clear-cut judgement.

[20] Chateauneuf et al. (1999) show that the same result holds for generalized rank-dependent utilitarian functions without imposing inequality aversion.

[21] See also Aaberge (2000).

[22] Obviously $F \succ_{\min} G$ could be expressed equivalently making use of $F_k^{-1}(p)$ and $G_k^{-1}(p)$ for any $k = 1, 2, ..$

3 Inequality comparisons

Until now we have discussed the relationship between welfare dominance and PPTS. If we restrict our attention to a Strong version of PPTS it is also possible to obtain results for inequality dominance. The Strong version of PPTS involves the definition of a generalization of the positional transfers covering situations in which both the amount of the transfer and the distance between the pairs of individuals involved may change. In order to formalize this principle we need to introduce the concept of *Favourable Composite Positional Transfer (FCPT).* We call this composite transfer *favourable* because its net effect is assumed to be inequality reducing, and *positional* because it is sensitive to changes in distances in the income ranking of the individuals involved.

Definition 9 (FCPT) *A Favourable Composite Positional Transfer is a combination of a rank-preserving progressive transfer and regressive transfer from the same donor, such that the Gini index is left unchanged.*

Formally, consider empirical distributions $F, \tilde{F} \in \mathcal{F}^n$, then $\tilde{F}$ is obtained from F through a FCPT[23] if it is obtained from a progressive transfer of a small amount $\delta > 0$ from a tiny fraction $dp > 0$ of the population at the $100j^{th}$ percentile of the distribution to a poorer fraction $dp > 0$ at the $100i^{th}$ percentile, and a regressive transfer of a small amount $\gamma > 0$ from the fraction dp of the population at the $100j^{th}$ percentile of the distribution to a richer fraction dp of individuals at the $100l^{th}$, where $l > j > i$ and such that $\Gamma(F) = \Gamma(\tilde{F})$.

We can now formalize the Strong version of PPTS.

Definition 10 *An inequality index obeys the Strong Principle of Positional Transfer Sensitivity (SPPTS) if and only if* $I(\tilde{F}) \leq I(F)$ *whenever* $\tilde{F}$ *is obtained from* F *applying a FCPT.*

This condition implies that if we apply a sequence of progressive transfers and/or favourable composite positional transfers, we obtain a distribution which must be considered more equal according to any inequality index satisfying the PT and SPPTS.

[23] Placing attention to $F, \tilde{F} \in \mathcal{F}^n$ we can restate the definition considering rank-preserving transfers of amounts $\delta, \gamma > 0$ involving individuals at the j^{th}, i^{th}, and l^{th} positions in the illfare ranked income distribution, where $l > j > i$ and such that $\Gamma(F) = \Gamma(\tilde{F})$.

Moreover any welfare function $W(.)$ consistent with such indices should satisfy $W(\tilde{F}) \geq W(F)$. As we will show[24], SPPTS is equivalent to PPTS when social welfare is evaluated according to YSWFs such that $v'(p) \leq 0$ and $v''(p) \geq 0$, but generally speaking it is a stronger condition than PPTS. It does not impose any restriction concerning the extent of the transfers and the distances between the individuals involved. Moreover, although FCPT restricts attention to the case in which the donor is the same both for the progressive and regressive transfer, this simplification is not weakening the impact of the SPPT. Any composite transfer where the two donors have incomes between those of the recipients can be obtained through a sequence of FCPTs.[25]

The following is the *main proposition* in the paper, it highlights the normative implications of making inequality or welfare comparisons using the 3ISD criterion. The result is dual to that presented in Shorrocks and Foster (1987)[26] concerning standard third degree stochastic dominance.

Proposition 11 *Given two income distributions $\tilde{F}$ and $F \in \mathcal{F}^n$ with equal means, the distribution $\tilde{F}$ can be obtained from F through a finite sequence of progressive transfers and/or favourable composite positional transfers if and only if $\tilde{F} \succcurlyeq_3^{-1} F$.*

Proof: See appendix.

This result has been hinted at in Zoli (1997). Chateauneuf and Wilthien (2000) independently obtain a similar characterization of 3ISD for the n-dimensional empirical case[27]. An interesting aspect which differentiate this result from those of Shorrocks and Foster (1987) and Chateauneuf and Wilthien (2000) is that in order to prove the equivalence between 3ISD and the sequence of progressive transfers and FCPTs, instead of concentrating on the effects of such transfers on the income profiles we exploit the result of equivalence between 3ISD and welfare dominance in Proposition 7. We therefore look at the restrictions on the set of transfers which are consistent with welfare dominance for all inequality averse YSWFs satisfying PPTS. This approach makes clear the intuition behind the result[28]. First notice that the

[24]See the only if part of the proof of Proposition 11.

[25]See Lemma 23 in the appendix.

[26]See also Menezes et al. (1980).

[27]See also Aaberge (2000) where similar results are hinted at.

[28]See Steps 1 and 2 in the proof. These are conceptually the most relevant steps of the proof and are particularly simple. They look at restrictions on simple composite transfers

FCPT is set such that it only concentrates on comparisons of effects of the progressive and regressive transfer, because these involve the same individual as the donor. Since the Gini based welfare function is the least transfer sensitive indicator within the class of welfare functions considered, then at least any welfare improving combination of transfers should not increase the Gini index. Any evaluator who is inequality averse but neutral w.r.t. transfer sensitivity, will consider the progressive transfer to the poor individual sufficient to offset the negative impact of the regressive transfer. Since this is an extreme position, then all evaluators minimally concerned with transfer sensitivity will give to the progressive transfer higher value compared to the regressive, then the FCPT will be considered welfare enhancing or inequality reducing.[29]

It is well known that given two distributions with equal means $\tilde{F}$ and F, the former Lorenz dominates the latter if and only if it is more equal according to any inequality index satisfying PT. The third order inverse stochastic dominance criterion extends the Lorenz dominance criterion over distributions with equal means when we refine the notion of inequality requiring that the inequality indices also satisfy SPPTS. In other words, we obtain:

Corollary 12 *If $\mu(F) = \mu(\tilde{F})$, then $I(\tilde{F}) \leq I(F)$ for any inequality index satisfying PT and SPPTS if and only if $\tilde{F} \succcurlyeq_3^{-1} F$.*

This implies that, over distributions with equal means, when the Lorenz curves of the two distributions cross once, we can apply the result in Proposition 8 for inequality comparisons.

If we want to extend the inequality comparisons to distributions with differing means it is necessary to specify the *Inequality Equivalence Criterion* applied. The inequality equivalence criterion specifies income transformations which leaves inequality unchanged[30]. In what follows we will consider

involving only a progressive and a regressive transfer. The remaining part of the proof, which is more technical, is devoted to show that any multiple transfer, involving more than one progressive and regressive transfer, in order to be welfare enhancing should be obtained through sequences of FCPTs and progressive transfers.

[29] A similar intuition could be applied for interpreting Shorrocks and Foster (1987) result, concerning composite transfers leaving the variance unchanged, within an utilitarian framework.

[30] If inequality is considered in terms of comparisons of relative differentials in income then scaling an income profile by a positive constant will leave the inequality unchanged. On the other hand if inequality evaluations are based on absolute income differentials, then

the parametric *Intermediate Inequality Equivalence* criterion suggested in Bossert and Pfingsten (1990) which reaches the absolute and relative inequality equivalence for the extreme values of the parameter $0 \leq \eta \leq 1$ representing the attitude toward inequality equivalence.

Axiom 13 (Intermediate Inequality Equivalence (IIE)) *Income profiles* $\mathbf{y}, \mathbf{y}' \in \mathcal{X}^n$ *are* η *inequality equivalent if*

$$\mathbf{y}' = \mathbf{y} + \lambda \left[\eta \mathbf{y} + (1-\eta) \mathbf{1} \right] \quad ; \quad \textit{for all } \lambda \in \mathbb{R} \textit{ such that } \mathbf{y}' \in \mathcal{X}^n, \tag{4}$$

where $0 \leq \eta \leq 1$, $\mathbf{1} = (1, 1, ..1)$

All distributions $\mathbf{y}' \in \mathcal{X}^n$ obtained from $\mathbf{y}$ through (4), for a given value of η, are inequality equivalent to $\mathbf{y}$ according to the intermediate criterion associated to η. When $\eta = 1$ then $\mathbf{y}' = \mathbf{y}(1+\lambda)$ we obtain the *relative inequality equivalence,* while if $\eta = 0$ then $\mathbf{y} = \mathbf{y}' + \lambda \mathbf{1}$ in (4), we reach the other extreme view of inequality, the *absolute inequality equivalence* originally proposed by Kolm (1976). While for all the other values of η we have intermediate positions.

Let $F_{(\eta)}(t)$ be the normalized distribution function, such that $F_{(\eta)}(\frac{\eta y + (1-\eta)}{\eta \mu(F) + (1-\eta)}) = F(y)$ if $0 \leq \eta < 1$, and $F_{(0)}(y - \mu(F)) = F(y)$ (in this last case $F_{(0)}$ has support $(-\infty, +\infty)$). Moreover, let $I_\eta(.)$ be an inequality index satisfying η equivalence. Then, the above equivalence condition requires that $I_\eta(F) = I_\eta(G)$ if $F_{(\eta)}(y) = G_{(\eta)}(y)$.[31]

We could therefore write the class of intermediate inequality indices which are consistent with YSWFs satisfying PT and SPPTS (or equivalently PPTS). These are transformations of the linear measures of inequality $I^M(.)$ defined in Mehran (1976):

$$I_\eta^M(F) = \int_0^1 w(p) \left(\frac{F^{-1}(p) - \mu(F)}{\eta \mu(F) + (1-\eta)} \right) dp,$$

traslating the initial income profile adding the same amount of income to every individual will not affect inequality.

[31] The class of S-Convex intermediate inequality indices has been characterized in Eichhorn (1988).

such that $w'(p) \geq 0$, $w''(p) \leq 0$, and $w(p)$ is normalized such that $\int_0^1 w(p)dp = 0$ and $w(1) = 1$.[32] The intermediate version of the Gini coefficient $\Gamma_\eta(.)$ can be obtained for $w(p) = 2(p-1)$, that is $\Gamma_\eta(F) = 2\int_0^1 (1-p)\left(\frac{\mu(F)-F^{-1}(p)}{\eta\mu(F)+(1-\eta)}\right) dp$.[33]

Moreover, is possible to specify "η Lorenz curves" $L_F^\eta(p)$:

$$L_F^\eta(p) = \frac{\mu(F)[L_F(p) - p]}{\eta\mu(F) + (1-\eta)}.$$

We will say that distribution $F \in \mathcal{F}$ *ηLorenz dominates* distribution $G \in \mathcal{F}$, $F \succcurlyeq_L^\eta G$ if and only if:

$$L_F^\eta(p) \geq L_G^\eta(p) \text{ for all } p \in [0,1]\,,\ 0 \leq \eta \leq 1.$$

As shown in Chakravarty (1988), $F \succcurlyeq_L^\eta G$ if and only if $I_\eta(F) \leq I_\eta(G)$ for all S-convex η intermediate inequality indices. For $\eta = 1$, we have the standard Lorenz dominance, while for $\eta = 0$ we obtain the absolute Lorenz dominance introduced in Moyes (1988).

In the remaining part of this section we will consider inequality comparisons according to $I_\eta(.)$ indices. For this class of indices in order to compare distributions with differing means we need only to make use of information on the "η Lorenz curve" and the η Gini index of the distributions, since both are invariant with respect to η inequality equivalent changes in the income distributions.[34]

Proposition 14 *If the "η Lorenz curves" for F and $G \in \mathcal{F}^n$ cross once, and $F_{(\eta)} \succ_{\min} G_{(\eta)}$, then $I_\eta(F) \leq I_\eta(G)$ for any η inequality index satisfying PT, and SPPTS if and only if $\Gamma_\eta(F) \leq \Gamma_\eta(G)$.*

Proof. Consider two distributions G and $\tilde{G}$ with equal means. Applying the results of Corollary 12 and Propositions 7 and 8, if the Lorenz curves

[32] Mehran (1976) discusses the relative index, that is $I_1^M(.)$.

[33] The intermediate inequality indices $\Gamma_{\theta,\eta}(.)$ associated to the Extended Gini welfare functions are obtained for $w(p) = 1 - \theta(1-p)^{\theta-1}$.

[34] The following results are analogous to those obtained in Zoli (1999) for welfare dominance in terms of YSWFs. They are dual to the results in Shorrocks and Foster (1987) and Davies and Hoy (1995) related to traditional third degree stochastic dominance associated to the utilitarian approach. These results involve comparison of variances of the populations or subgroups of the populations.

Zoli (1997), Chateauneuf and Wilthien (2000) and Aaberge (2000) provide results concerning relative inequality comparisons.

for G and $\tilde{G}$ cross once, and $\tilde{G} \succ_{\min} G$[35], then $\Gamma(\tilde{G}) \leq \Gamma(G)$ if and only if $I(\tilde{G}) \leq I(G)$ for any inequality index satisfying PT and SPPTS, i.e. $\tilde{G} \succcurlyeq_3^{-1} G$, or, in other words, $\tilde{G}$ is obtained from G through a finite sequence of progressive transfers and/or favourable composite positional transfers.

If we consider η *inequality indices* we need to concentrate on $\tilde{G}_{(\eta)}$ and $G_{(\eta)}$. We can set distribution F, where $\mu(F)$ possibly differs from $\mu(G)$ such that $F_{(\eta)} = \tilde{G}_{(\eta)}$. It follows that by definition of the inequality equivalence criterion $I_\eta(F) = I_\eta(\tilde{G}) \leq I_\eta(G)$, for all η *inequality indices* satisfying PT, and SPPTS. Moreover $F_{(\eta)} \succ_{\min} G_{(\eta)}$, $\Gamma_\eta(F) = \Gamma_\eta(\tilde{G}) \leq \Gamma_\eta(G)$ and the η Lorenz curves of F and $\tilde{G}$ are unaffected. Which gives the initial conditions in the proposition. ■

We now extend the inequality comparisons to distributions whose Lorenz curves cross more than once[36]. Following Zoli (1999) we express 3ISD in terms of the means and the Gini coefficients associated with subgroups of the populations. Let F_p and G_p denote the distributions of incomes of the first p^{th} fraction of individuals induced by distributions F and G respectively, such that $F_p(y) = F(y)/p$ for $y \leq F^{-1}(p)$ and $F_p(y) = 1$ for all $y > F^{-1}(p)$. As shown in Zoli (1999):

Lemma 15 $F \succcurlyeq_3^{-1} G \Leftrightarrow \mu(F_p)\left[1 - \Gamma(F_p)\right] \geq \mu(G_p)\left[1 - \Gamma(G_p)\right] \; \forall p \in [0,1]$.

Since we are considering inequality comparisons, we restrict the analysis to distributions with equal means $\mu(F) = \mu(G)$ and then extend the considerations to distributions with differing means by making use of the inequality equivalence criterion. We consider the case of Lorenz curves L_F and L_G crossing c times.

Defines percentiles points p_j as follows: $p_0 = 0$, p_j is the percentile point at which the $2j^{th}$ intersection of L_F and L_G takes place, where j runs from 1 to k, and $p_{k+1} = 1$.

The i^{th} subpopulation consists of those income recipients belonging to the poorest $100p_i$ percent but not to the poorest $100p_{i-1}$ percent of the population. F^i and G^i are the distribution functions of the i^{th} subpopulation which are induced by F and G respectively, such that $F^i(y) = 0$ for all $y \leq F^{-1}(p_{i-1})$, $F^i(y) = 1$ for all $y \geq F^{-1}(p_i)$ and $F^i(y) = \frac{F(y)-p_{i-1}}{p_i-p_{i-1}}$ for

[35]Notice that in this case, because $\mu(\tilde{G}) = \mu(G)$, then $\tilde{G} \succ_{\min} G \Leftrightarrow \tilde{G}_{(\eta)} \succ_{\min} G_{(\eta)}$. and the crossing condition is also satisfied by the η Lorenz curves.

[36]The following results are analogous to those obtained in Zoli (1999) for welfare dominance in terms of YSWFs. They are dual to the results in Davies and Hoy (1995).

$F^{-1}(p_{i-1}) \leq y \leq F^{-1}(p_i)$; that is: $F_{p_1} \equiv F^1$, $F_{p_j} \equiv \sum_{i=1}^{j} \frac{(p_i - p_{i-1})F^i}{p_j}$, and $F_{p_{k+1}} \equiv F$.

We can now state the following proposition which generalizes the result in the previous proposition concerning single crossing of Lorenz curves.

Proposition 16 *If the "η Lorenz curves" for F and $G \in \mathcal{F}^n$ cross c times, and $F_{(\eta)} \succ_{\min} G_{(\eta)}$, then $I_\eta(F) \leq I_\eta(G)$ for any η inequality index satisfying PT and SPPTS if and only if:*
(**A**) *For c even (i.e. $c = 2k$): $\Gamma_\eta(F_{p_j}) \leq \Gamma_\eta(G_{p_j})$ for all $j \leq k$.*
(**B**) *For c odd (i.e. $c = 2k+1$): $\Gamma_\eta(F_{p_j}) \leq \Gamma_\eta(G_{p_j})$ for all $j \leq k+1$ (i.e. $\Gamma_\eta(F_{p_j}) \leq \Gamma_\eta(G_{p_j})$ for all $j \leq k$ and $\Gamma_\eta(F) \leq \Gamma_\eta(G)$).*

Proof. Consider two distributions G and $\tilde{G}$ with equal means $\mu(\tilde{G}) = \mu(G) = \mu$, such that $\tilde{G} \succ_{\min} G$ and the Lorenz curves for G and $\tilde{G}$ cross an *even* number of times (A). Then it is immediate to check that $\tilde{G}^i \succ_{\min} G^i$ for all $i \in N$ and the Lorenz curve of $\tilde{G}^{k+1}$dominates that of G^{k+1}. Then the points p_j for $j \leq k$ are local minima of the function $X(p) = \tilde{G}_3^{-1}(p) - G_3^{-1}(p) = \mu \int_0^p L_{\tilde{G}}(t) - L_G(t)dt$ for $0 \leq p \leq 1$. Then $\tilde{G} \succcurlyeq_3^{-1} G$ if and only if $X(p_j) \geq 0$, for all $j \leq k$.

Notice that $\mu(G_{p_j}) = \int_0^1 G_{p_j}^{-1}(t)dt = \int_0^1 G^{-1}(tp_j)dt = \frac{1}{p_j}\int_0^{p_j} G^{-1}(q)dq$ that is $\mu(G_{p_j}) = \frac{1}{p_j}\mu(G)L_G(p_j)$. Therefore since $L_G(p)$ and $L_{\tilde{G}}(p)$ cross at p_j and $\mu(\tilde{G}) = \mu(G) = \mu$, then $\mu(\tilde{G}_{p_j}) = \mu(G_{p_j})$ if $j \leq k$. Recalling Lemma 15, then $X(p_j) \geq 0$, for all $j \leq k$ if and only if all the first k subgroups with distributions $\tilde{G}_{p_j}$ and G_{p_j} satisfy the condition $\Gamma(\tilde{G}_{p_j}) \leq \Gamma(G_{p_j})$. It follows that $\tilde{G}$ is more equal than G according to any inequality index satisfying PT and SPPTS if and only if $\Gamma(\tilde{G}_{p_j}) \leq \Gamma(G_{p_j})$ for all $j \leq k$.

(B) If the Lorenz curves for G and $\tilde{G}$ cross an odd number of times, and $\tilde{G} \succ_{\min} G$, then the Lorenz curves of $\tilde{G}^{k+1}$and G^{k+1} cross once and $\tilde{G}^{k+1} \succ_{\min} G^{k+1}$. Then not only the points p_j for $j \leq k$ are local minima of $X(p)$, but also $p_{k+1} = 1$ is a local minimum. Then $\tilde{G} \succcurlyeq_3^{-1} G$ if and only if $X(p_j) \geq 0$, for all $j \leq k+1$; this condition is satisfied if and only if $\mu(F_{p_j})\left[1 - \Gamma(F_{p_j})\right] \geq \mu(G_{p_j})\left[1 - \Gamma(G_{p_j})\right]$ for all $j \leq k+1$, i.e. $\Gamma(F_{p_j}) \leq \Gamma(G_{p_j})$ for all $j \leq k$, and also $\Gamma(F) \leq \Gamma(G)$.

In order to extend this proof to distributions with unequal means when the indices of inequality are η invariant one can argue in the same way as done in the final part of the proof of previous proposition. ■

3.0.1 Extended Ginis and inverse stochastic dominance

As mentioned before, a single parametric generalization of the Gini based welfare index, the Extended Gini (EG) index

$$\Xi_\theta(F) = \theta \int_0^1 (1-p)^{\theta-1} F^{-1}(p)dp. \tag{5}$$

is well known in literature on economic inequality and welfare measurement. It has been independently proposed by Kakwani (1980) and Donaldson and Weymark (1980, 1983). EG satisfies PT if $\delta \geq 1$, and SPPTS (also PPTS) if $\delta \geq 2$. For $\delta = 1$ the associated ordering is inequality neutral $\Xi_1(F) = \mu(F)$. For $\delta = 2$ EG becomes the social welfare index $\mu(F)\left[1 - \Gamma(F)\right]$. For $\delta \to \infty$ the associated welfare ordering approaches the leximin ordering.

Yitzhaki (1982, 1983) investigates the relationships between the welfare ordering induced by this family of indices and the criteria of first and second order stochastic dominance, providing *sufficient conditions* for stochastic dominance in terms of dominance according EGs evaluated for differing values of the parameters. Muliere and Scarsini (1989) argue that the EGs are also closely linked to the inverse stochastic dominance criteria, and show that k^{th} degree inverse stochastic dominance is a *necessary condition* for dominance according to EGs evaluated for integer values of the parameter not lower than $k-1$.

The aim of this section is to extend the set of sufficient conditions introduced in Yitzhaki (1982, 1983), providing sufficient conditions for k^{th} degree inverse stochastic dominance. This shows, as hinted in Muliere and Scarsini (1989), that the natural way of linking EG orderings and stochastic dominance is by means of the inverse stochastic approach and not the direct one. Furthermore, as already pointed out, since EGs are a specification of the YSWF, the obtained dominance results involving EGs will be compared with those obtained for YSWFs dominance.

Let $\mathbb{N}$ the set of natural numbers and $F, G \in \mathcal{F}$. The two fundamental propositions linking dominance according to EGs and inverse stochastic dominance are, firstly:

Proposition 17 (Muliere and Scarsini (1989)) $F \succcurlyeq_k^{-1} G \Rightarrow \Xi_\theta(F) \geq \Xi_\theta(G)$ $\forall \theta \geq k-1$, *(with at least one strict inequality)* $(\theta \in \mathbb{N})$.

And secondly, the following result, due to Yitzhaki (the original result is in terms of standard stochastic dominance, here it has been rewritten in terms of inverse distributions and inverse stochastic dominance criteria, in order to highlight the links with the other results in this section):

Proposition 18 (Yitzhaki (1982, 1983)) *Let* $F^{-1}(p)$ *and* $G^{-1}(p)$ *be two inverse distribution functions which intersect at most once. Then,* $\Xi_\theta(F) \geq \Xi_\theta(G) \quad \forall \theta \geq 1$ *(with at least one strict inequality)* ($\theta \in \mathbb{N}$)*, is a sufficient condition for* $F \succcurlyeq_2^{-1} G$*.*

We now introduce a lemma presented in Muliere and Scarsini (1989), which will become useful for the proof of next proposition.

Lemma 19 $\Xi_k(F) \geq \Xi_k(G)$ *if and only if* $F_{k+1}^{-1}(1) \geq G_{k+1}^{-1}(1)$*.*

Proof: Integrating by parts in (5) gives $\Xi_\theta(F) = \theta(\theta-1)\int_0^1 (1-p)^{\theta-2} F_2^{-1}(p)dp$. After further integrations by part we obtain $\Xi_\theta(F) = [\prod_{i=0}^{k-1}(\theta-i)]\int_0^1 (1-p)^{\theta-k} F_k^{-1}(p)dp$ for $\theta \geq k$. Thus:

$$\Xi_\theta(F) - \Xi_\theta(G) = [\prod_{i=0}^{k-1}(\theta-i)] \int_0^1 (1-p)^{\theta-k}[F_k^{-1}(p) - G_k^{-1}(p)]dp, \quad \theta \geq k. \tag{6}$$

It follows that $\Xi_\theta(F) \geq \Xi_\theta(G)$ for $\theta = k$ if and only if:

$$k![F_{k+1}^{-1}(1) - G_{k+1}^{-1}(1)] \geq 0, \tag{7}$$

that is equivalently $F_{k+1}^{-1}(1) \geq G_{k+1}^{-1}(1)$. ■

The requirement that distribution F dominates G according to the criterion of $(k+1)^{th}$ degree inverse stochastic dominance at the upper level of income (i.e. at $p = 1$) is a necessary and sufficient condition for dominance according to EGs with $\theta = k$.

The following result extends the sufficient conditions in Proposition 18.[37]

Proposition 20 *Let* $F_k^{-1}(p)$ *and* $G_k^{-1}(p)$ *intersect at most once. Then* $\Xi_\theta(F) \geq \Xi_\theta(G) \quad \forall \theta \geq k$ *(with at least one strict inequality), is a sufficient condition for* $F \succcurlyeq_{k+1}^{-1} G$*.*

[37] Yitzhaki (1983, p.621-622) presents a similar result when $n = 2$ and $\mu(F) = \mu(G)$.

Proof: It will be sufficient to prove the sufficiency for the extreme values of θ and then extend the result to the intermediate values.

If $\Xi_\theta(F) \geq \Xi_\theta(G)$ when $\theta = k$ from Lemma 19 we know that $F_{k+1}^{-1}(1) \geq G_{k+1}^{-1}(1)$. If $\Xi_\theta(F) \geq \Xi_\theta(G)$ when $\theta \to \infty$, we obtain that $F \succ_{\min} G$, that is, there exists an interval $(0, p^*)$ on which $F^{-1}(p) \neq G^{-1}(p)$ and $F^{-1}(p) \geq G^{-1}(p)$. Notice that, in this case, also in the interval $(0, p^*)$ $F_k^{-1}(p) \neq G_k^{-1}(p)$ and $F_k^{-1}(p) \geq G_k^{-1}(p)$ will be true for any k.

If $F_k^{-1}(p)$ and $G_k^{-1}(p)$ intersect at most once, then also their integrals $F_{k+1}^{-1}(p)$ and $G_{k+1}^{-1}(p)$ intersect at most once. We know that $F_{k+1}^{-1}(1) \geq G_{k+1}^{-1}(1)$ and also that in the interval $(0, p^*)$ $F_{k+1}^{-1}(p) \neq G_{k+1}^{-1}(p)$ and $F_{k+1}^{-1}(p) \geq G_{k+1}^{-1}(p)$ so, if $F_{k+1}^{-1}(p)$ dominates $G_{k+1}^{-1}(p)$ at the extremes $p = 1$ and the interval $(0, p^*)$, and the two functions intersect at most once, then $F_{k+1}^{-1}(p)$ dominates $G_{k+1}^{-1}(p)$ for all values of p, i.e. $F \succcurlyeq_{k+1}^{-1} G$. From Proposition 17 we know that if $F \succcurlyeq_{k+1}^{-1} G$ then $\Xi_\theta(F) \geq \Xi_\theta(G) \, \forall \theta \geq k$, then, welfare dominance in terms of $\Xi_\delta(.)$ for $\delta > k$ is implied by welfare dominance for $\theta = k$ and $\theta \to \infty$. ■

Not surprisingly the dominance in terms of EGs is not sufficient to guarantee inverse stochastic dominance; the result shows under which conditions it is. The next proposition follows from the results in Proposition 17 and Proposition 20 (the proof is omitted since it is straightforward).

Proposition 21 *Let $F_k^{-1}(p)$ and $G_k^{-1}(p)$ intersect at most once, and $F \succ_{\min} G$. Then $\Xi_k(F) \geq \Xi_k(G)$ if and only if $F \succcurlyeq_{k+1}^{-1} G$.*

A special case of this proposition which may be of interest occurs when $k = 2$. Recalling that $F_2^{-1}(p)$ and $G_2^{-1}(p)$ are the generalized Lorenz curves of F and G, and that $\mu(F)[1 - \Gamma(F)] = \Xi_2(F)$ we have:[38]

Corollary 22 *If the generalized Lorenz curves of distributions F and $G \in \mathcal{F}$ intersect at most once, and $F \succ_{\min} G$, then $\mu(F)[1-\Gamma(F)] \geq \mu(G)[1-\Gamma(G)]$ if and only if $F \succcurlyeq_3^{-1} G$.*

4 Concluding remarks

We discuss the normative justifications for supporting third degree inverse stochastic dominance in inequality comparisons when Lorenz curves intersect. The relevant aspect is captured by a positional version of the transfers

[38] This result is equivalent to those obtained in Propositions 3 and 4 in Zoli (1999) and also highlighted in this paper in Proposition 8 for the case in which $\mu(F) = \mu(G)$.

sensitivity axiom which requires a combination of a progressive and a regressive transfer from the same donor which leaves the Gini index unchanged to be inequality reducing. A combination of this property with the Principle of Transfers leads to the 3ISD partial ordering.

Inequality comparisons consistent with 3ISD are extended over distributions with different total income using the intermediate inequality equivalence criterion. When intermediate Lorenz curves intersect once, the distribution whose Lorenz curve intersects the other from above, is always less unequal according to the transfer sensitivity property discussed if and only if it exhibits a lower intermediate Gini coefficient. This result has been also extended to the case of multiple crossings.

The relationships between the generalized Gini indices and inverse stochastic dominance is finally investigated, and extensions of results of Muliere and Scarsini (1989) and Yitzhaki (1982, 1983) are provided.

5 Appendix

Proposition Given two income distributions $\tilde{F}$ and $F \in \mathcal{F}^n$, with equal means, the distribution $\tilde{F}$ can be obtained from F through a finite sequence of progressive transfers and/or favourable composite positional transfers, if and only if $\tilde{F} \succcurlyeq_3^{-1} F$.

Proof.

Given the previous result in Proposition 7 concerning equivalence between 3ISD and dominance in terms of inequality averse YSWFs satisfying PPTS we have to prove that if $\tilde{F}$ is obtained from F through a finite sequence of progressive transfers and/or favourable composite positional transfers, then $\tilde{W}(\tilde{F}) \geq \tilde{W}(F)$ for all YSWFs such that $v'(p) \leq 0$ and $v''(p) \geq \dot{0}$ (this is the only if part). And that if $\tilde{W}(\tilde{F}) \geq \tilde{W}(F)$ for all such evaluation functions then $\tilde{F}$ is necessarily obtained from F through a finite sequence of progressive transfers and/or favourable composite positional transfers (if part).

The structure of the proof is as follows: we first concentrate on *Elementary Composite Transfers (ECTs)*, that is the combination of a progressive and a regressive transfer, and we show that the FCPTs are indeed the subset of such transfers which together with simple progressive transfers induce 3ISD[39]. Then, we move to the general case of *Multiple Composite Transfers*

[39] Most of this proof follows from the proof of Proposition 1 in Zoli (1999) and Proposition

(MCT), that is combinations of more than a progressive and a regressive transfer (not necessarily obtainable through a sequence of ECTs) and we show that welfare dominance consistent with 3ISD selects only MCTs which can be obtained through a finite sequence of FCPTs.

Only if :

By definition of F and $\tilde{F}$, we have $\mu(F) = \mu(\tilde{F}) = \mu$. All YSWFs such that $v'(p) \leq 0$ satisfy PT, therefore a progressive transfer is not welfare worsening, we need to check for the impact of a *Favorable Composite Positional Transfer*. Consider a progressive transfer of a small amount $\delta > 0$ from a tiny fraction $dp > 0$ of the population at the $100(i+\rho)^{th}$ percentile of the distribution to a poorer fraction $dp > 0$ at the $100i^{th}$ percentile, and a regressive transfer of a small amount $\gamma > 0$ from the fraction dp of the population at the $100j^{th}$ percentile of the distribution to a richer fraction dp of individuals at the $100(j+\alpha)^{th}$ percentile, and $j > i$.[40] According to (2) we have that $\Gamma(\tilde{F}) = \Gamma(F)$ if and only if:

$$\frac{2}{\mu}\left\{\delta\left[(1-i-\rho)-(1-i)\right]-\gamma\left[(1-j-\alpha)-(1-j)\right]\right\}dp = 0 \qquad (8)$$

or, equivalently, if and only if

$$\alpha\gamma = \rho\delta. \qquad (9)$$

Next, we have $\Delta\tilde{W} = \tilde{W}(\tilde{F}) - \tilde{W}(F) \geq 0$ if and only if:

$$\delta\left[v(i)-v(i+\rho)\right]dp \geq \gamma\left[v(j)-v(j+\alpha)\right]dp \geq 0. \qquad (10)$$

Now, dividing the left hand side of (10) by $\rho\delta$ and the right hand side by $\alpha\gamma$ we get

$$\frac{1}{\rho}\left[v(i)-v(i+\rho)\right]dp \geq \frac{1}{\alpha}\left[v(j)-v(j+\alpha)\right]dp \qquad j > i \qquad (11)$$

that is, when $i+\rho < j+\alpha$, then $v''(p) \geq 0$ (if $v(p)$ is twice differentiable) or generally $v(p)$ is convex. Which shows that all inequality averse YSWFs satisfying the PPTS also satisfy the Strong version of PPTS. Therefore $\Delta\tilde{W} \geq 0$ for all these functions, which implies $\tilde{F} \succcurlyeq_3^{-1} F$.

7 in Zoli (1997).

[40] Notice that these transfers are more general than FCPTs, in that we do not restrict $i+\rho = j$.

If part: We divide this part of the proof into various steps. (Step 1) We first consider ECTs and show the restrictions on these transfers imposed by consistency with welfare dominance. What we select are *General FCPTs*, that is FCPTs where the donor is not necessarily the same, that is they satisfy $i < j \leq i+\rho < j+\alpha$. Then (Step 2) we prove that all General FCPTs can be obtained through a sequence of FCPTs. We will move afterwards to consider (Step 3) the restrictions on MCTs imposed by consistency with welfare dominance equivalent to 3ISD, and show that these restrictions are equivalent to those induced considering MCTs as obtained through sequences of FCPTs. This will conclude the proof. For completeness we add Step 4 where an algorithm for decomposing the obtained MCTs consistent with 3ISD into sequences of FCPTs is introduced. We conclude with an example of the application of the algorithm.

STEP 1: Consider distributions H and $F \in \mathcal{F}^n$ such that $\mu(F) = \mu(H) = \mu$, and $H \succcurlyeq_3^{-1} F$. Notice that 3ISD is a finer partial order than second degree ISD dominance (i.e. $H \succcurlyeq_2^{-1} F \Rightarrow H \succcurlyeq_3^{-1} F$). Since $H \succcurlyeq_2^{-1} F$ is satisfied by distributions H and F whenever H is obtained from F through a finite sequence of progressive transfers (see Fields and Fei, 1978), in order to satisfy $H \succcurlyeq_3^{-1} F$, but not $H \succcurlyeq_2^{-1} F$ it should be that H is obtained from F also through some regressive transfers. From the previous part we already know that sequences of FCPTs imply 3ISD, we need to show that together with progressive transfers they are the only transfers which allows to obtain this dominance result.

Consider all YSWFs satisfying $v'(p) \leq 0$ and $v''(p) \geq 0$. For all such functions it should be

$$\frac{v(i) - v(i+\rho)}{\rho} \geq \frac{v(j) - v(j+\alpha)}{\alpha} \geq 0 \qquad \text{for all } j > i, i+\rho < j+\alpha. \quad (12)$$

Then, consider a combination of a progressive and a regressive transfer both rank preserving[41], without loss of generality suppose the amount of the incomes transferred are δ, and $-\gamma$, where $\delta\gamma > 0$, that is δ and γ are either both positive or both negative. Moreover, suppose the individuals involved in the transfers are situated at the $100i^{th}$, $100(i+\tilde{\rho})^{th}$, $100j^{th}$ and $100(j+\alpha)^{th}$ percentiles, where $j > i$, but $i+\tilde{\rho}$ is not restricted to be lower than $j+\alpha$. The

[41] Since we are interested in selecting the most elementary transfers supporting the dominance condition, we restrict our attention to rank-preserving transfers. Under anonimity it can be shown that any set of transfers can be interpreted as a sequence of rank-preserving transfers.

combined welfare effect of these transfers $\Delta\tilde{W}$ evaluated in terms of YSWFs is given by $\Delta\tilde{W} = \delta\left[v(i) - v(i+\tilde{\rho})\right] dp - \gamma\left[v(j) - v(j+\alpha)\right] dp$. These transfers induce a welfare improvement iff

$$\delta\left[v(i) - v(i+\tilde{\rho})\right] \geq \gamma\left[v(j) - v(j+\alpha)\right]. \tag{13}$$

We need to check the restrictions on δ, γ, and $\tilde{\rho}$ in (13) imposed by (12). Notice that if δ and γ are positive then the progressive transfer is in favor of the poorer individual i,[42] while if δ and γ are negative, the poorer individual is the donor involved in a regressive transfer. Suppose $\tilde{\rho} < j + \alpha - i$, that is $\tilde{\rho}$ shows the same restriction as ρ in (12). We can rewrite (12) as

$$\frac{\alpha}{\rho}\left[v(i) - v(i+\rho)\right] \geq \left[v(j) - v(j+\alpha)\right] \geq 0, \tag{14}$$

if δ and γ are negative, from (13) we have $\frac{\delta}{\gamma}\left[v(i) - v(i+\tilde{\rho})\right] \leq \left[v(j) - v(j+\alpha)\right]$ which contradicts (14). It is possible to specify $v(p)$ such that (14) is satisfied[43] for $v(j) - v(j+\alpha) = 0$, and $v(i) > v(i+\rho)$, in which case according to (14) $v(i) - v(i+\tilde{\rho}) > 0$ which contradicts $\frac{\delta}{\gamma}\left[v(i) - v(i+\tilde{\rho})\right] \leq 0$.

Only if $\delta, \gamma > 0$ and if $\frac{\delta}{\gamma} \geq \frac{\alpha}{\rho} > 0$ then (14) is satisfied. It follows that the progressive transfer should be made in favor of the poorer of the individuals involved in the composite transfer. Moreover the condition $\frac{\delta}{\gamma} \geq \frac{\alpha}{\rho}$ could be reinterpreted, as $\delta\rho \geq \alpha\gamma$, which according to (9) means that the ECT should decrease the Gini index or at the limit leave the index unchanged. Since we are considering ECTs together with simple progressive transfers, then the condition boils down to $\delta\rho = \alpha\gamma$. A combination of rank-preserving transfers reducing the Gini coefficient can be seen as the sum of a progressive transfer involving the pair of individuals at the bottom of the distribution and a ECT leaving the Gini index unchanged. That is, we can consider $\delta = \delta_1 + \delta_2 > 0$, where $\delta_1, \delta_2 > 0$ such that $\delta_1\rho = \alpha\gamma$.

[42] With a little abuse of terminology we will call individual i the fraction dp of individuals at the $100i^{th}$ percentile. This does not in principle create any lack of consistency, if $dp = 1/n$, then the income associated to population at the $100i^{th}$ percentile is actually that belonging to the individual ranked i within the illfare ranked income profile.

[43] Consider for instance the function $v(p)$:

$$v(p) = \begin{cases} (j-p)^{\theta} + \beta & \text{if } p \leq j \qquad \theta > 2, \beta \geq 0 \\ \beta & \text{if } \qquad p > j. \end{cases}$$

We need now to show that $\tilde{\rho}$ could not be larger than $j+\alpha-i$, that is the progressive transfer within the ECT cannot involve a donor which is richer than the recipient in the regressive transfer. We call General FCPT the ECT where the progressive transfer occurs at the bottom and $i+\rho \leq j+\alpha$. Of course, given the previous result we can always think at a sequence of a progressive transfer from an individual at $i+\tilde{\rho} > j+\alpha$ to someone at $i+\rho \leq j+\alpha$, and a General FCPT where this last individual at $i+\rho$ is involved. Such a transfer will belong to the set of those shown being consistent with welfare dominance. What we need to consider is whether there exist transfers that cannot be interpreted in this way, but can improve welfare if the donor of the progressive transfer is at $i+\tilde{\rho} > j+\alpha$.

We will show that there exist classes of weight functions $v(p)$ (see (16)) satisfying $v'(p) \leq 0$, $v''(p) \geq 0$, such that all these transfers are welfare reducing.

Consider $i+\tilde{\rho} > j+\alpha$, let $i+\tilde{\rho}-(j+\alpha) = \sigma > 0$, and $j-i = \beta > 0$, therefore $\tilde{\rho} = \alpha+\beta+\sigma > 0$. Since the YSWFs with linear weighting functions belong to the set of welfare functions we consider (where $v''(p) = 0$) it is evident that the transfer in order to not reduce welfare should at least leave the Gini coefficient unchanged, or reduce it, that is

$$\delta\left(\alpha+\beta+\sigma\right) \geq \alpha\gamma.$$

Moreover, it should be such that the composite transfer cannot be considered as obtained through a sequence of a progressive transfer and a General FCPT satisfying $\delta\left(\alpha+\beta\right) = \alpha\gamma$ (that is such that $\sigma = 0$), it follows that we need to require that δ and γ are such that the Gini index is increased, that is $\delta < \frac{\alpha\gamma}{\alpha+\beta}$. We need therefore to considered all possible transfers such that

$$\frac{\alpha\gamma}{\alpha+\beta} > \delta \geq \frac{\alpha\gamma}{\alpha+\beta+\sigma}.$$

This condition can be formalized as

$$\delta\left(\alpha+\beta+\tau\sigma\right) = \alpha\gamma, \quad \text{where } \ 1 \geq \tau > 0. \tag{15}$$

Consider the following weighting function:

$$v(p) = \begin{cases} (j+\alpha)^{-\varepsilon}\left[1+\varepsilon\frac{j+\alpha-p}{j+\alpha}\right] & \text{if } 0 \leq p \leq j+\alpha \\ p^{-\varepsilon} & \text{if } \ p > j+\alpha \end{cases} \tag{16}$$

where $\varepsilon > 0$.

This function satisfies the restrictions on the weighting functions discussed above, it is continuous, $v'(p) < 0$, and $v''(p) \geq 0$. A composite transfer in order to lead to a welfare improvement should satisfy (13), which in our case could be rewritten as $\delta\left[v(j-\beta) - v(j+\alpha+\sigma)\right] \geq \gamma\left[v(j) - v(j+\alpha)\right]$. Substituting from (16) we have:

$$v(j-\beta) - v(j+\alpha+\sigma) = (j+\alpha)^{-\varepsilon}\left[1 + \varepsilon\frac{\alpha+\beta}{j+\alpha} - \left(1 + \frac{\sigma}{j+\alpha}\right)^{-\varepsilon}\right]$$

and $v(j) - v(j+\alpha) = (j+\alpha)^{-\varepsilon}\,\varepsilon\frac{\alpha}{j+\alpha}$.

After substituting from (15) we can check the dominance condition (13) considering the restriction on the extent of the transfers: $\delta\frac{(\alpha+\beta+\tau\sigma)}{\alpha} = \gamma$. It follows that $\Delta\tilde{W} \geq 0$ iff

$$\delta\,(j+\alpha)^{-\varepsilon}\left[1 + \varepsilon\frac{\alpha+\beta}{j+\alpha} - \left(1 + \frac{\sigma}{j+\alpha}\right)^{-\varepsilon}\right] \geq \delta\frac{(\alpha+\beta+\tau\sigma)}{\alpha}\,(j+\alpha)^{-\varepsilon}\,\varepsilon\frac{\alpha}{j+\alpha}.$$

After rearranging and simplifying, we get

$$1 - \left(1 + \frac{\sigma}{j+\alpha}\right)^{-\varepsilon} \geq \frac{\sigma}{j+\alpha}\varepsilon\tau.$$

Letting $\frac{\sigma}{j+\alpha} = k > 0$, we can rewrite

$$1 - \varepsilon\tau k \geq (1+k)^{-\varepsilon}. \tag{17}$$

If $\tau = 1$, the condition becomes $1 - \varepsilon k \geq (1+k)^{-\varepsilon}$, but indeed for all $k > 0$, $\varepsilon > 0$ we have $1 - \varepsilon k < (1+k)^{-\varepsilon}$ which contradicts the welfare dominance condition for all $j + \alpha > 0, \sigma > 0$. That is, if the composite transfer leaves the Gini index unchanged it turns out to be welfare reducing. In general for $1 \geq \tau > 0$ we can rearrange (17) such that the welfare dominance requires:

$$\frac{1 - (1+k)^{-\varepsilon}}{\varepsilon k} \geq \tau \tag{18}$$

for all $k > 0$, $\varepsilon > 0$. We now show that this condition is not satisfied. Let

$$G(\infty) = \lim_{\varepsilon \to \infty} G(\varepsilon) = \lim_{\varepsilon \to \infty} \frac{1 - (1+k)^{-\varepsilon}}{\varepsilon k} = 0,$$

it is evident that, since $1 \geq \tau > 0$, then $\tau > G(\infty) = 0$ which contradicts (18).

We have found the elementary set of transfers which are welfare improving, they are rank-preserving combinations of a progressive transfer of the amount δ incurring between individuals at $100i^{th}$ and $100(i+\rho)^{th}$ percentiles, and a regressive transfer γ involving individuals at $100j^{th}$ and $100(j+\alpha)^{th}$ percentiles, these transfers are General FCPTs, the Gini coefficient is left unchanged and $i < j \leq i + \rho < j + \alpha$.

STEP 2: We need to prove that such transfers are indeed combinations of FCPTs.

Lemma 23 *Any General FCPT can be obtained through a sequence of FCPTs.*

Proof: Let $j - i = a_1$, $i + \rho - j = a_2$, $j + \alpha - i - \rho = a_3$, then the condition on the Gini coefficient requires:

$$\delta (a_1 + a_2) = \gamma (a_2 + a_3). \tag{19}$$

Denote with (δ, i) the transfer of an amount δ occurring to an individual at the $100i^{th}$ percentile. The set of net transfers is

$$T = [(\delta, i), (-\gamma, i + a_1), (-\delta, i + a_1 + a_2), (\gamma, i + a_1 + a_2 + a_3)].$$

Consider now the following FCPTs (we omit the ranks for simplicity):

$$\hat{T} = \left[(\gamma \frac{a_2 + a_3}{a_1 + a_2 + a_3}); (-\gamma); (0); (\gamma \frac{a_1}{a_1 + a_2 + a_3}) \right]$$

and

$$\tilde{T} = \left[(\delta \frac{a_3}{a_1 + a_2 + a_3}); (0); (-\delta); (\delta \frac{a_1 + a_2}{a_1 + a_2 + a_3}) \right].$$

Notice that all these transfers are leaving the Gini index unaffected, indeed $(\gamma \frac{a_2 + a_3}{a_1 + a_2 + a_3}) a_1 = (\gamma \frac{a_1}{a_1 + a_2 + a_3}) (a_2 + a_3)$ in $\hat{T}$, and similarly in $\tilde{T}$, and also since those leading to T were rank-preserving these are also. After substituting from (19) we notice that $\gamma\frac{a_2+a_3}{a_1+a_2+a_3} = \delta\frac{a_1+a_2}{a_1+a_2+a_3}$, which makes evident that $\hat{T}+\tilde{T}=T$. The solution does not depend on whether $\delta \lesseqqgtr \gamma$ or $a_3 \lesseqqgtr a_1$. ■

Therefore the FCPTs are the elementary transfers consistent with YSWFs dominance when $v'(p) \leq 0$ and $v''(p) \geq 0$, then, given the equivalence with 3ISD the FCPTs are the simplest basis of this dominance condition.

STEP 3: We need now to show that FCPTs are the only basis of 3ISD (together of course with simple progressive transfers). This requires to show that there is no MCT (multiple composite transfer) consistent with 3ISD which cannot be obtained as a sequence of FCPTs. For this purpose we modify the welfare dominance condition equivalent to 3ISD. Next lemma can be seen as a result alternative to that in Proposition 7. Consider the following set of weighting functions:

$$v_q(p) = \begin{cases} \kappa(q-p) & \text{if } p < q \\ 0 & \text{if } p \geq q. \end{cases}$$

where κ is an arbitrary constant $\kappa > 0$. Denote $\tilde{\mathcal{W}}_q$ the set of YSWFs such that $v(p) = v_q(p)$ for all $\kappa > 0$, $q \in (0,1]$.

Lemma 24 *Consider distributions $F, G \in \mathcal{F}$, $\tilde{W}(F) \geq \tilde{W}(G)$ for all YSWFs in $\tilde{\mathcal{W}}_q$ if and only if $F \succcurlyeq_3^{-1} G$.*

Proof: Let $\Delta\tilde{W} = \tilde{W}(F) - \tilde{W}(G) = \int_0^1 v_q(p)[F^{-1}(p) - G^{-1}(p)]dp$, given the definition of $v_q(p)$, $\Delta\tilde{W} \geq 0$ is equivalent to

$$\int_0^q v_q(p)[F^{-1}(p) - G^{-1}(p)]dp \geq 0 \quad \text{for all } q \in (0,1].$$

Integrating by parts we get

$$\int_0^q \kappa[F_2^{-1}(p) - G_2^{-1}(p)]dp + \left[v_q(p)[F_2^{-1}(p) - G_2^{-1}(p)]\right]_0^q \geq 0 \quad \forall q$$

that is $\kappa \int_0^q [F_2^{-1}(p) - G_2^{-1}(p)]dp \geq 0 \;\; \forall q \in (0,1]$. Which is equivalent to $F_3^{-1}(q) - G_3^{-1}(q) \geq 0 \;\; \forall q \in (0,1]$, that is $F \succcurlyeq_3^{-1} G$. ■

We have therefore found a new equivalence between 3ISD and YSWFs dominance, where weighting functions are not differentiable, but are both non-negative, non increasing and convex, as in the case of differentiable functions. We will now use this class of functions in order to investigate the restrictions on the set of transfers supporting 3ISD. Our aim is to show that all these transfers can be obtained as sequences of FCPTs, the elementary transfers which we have shown being implied by third 3ISD.

Consider a set of transfers such that, the associated vector of net income changes is denoted $T = [(t_1, p_1); (t_2, p_2); (t_3, p_3)..(t_i, p_i)...(t_n, p_n)]$ with $n \geq 3$, where t_i is the change in income of a small fraction of population at the $100p_i^{th}$ percentile. Let $p_i < p_{i+1}$, and $t_i \lesseqgtr 0$, and obviously $\sum_{i=1}^{n} t_i = 0$.

We now investigate which conditions should this vector of transfers satisfy in order to be consistent with welfare dominance for all $\tilde{W} \in \tilde{\mathcal{W}}_q$ and therefore also with 3ISD.

Restrict attention to $v_q(p)$ where $p_{j+1} \geq q > p_j$. In order for T to not lead to a welfare decrease, it should be that

$$\sum_{i=1}^{n} t_i v_q(p_i) \geq 0 \Leftrightarrow \kappa \sum_{i=1}^{j} t_i(q - p_i) \geq 0,$$

that is $\sum_{i=1}^{j} t_i(q - p_i) \geq 0$, for all q such that $1 \geq p_{j+1} \geq q > p_j > 0$.

Since the above condition should be satisfied for all $q \in (0, 1]$, it follows that

$$\begin{aligned} S(q) &= \sum_{i=1}^{j} t_i(q - p_i) \geq 0 \\ \forall\ q \text{ such that } 1 &\geq p_{j+1} \geq q > p_j > 0, \text{ and } \forall j = 1, 2, ..n \end{aligned} \quad (20)$$

where $p_{n+1} = 1$. We call this condition A and define $S(q) = 0$ if $p_1 > q > 0$.

Let $q = 1$, then, from condition A follows $\sum_{i=1}^{n} t_i(1 - p_i) \geq 0$, recalling that $\sum_{i=1}^{n} t_i = 0$, then $\sum_{i=1}^{n} t_i p_i \leq 0$, that is condition A requires that the Gini coefficient should not be increased by the transfers.

Since we are restricting our attention to composite transfers forming the basis of 3ISD, then any set of transfers reducing the Gini index could be considered as a composition of a progressive transfer and transfers leaving unchanged the Gini. We follow therefore the same line of reasoning as for the ECT as discussed in Step 1, and require as minimal condition that the Gini index is left unchanged, therefore we have condition B:

$$\sum_{i=1}^{n} t_i p_i = 0. \quad (\mathbf{B})$$

Our aim is to make explicit the restrictions, implied by A and B, and then check whether sequences of FCPTs lead exactly to the same restrictions.

First notice that from A and B:

Remark 25 $t_1 > 0$, $t_n > 0$.

Proof: The first consideration follows from A when we let $p_2 \geq q > p_1$, in this case in order to have $S(q) \geq 0$ (as in A) for all $q \in (p_1, p_2]$ it must be that $t_1 > 0$. While the second result ($t_n > 0$) follows from A, B and $\sum_{i=1}^{n} t_i = 0$. Let $p_{n-1} < q < p_n$, then $\sum_{i=1}^{n-1} t_i(q - p_i) \geq 0$, since from $\sum_{i=1}^{n} t_i = 0$ we have $\sum_{i=1}^{n-1} t_i = -t_n$ then $-t_n q \geq \sum_{i=1}^{n-1} t_i p_i$, adding $t_n p_n$ to both sides we get $t_n(p_n - q) \geq \sum_{i=1}^{n} t_i p_i = 0$, that is $t_n > 0$ (since $t_i \neq 0$ for all i). ■

Next lemma shows how we can partition the interval $\mathcal{I} \equiv [p_1, p_n]$ into sub intervals where $S(q) > 0$. The condition $S(q) > 0$ is important in order to show that sequences of FCPTs are imposing exactly the same restrictions as in condition A, B and in the previous remark.

Lemma 26 *The interval* $\mathcal{I} \equiv [p_1, p_n] \subseteq (0, 1]$, *could be partitioned into at most* m *subintervals* $\mathcal{I}_j \equiv [p_{s_j}, p_{r_j}]$ $j = 1, 2, ..m$ $(m = (n-1)/2$ *if* n *is odd or* $m = n/2 - 1)$ *where* $r_j - s_j \geq 2$, *such that either* $r_j = s_{j+1}$ *or* $r_j + 1 = s_{j+1}$, *and* $r_m = n$, $s_1 = 1$. *For all* $\mathcal{I}_j$ *we have:*
a) $S(p_{s_j}) = S(p_{r_j}) = 0$
b) $S(q) > 0$ *for all* $q \in (p_{s_j}, p_{r_j})$ *and for all* $j = 1, 2, ..m$.
c) $S(q) = 0$ *for all* $q \in [p_n, 1]$
d) $t_{s_j} > 0, t_{r_j} > 0$ *for all* $j = 1, 2, ..m$
e) if $r_j + 1 = s_{j+1}$ *then* $\sum_{i=1}^{r_j} t_i p_i = 0$
f) if $r_j = s_{j+1}$ *then we can split* $t_{r_j} = t'_{r_j} + t''_{r_j}$, *such that* $\sum_{i=1}^{r_j - 1} t_i p_i + t'_{r_j} p_{r_j} = 0$.

Proof: Notice that by definition $S(q) = 0$ for all q such that $p_1 \geq q > 0$, while, since $t_1 > 0$, $S(q) > 0$ for all q such that $p_2 \geq q > p_1$. Moreover (part **c**), $S(q) = \sum_{i=1}^{n} t_i(q - p_i) = \sum_{i=1}^{n} t_i q - \sum_{i=1}^{n} t_i p_i = 0$ for all q such that $1 \geq q \geq p_n$ because we consider transfers such that $\sum_{i=1}^{n} t_i = 0 = \sum_{i=1}^{n} t_i p_i$ (B). Since $t_n > 0$ let q be such that $p_n > q \geq p_{n-1}$, then (part **b**)

$$\begin{aligned} S(q) &= \sum_{i=1}^{n-1} t_i(q - p_i - p_n + p_n) \\ &= \sum_{i=1}^{n-1} t_i(p_n - p_i - (p_n - q)) + t_n(q - p_n) + t_n(p_n - q) \\ &= \sum_{i=1}^{n} t_i(p_n - p_i - (p_n - q)) + t_n(p_n - q) \\ &= \sum_{i=1}^{n} t_i q - \sum_{i=1}^{n} t_i p_i + t_n(p_n - q) = t_n(p_n - q) > 0. \end{aligned}$$

Therefore, $S(q)$ is strictly positive for all q such that $p_n > q \geq p_{n-1}$, and $p_2 \geq q > p_1$, and in general in some interval of (p_1, p_n).

The question is whether $S(q) = 0$ for some $q \in (p_2, p_{n-1})$, if not $S(q) > 0$ in all $q \in (p_1, p_n)$.

Suppose $S(p_s) = 0$, for some $p_s \in (p_2, p_{n-1})$, that is $\sum_{i=1}^{s} t_i(p_s - p_i) = 0$, then consider $p_s < q < p_{s+1}$, it should be $\sum_{i=1}^{s} t_i(q - p_i) \geq \sum_{i=1}^{s} t_i(p_s - p_i) = 0$, that is $\sum_{i=1}^{s} t_i(q - p_s) \geq 0$, from which, since $q - p_s > 0$, we have $\sum_{i=1}^{s} t_i \geq 0$.

In order to know the behavior of $S(q)$ for $p_s < q < p_{s+1}$, we are facing two possible alternatives either (i) $\sum_{i=1}^{s} t_i = 0$ or (ii) $\sum_{i=1}^{s} t_i > 0$. Consider the first alternative (i): $\sum_{i=1}^{s} t_i = 0$. In this case $S(q) = 0$ for all $q \in [p_s, p_{s+1}]$. The reason is that, since $\sum_{i=1}^{s} t_i(q - p_i) = 0$, then $\sum_{i=1}^{s} t_i = 0$ implies $\sum_{i=1}^{s} t_i p_i = 0$, therefore the condition $\sum_{i=1}^{s} t_i(q' - p_i)$ will be always satisfied for all q' such that $p_s \leq q' \leq p_{s+1}$. In this case $r_j + 1 = s_{j+1}$. It is important to notice that $\sum_{i=1}^{s} t_i p_i = 0$ means that the transfers leading to the income gaps t_i for $i = 1, 2, ..s$ are set such that the Gini coefficient is left unchanged (part **e**).

If this is the case then it is evident that $t_{s+1} > 0$ in order to have $S(q) \geq 0$, for all q such that $q > p_{s+1}$ (part **d**). Indeed since $t_{s+1} > 0$, then $S(q) > 0$ for $p_{s+2} \geq q > p_{s+1}$ (part **b**).

Because by definition $t_i \lesseqgtr 0$ it is clear that $S(q)$ cannot be 0 in two or more adjacent intervals. Therefore it should be that $S(q) > 0$, for $q \in [p_{s-1}, p_s)$ (part **b**). From this we have that $S(q)$ is decreasing in the interval $[p_{s-1}, p_s)$, in which case consider q and q' such that $p_{s-1} < q < q' < p_s$ we have $\sum_{i=1}^{s-1} t_i(q - p_i) > \sum_{i=1}^{s-1} t_i(q' - p_i)$, which gives $0 > \sum_{i=1}^{s-1} t_i(q' - q) = (q' - q) \sum_{i=1}^{s-1} t_i$ that is $0 > \sum_{i=1}^{s-1} t_i$. Then, since we have shown that $\sum_{i=1}^{s} t_i \geq 0$, it follows $t_s > 0$. We can therefore conclude that whenever $S(p_s) = 0$, the associated t_s is positive (part **d**).

Consider now the alternative (ii): $\sum_{i=1}^{s} t_i > 0$. Notice that $\sum_{i=1}^{s-1} t_i < 0$, it is therefore possible to consider t_s as a composite transfer such that $t_s = t_s' + t_s'' > 0$, where both terms are positive and $t_s' = -\sum_{i=1}^{s-1} t_i$. In which case we can interpret the amount transferred t_s as obtained from two transfers, the first such that $\sum_{i=1}^{s} t_i = 0$ implies $\sum_{i=1}^{s} t_i p_i = 0$, and the second as an effect of a progressive transfer from individuals placed above p_s (part **f**).

Concluding, the percentiles where $S(q) = 0$ partition $[p_1, p_n]$ into open intervals where $S(q) > 0$.

Suppose that whenever $S(p_s) = 0$ for some $p_s \in (p_2, p_{n-1})$ we have $\sum_{i=1}^{s} t_i > 0$ (this is the case in which we can have the largest possible number of sub intervals), it follows that since $S(q)$ is increasing at the beginning of the interval where $S(q) > 0$ and decreases in the final part. Thus, we need, at least, to have a position within the subinterval. Therefore the maximum number of subgroups is $(n-1)/2$ if n is odd otherwise $n/2-1$, and in any case, we need at least $n \geq 3$ as stated initially. ■

Notice that because of parts e and f in previous lemma, then: *for any suitable subinterval the associated net transfers do not induce any change in the Gini coefficient.*

From previous lemma we know that is possible to partition any set of net transfers which are consistent with all YSWFs in $\tilde{\mathcal{W}}_q$, into non overlapping subsets such that the Gini index is left unchanged.

We need to show that FCPTs are the proper basis of these subsets of transfers, that is all such transfers can be obtained through finite sequences of FCPTs.

Consider a FCPT where the combined progressive and regressive transfer have the same portion of the population as donor.

As a result of such a transfer we have $T_{1,2,3} = [(t_1, p_1); (t_2, p_2); (t_3, p_3)]$ where $p_1 < p_2 < p_3$, $t_1 > 0$, $t_2 < 0, t_3 > 0$, $\sum_{i=1}^{3} t_i = 0$, and $\sum_{i=1}^{3} t_i p_i = 0$ because the transfer leaves the Gini index unchanged. Moreover let $S_{1,3}(q) = \sum_{i=1}^{j} t_i(q - p_i)$ for all q such that $p_j \leq q \leq p_{j+1}$, for $j = 1, 2$. Notice that $S_{1,3}(q) > 0$ for all $q \in (p_1, p_3)$, indeed $0 < S_{1,3}(q) < S_{1,3}(q')$ for all q, q' such that $p_1 < q < q' < p_2$ and $S_{1,3}(q''') > S_{1,3}(q'') > 0$ for all q', q'' such that $p_2 > q' > q'' > p_3$. That is $S_{1,3}(p_2) = t_1(p_2 - p_1)$ is the maximum of $S_{1,3}(q)$.

Consider now the generic set $T = [(t_1, p_1); (t_2, p_2); (t_3, p_3)..(t_i, p_i)...(t_n, p_n)]$ introduced before, and suppose w.l.o.g. that $S(q) > 0$ for all $q \in (p_1, p_n)$. We can think of the set as one of the subsets associated to the subintervals investigated in the previous lemma.

The conditions which we need to add to $S(q) > 0$ for all $q \in (p_1, p_n)$ in order to lead to welfare dominance are $\sum_{i=1}^{n} t_i = 0$, $\sum_{i=1}^{n} t_i p_i = 0$ and $t_1 > 0, t_n > 0$. Moreover, remember that $S(q) = \sum_{i=1}^{j} t_i(q - p_i) = \sum_{i=1}^{j} t_i q - \sum_{i=1}^{j} t_i p_i$, then $dS(q)/dq = \sum_{i=1}^{j} t_i$.

Make the following considerations:

a) a FCPT is such that $t_1 > 0$, $t_2 < 0, t_3 > 0$, therefore for any sequence of such transfers we have that the obtained extreme values are positive.

b) a FCPT leaves Gini index unaffected, then any sequence of FCPTs leaves it unaffected, i.e. $\sum_{i=1}^{n} t_i p_i = 0$

c) a FCPT inducing $T_{k,j,l} = [(t_k, p_k); (t_j, p_j); (t_l, p_l)]$ gives $S_{k,l}(q) > 0$ for $q \in (p_k, p_l)$ and $S(q) = 0$ for $q \in (0, p_k] \cup [p_l, 1]$, therefore any combination of this FCPT with, suppose a FCPT inducing $T_{r,z,h}$, will give $S(q) = S_{k,j,l}(q) +$

$S_{r,z,h}(q) > 0$ for $q \in (\min(p_k, p_r), \max(p_l, p_h))$ and $S(q) = 0$ otherwise.

d) For a FCPT inducing $T_{k,j,l} = [(t_k, p_k); (t_j, p_j); (t_l, p_l)]$, $dS(q)/dq = t_k$ if $p_k < q < p_j$, and $dS(q)/dq = t_k + t_j$ if $p_j < q < p_l$, and in general any sequence of FCPTs gives $dS(q)/dq = \sum_{i=1}^{j} t_i$ if $p_j < q < p_{j+1}$.

We have shown that not only combinations of FCPTs satisfy the conditions for welfare dominance but that they induce conditions which are exactly the same (i.e. the conditions over set of welfare improving transfers are not weaker than those satisfied by combinations of FCPTs)[44] which means that the FCPTs are the basis of the welfare improving transfers, and therefore of 3ISD. Which concludes the proof of the proposition. ■

STEP 4: Finally we show that the decomposition is implementable. We suggest an algorithm which shows a procedure for decomposing exhaustively a set of net transfers consistent with 3ISD into net transfers obtained through a sequence of FCPTs.

Consider the vector of net transfers

$$T = [(t_1, p_1); (t_2, p_2); (t_3, p_3)..(t_i, p_i)...(t_n, p_n)].$$

Let $N = \{p_i : t_i < 0\}$, $P = \{p_i : t_i > 0\}$, $N(p_h) = \{p_i \in N : p_i > p_h\}$, and $P(p_h) = \{p_i \in P : p_i > p_h\}$.

i) Consider t_i such that $p_i = \min N$. Compare t_i with t_j such that $p_j = \min P$ and t_k such that $p_k = \min P(p_i)$. Notice that $p_j < p_i < p_k$, the second inequality is obtained by definition, while the first come from the fact that $p_j = p_1$ in this first stage. We now face three alternative situations (a), (b), and (c).

ii) (iia) If $-t_i \frac{p_k - p_i}{p_k - p_j} \leq t_j$, and $-t_i \frac{p_i - p_j}{p_k - p_j} \leq t_k$, then implement the FCPT

$$T^1_{j,i,k} = \left[(-t_i \frac{p_k - p_i}{p_k - p_j}, p_j); (t_i, p_i); (-t_i \frac{p_i - p_j}{p_k - p_j}, p_k)\right]$$

which eliminates the impact of t_i from T. Otherwise consider $\min(t_j (p_i - p_j),$ $t_k (p_k - p_i))$, suppose $t_j (p_i - p_j)$ is the minimum (iib), and implement the following FCPT

$$T^2_{j,i,k} = \left[(t_j, p_j); (-t_j \frac{p_k - p_j}{p_k - p_i}, p_i); (t_j \frac{p_i - p_j}{p_k - p_i}, p_k)\right]$$

[44] This is why we needed the previous lemma, we have been able to show that any interval can be partitioned into subintervals where $S(q) > 0$ and the Gini coefficient is left unchanged by the net transfers within any subinterval. Otherwise the condition $S(q) \geq 0$, would have been weaker than $S(q) > 0$ obtained implementing sequences of FCPTs.

eliminating the impact of t_j from T. Alternatively (iic) consider the FCPT

$$T^3_{j,i,k} = \left[(t_k \frac{p_k - p_i}{p_i - p_j}, p_j); (-t_k \frac{p_k - p_j}{p_i - p_j}, p_i); (t_k, p_k) \right]$$

eliminating the impact of t_k from T.

iii) Notice that in both cases after the FCPT (a) $\min P < \min N$, and (b) $\max P > \max N$.

iv) iterate the procedure.

The claim in iii) is important, because it ensures that after having eliminated the effect of a FCPT in (ii) on T, the remaining net vector satisfies the initial conditions, we now prove it.

Claim 27 *After the FCPT in (ii):*
(a) $\min P < \min N$, *and*
(b) $\max P > \max N$.

Proof: We will prove claims (a) and (b) for all FCPTs in (ii).

We prove claim (a) for (iia):

Suppose we implement $T^1_{j,i,k}$, then (a) is violated only if $t_j = -t_i \frac{p_k - p_i}{p_k - p_j}$, $i = j + 1$, and $t_{i+1} < 0$. This cannot happen because in this case we have $t_j (p_k - p_j) + t_i (p_k - p_i) = 0$. So either $k = i + 1$, in which case this violates $S(p_k) > 0$, since it is $S(p_k) = t_j (p_k - p_j) + t_i (p_k - p_i) = 0$, or $k > i + 1$, then $S(p_k) > 0$ if and only if $t_{i+1} (p_k - p_{i+1}) + ... + t_{k-1} (p_k - p_{k-1}) > 0$ but since by definition $p_k = \min P(p_i)$ then all the values $t_{i+1}, t_{i+2}, ...t_{k-1}$ are negative which contradicts $S(p_k) > 0$.

Consider claim (b) for (iia).

If we implement $T^1_{j,i,k}$, then (b) is violated if and only if $p_k = \max P$, $t_k = -t_i \frac{p_i - p_j}{p_k - p_j}$ and $t_{k-1} < 0$, for $k - 1 > i$. We show that this is not the case.

First, notice that $T^1_{j,i,k}$ is implemented if $t_k (p_k - p_j) = -t_i (p_i - p_j)$, and if $-t_i \frac{p_k - p_i}{p_k - p_j} \leq t_j$, which gives $t_k (p_k - p_i) \leq t_j (p_i - p_j)$. Then, consider the condition $\sum_{l=1}^{k} t_l p_l = 0$ concerning invariance of the Gini coefficient over the set of starting net transfers. Notice that by definition $t_l > 0$ if $i > l \geq j$ and $l = k$, while $t_l < 0$ if $k > l \geq i$, then the minimum value that $\sum_{l=i}^{k-1} -t_l p_l$ could get is $\sum_{l=i}^{k-1} -t_l p_i$, that is $\sum_{l=i}^{k-1} t_l p_l \leq p_i \sum_{l=i}^{k-1} t_l < 0$. On the other hands, the maximum value that $\sum_{l=j}^{i-1} t_l p_l + t_k p_k$ (the weighted average of positive t_l) could reach is when all t_l, for all l such that $i > l > j$ have weight p_i, or when $i = j + 1$, that is $\sum_{l=j+1}^{i-1} t_l p_i + t_k p_k + t_j p_j$, from which $p_i \sum_{l=j+1}^{i-1} t_l \geq \sum_{l=j+1}^{i-1} t_l p_l$.

It is now clear that, since the condition $\sum_{l=1}^{k} t_l p_l = 0$ could be rewritten as $\sum_{l=i}^{k-1} t_l p_l + \sum_{l=j}^{i-1} t_l p_l = -t_k p_k - t_j p_j$ this implies that, since as shown above $p_i \sum_{l=i}^{k-1} t_l + p_i \sum_{l=j+1}^{i-1} t_l \geq \sum_{l=i}^{k-1} t_l p_l + \sum_{l=j}^{i-1} t_l p_l$, we have:

$$p_i \sum_{l=j+1}^{k-1} t_l \geq -t_k p_k - t_j p_j.$$

Notice, that also the condition $\sum_{l=1}^{k} t_l = 0$ should be satisfied, from which $t_j + \sum_{l=j+1}^{i-1} t_l + \sum_{l=i}^{k-1} t_l + t_k = 0$ which gives $\sum_{l=j+1}^{k-1} t_l = -t_k - t_j$. Substituting into the previous condition we get

$$p_i(t_k + t_j) \leq t_k p_k + t_j p_j, \tag{21}$$

which is consistent with the condition $t_k (p_k - p_i) \leq t_j (p_i - p_j)$ stated at the beginning, only if $t_k (p_k - p_i) = t_j (p_i - p_j)$. In this case $\sum_{l=j+1}^{i-1} t_l = 0 = \sum_{l=i+1}^{k-1} t_l$, that is the only $t_l \neq 0$ are t_k, t_j, and t_i, and the FCPT is the final of the sequence originated by the algorithm.

Consider now claim (a) for (iib).

For $T_{j,i,k}^2$ to violate (a) it should be that, given the definition of t_i, $j + 1 = i$. Since $t_j (p_i - p_j) < t_k (p_k - p_i)$ we have that at least $-t_i (p_k - p_i) > t_j (p_k - p_j)$, remember that by definition $t_l < 0$ for all l such that $k > l > i$ if there are any, then it is evident that the condition $S(p_k) > 0$ is violated, since $0 > t_j (p_k - p_j) + t_i (p_k - p_i)$ and by definition all t_l such that $k > l > i$ are negative, therefore it should be the case that $j + 1 < i$ and $t_l > 0$ for all l such that $i > l > j + 1$.

Claim (b) for (iib).

Notice that $T_{j,i,k}^2$ can violate condition (b), only if $t_j \frac{(p_i - p_j)}{(p_k - p_i)} = t_k$ and $p_k = \max P$, if this is the case then $-t_i \geq \frac{(p_k - p_j)}{(p_i - p_j)} t_k = \frac{(p_k - p_j)}{(p_k - p_i)} t_j$. We just need to follow the same considerations adopted when we proved that $T_{j,i,k}^1$ cannot violate (b). In which case we get (21), which is consistent with $t_j (p_i - p_j) = t_k (p_k - p_i)$ only if $\sum_{l=j+1}^{i-1} t_l = 0 = \sum_{l=i+1}^{k-1} t_l$. The same considerations made before for claim (b) hold if applied to (iia).

Claim (a) for (iic).

Consider now $T_{j,i,k}^3$, the implementation of this FCPT violates (a) only if $t_k \frac{p_k - p_i}{p_i - p_j} = t_j$, $t_i < -t_k \frac{p_k - p_j}{p_i - p_j}$, and there exist $t_l < 0$ for all l such that

$i < l < k$. It follows $t_i < -t_j \frac{p_k - p_j}{p_k - p_i}$, which gives $t_i (p_k - p_i) + t_j (p_k - p_j) < 0$ which violated $S(p_k) > 0$ as discussed for the $T^2_{j,i,k}$ violation of (a).

Claim (b) for (iic).

Finally we have to check that implementation of $T^3_{j,i,k}$ cannot violate (b). This could happen only if $p_k = \max P$ and there exist $t_l < 0$ for all l such that $i < l < k$, or if $t_i < -t_k \frac{p_k - p_j}{p_i - p_j}$, the last condition gives $-t_i (p_i - p_j) > t_k (p_k - p_j)$. Following the same argument as in the previous discussion of violation of (b) we get the condition $p_i(t_k + t_j) \leq t_k p_k + t_j p_j$ as in (21), remember that we have implemented $T^3_{j,i,k}$ because $t_j (p_i - p_j) \geq t_k (p_k - p_i)$, which lead us to the same conclusions as in the previous cases. ■

It is finally time to conclude. What we have suggested is an algorithm such that for any set of net transfers T which are consistent with welfare dominance in terms of YSWFs in $\tilde{\mathcal{W}}_q$, and therefore consistent with 3ISD, allows to recover a sequence of FCPTs which originates it. The different steps of the decomposition are such that the resulting net transfers still satisfy the requirements of consistency with YSWFs in $\tilde{\mathcal{W}}_q$. Moreover, what we have left after the first stage is a set of net transfers including at the most $n - 1$ elements. Continuing with the procedure the whole set of transfers could be therefore covered in a finite number of steps.

The following example make use of the described algorithm:

Example 28 *Consider*

$$T = [(2, .1); (-3, .2); (2, .3); (5, .4); (-8, .5); (-1, .6); (0, .7); (3, .8)]$$

which satisfies the above conditions for welfare dominance. It can be decomposed through the following FCPTs:

$$\begin{aligned} T_{1,2,3} &= [(1.5, .1); (-3, .2); (1.5, .3)] \\ T_{1,5,8} &= [(.5, .1); (-7/6, .5); (2/3, .8)] \\ T_{3,5,8} &= [(.5, .3); (-5/6, .5); (1/3, .8)] \\ T_{4,5,8} &= [(4.5, .3); (-6, .5); (1.5, .8)] \\ T_{4,6,8} &= [(.5, .4); (-1, .6); (.5, .8)]. \end{aligned}$$

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