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Asymmetric Contests with General Technologies

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Abstract

We investigate the Nash equilibria of asymmetric, winner-take-all, imperfectly discriminating contests, focussing on existence, uniqueness and rent dissipation. When the contest success function is determined by a production function with decreasing returns for each contestant, equilibria are unique. If marginal product is also bounded, limiting total expenditure is equal to the value of the prize in large contests even if contestants differ. Partial dissipation can occur only when infinite marginal products are permitted. Our analysis relies heavily on the use of 'share functions' and we discuss their theory and application. Increasing returns typically introduces multiple equilibria and requires an extension of share functions to correspondences. We describe the appropriate theory and apply it to the characterisation of all equilibria of contests employing the asymmetric generalisation of a widely-used symmetric contest success function.

Keywords: Contests, rentseeking, noncooperative games, share functions, share correspondences.

JEL classifications: C72, D72

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1 Introduction

There is now a large and growing literature on the theory and application of contests. At its simplest a contest is a game in which players expend effort to increase their chance of winning a prize. An early contribution by Tullock[29] assumes identical risk neutral contestants who expend resources competing for an exogenous prize. Contestant i wins the prize with probability $p_i = x_i / \sum_j x_j$, where x_i is the level of resources expended by i. This formulation of the 'contest success function' implies that, if at least one contestant expends a strictly positive level of resources, the prize will be won by one of the players with certainty. However, the outcome is not well-defined if $x_i = 0$ for all contestants, and in that case it is typically assumed that there is no contest, and the prize is not won. Tullock was principally interested in the extent to which monopoly rents were dissipated in the act of rentseeking. Since Tullock's contribution, the basic model has been extended in many directions. For example, Hillman and Katz[11], Hillman and Samet[13], Skaperdas and Gan[26] and Konrad and Schlesinger[15] examine the implications of strict risk aversion among contestants. Perez-Castrillo and Verdier[20] explore the implications of a contest success function of the form $p_i = x_i^r / \sum_j x_j^r$ where $r > 0$ is an exogenous discrimination factor, and Nti[19] considers contests in which there is a positive probability that no player wins the rent. Corchon[3] allows prior probabilities that the prize will go to one of the agents even if no resources are expended on the contest. The resources expended by contestants modify the prior probability. Dixit[5] analyses a Stackelberg formulation in which one player is able to precommit.

An important motivation for these and various other extensions is the recognition that a wide variety of problems are contests of some kind. They have been used to model rent-seeking (Tullock[29], Hillman[10]), conflict and appropriation (Garfinkel and Skaperdas[9]), R&D and patent races (Loury[16], Beath et al[1], Nti[19]), nonprice competition (Huck et al[14]), the choice between lobbying and litigation (Rubin et al[23]), the periodic contests between cities and countries to host prestigious events such as the Olympic Games (Corchon[3]) and status games (Frank[6], Frank and Cook[7]). There remain many further problems that invite modelling as contests - for example, the competition between universities for students, and the competitive application for grants by researchers.

However, in spite of the many extensions and applications of the basic framework - ably surveyed by Nitzan[17] and Tollison[28] - and the prospects of many further applications, certain fundamental modelling issues have been only incompletely addressed. For example, the characterisation of Nash equilibria when the technology for transforming effort into probability of winning the contest exhibits increasing returns has not been investigated for heterogeneous contests. Another issue arises from the finding in the early literature on rent-seeking that with free entry the whole rent will be dissipated in rent seeking at least when potential contestants have access to the same linear 'technology'. The claim that this appears to predict higher rent-seeking expenditures than are typically observed has been christened the "Tullock paradox" by Riley[22] and motivates a search for modifications leading to partial rent dissipation. Potential candidates are high entry costs, modified technologies and heterogeneous contestants. One possible reason why these issues remain unaddressed is the difficulty of handling non-identical players by conventional means, which treat Nash equilibrium as a fixed point of the best response mapping. This entails working in a space of dimension equal to the number of players. In this paper, we adopt an alternative approach introduced in Cornes and Hartley [4] which allows us to work entirely with functions of a single variable, considerably simplifying the analysis. A subsidiary aim of this paper is to illustrate the power of this technique, whose scope for application to this area is by no means exhausted in our discussion.

The present paper follows Perez-Castrillo and Verdier[20] and most of the literature in confining attention to contests played by risk neutral contestants. In return for this self-imposed restriction, we are able to focus on the role played by the technology that transforms efforts into winning probabilities. In particular we scrutinise the implications of representing this technology by a homogeneous function, as suggested by Tullock and adopted by others.

Section 2 describes the basic contest. Section 3 analyses the simplest situation in which the technology available to each contestant is linear and also allows us to introduce our approach in a relatively uncomplicated setting. Section 4 extends the analysis to incorporate a general decreasing-returns technology and Section 5 analyses situations involving increasing returns.

2 The basic model of contests

There are n contestants, where $n \geq 2$, in a 'winner take all' contest. Contestant i has initial wealth I_i and chooses an effort level x_i. The greater is x_i, the greater is the probability that contestant i will be the sole winner of the exogenous prize R. The probability that contestant i wins the rent is

$$p_i = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)} \qquad (1)$$

where $f_i(\cdot)$ is an increasing function for all i. Some authors - for example, Szidarovszky and Okuguchi[27] - call $f_i(\cdot)$ contestant i's 'production function for lotteries'. Contest success functions such as (1) in which an increase in effort always increases the probability of winning are often called imperfectly discriminating - see Hillman and Riley[12] - in contrast to perfectly discriminating contests in which the prize is awarded to the contestant supplying the most effort. An axiomatic foundation for the form (1) was offered by Clark and Riis [2]. A particularly well-studied form for f_i is $f_i(x_i) = a_i x_i^{r_i}$, where $r_i > 1$ and $a_i > 0$, and we shall return to it at various points in the exposition. This form was introduced by Tullock[29] and given an axiomatic foundation in symmetric contests by Skaperdas[25].

It will prove convenient to change variables by setting $y_i = f_i(x_i)$ for each i. Then the function $f_i(\cdot)$ may be thought of as transforming individual effort x_i into an 'intermediate input' y_i. We will henceforth refer to x_i as the **effort**, and y_i as the **input**, of contestant i. Since f_i is monotonic, it has a well-defined inverse function, $g_i(y_i) = f_i^{-1}(y_i)$. The function $g_i(y_i)$ describes the total cost to contestant i of generating the level y_i of input. Writing Y for the aggregate input - $Y = \sum_{i=1}^{n} y_i$ - the probability that contestant i wins is $p_i = y_i/Y$, which is that contestant's share of the total input into the contest. This ratio plays a key role in subsequent analysis, and we denote it by σ_i.

We shall assume throughout the paper that contestants are risk neutral. Consequently, the payoff of contestant i is

$$p_i\left[I_i + R - x_i\right] + (1 - p_i)(I_i - x_i),$$

which can be written in transformed variables as

$$\pi_i\left(y_i, \tilde{Y}_i\right) = \left(\frac{y_i}{y_i + \tilde{Y}_i}\right) R + I_i - g_i(y_i), \tag{2}$$

where $\tilde{Y}_i = Y - y_i = \sum_{j \neq i} y_j$. The expression (2) applies provided at least one contestant makes a positive input. If $y_i = 0$ for all i we assume that no player wins the rent so that $\pi_i(0, 0) = I_i$. This defines a simultaneous-move game and the solution concept we use throughout the paper is that of a pure-strategy Nash equilibrium of this game: a vector of input levels in which each player's effort maximises his payoff given the effort levels of all his opponents.

The simplest technology that is encountered in the literature is the linear technology, under which effort is transformed into input under constant returns to scale: $y_i = a_i x_i$, where $a_i > 0$ is a multiplicative parameter reflecting i's efficiency in this transformation. This technology is analysed in Section 3. In particular, we show that there is a unique Nash equilibrium and investigate large contests. Section 4 extends the model by considering a decreasing returns to scale technology. As in Section 3, we prove that a unique Nash equilibrium exists. We also subject to critical scrutiny the suggestion made by Tullock and adopted by a number of later writers of using a homogeneous production function, $f_i(x_i) = a_i x_i^r$, $0 < r < 1$. We argue that it is not entirely innocuous in this context. Although it is indeed a very simple way of introducing decreasing returns, it also implies unbounded marginal product as an individual's effort level approaches zero. In the rentseeking context this feature has significant implications in large contests. We also suggest how decreasing returns may be modelled to avoid this difficulty and investigate the corresponding large-game limit. In Section 5, we extend the analysis to contests with increasing returns to scale technology. This has profound effects: there may be multiple equilibria or even no equilibria at all and the results for large contests found when returns to scale are constant or decreasing do not extend to this case. However, we are able to characterise the set of Nash equilibria, even in the form of an analytic solution when the contest is symmetric.

3 Contests with linear technologies

In this section we consider the case in which all contestants use a linear technology for converting effort into input. We single this case out for special study as it will allow us to introduce our approach in a relatively uncomplicated setting where explicit formulae are available. We assume that $f_i(x) = a_i x$, where $a_i > 0$ is a multiplicative efficiency factor for contestant i. If $a_i > a_j$, contestant i is inherently more productive than contestant j and has to apply less effort to obtain a given incremental increase in the probability of winning.

3.1 Share functions

With linear f_i, equation (2) becomes

$$\pi_i\left(y_i, Y_i\right) = \frac{y_i}{y_i + Y_i} R + l_i - \frac{y_i}{a_i} \tag{3}$$

which is strictly concave in y_i. So $y_i > 0$ is a unique best response to Y_i if and only if

$$y_i^2 + 2Y_i y_i + \left(Y_i^2 - a_i R Y_i\right) = 0. \tag{4}$$

Restricting attention to the positive solution of this quadratic in y_i, and taking account of the requirement that the input level must be non-negative, the best response $\hat{x}_i$ to $\hat{Y}_{-i}$ satisfies

$$\hat{x}_i = \max\left\{\sqrt{a_i R \hat{Y}_{-i}} - \hat{Y}_{-i}; 0\right\}: \tag{5}$$

Rather than work directly with best responses, which can become very messy when contestants are heterogeneous, a considerable simplification can be achieved by an alternative approach. Player i's **replacement function**[1] $r_i(Y)$ is defined for any $Y > 0$ by requiring $r_i(Y)$ to be the best response to to $\hat{Y}_{-i} = Y - r_i(Y)$ so that, from (5), we obtain

$$r_i(Y) = \max\left\{Y - \frac{Y^2}{a_i R}; 0\right\}. \tag{6}$$

Rather than use the replacement function directly, it proves convenient to divide both sides of (6) by $Y > 0$. We refer to the resulting function as contestant i's **share function**, $s_i(Y)$. Note that $s_i(Y)$ is the probability that contestant i wins so i's share function can be thought of as her probability of winning when equilibrium aggregate input is equal to Y. Since this function is the key to subsequent analysis, we display it explicitly in the following proposition.

Proposition 3.1 If $f_i(x) = a_i x$ for contestant i, a share function exists for that contestant and satisfies

$$s_i(Y) = \max\left\{1 - \frac{Y}{a_i R}; 0\right\} \quad \text{for all } Y > 0: \tag{7}$$

Figure 1 shows the graphs of the best response, replacement and share functions of a risk neutral contestant with linear technology. Observe that the best response function graphed in Panel (a) is not monotonic. Therefore proofs of results that rely on monotonicity of best responses are not available to us in this model. The graph of the replacement function in Panel (b) is easily obtained from that of the best response function. At each point - for example $\left(\hat{Y}^0; \hat{x}\left(\hat{Y}^0\right)\right)$ - on the graph of the best response function, construct a square with that point as its top left hand corner. Then its top

[1] If aggregate input level is Y and $r_i(Y)$ is removed from this input, player i's best response is to replace this shortfall to restore the level to Y. This explains our terminology. The same function is referred to as the 'backward best response function' by Novshek [18]. Selten[24] calls it the 'Einpassungsfunktion', which Wolfstetter[31] translates as the 'inclusive reaction function', and Phlips[21] calls it the 'fitting-in function'. Brief references to this function can also be found in Friedman[8] and Vives[30].

right hand corner represents the corresponding point on the graph of the replacement function. This construction simply adds the quantity $\tilde{\phi}(\tilde{Y}^0)$ horizontally to the quantity $\tilde{Y}^0$, so that the graph in panel (b) maps $\tilde{\phi}(\tilde{Y}^0)$ against $Y = \tilde{Y} + \tilde{\phi}(\tilde{Y}^0)$. A further purely geometric device enables us to construct the graph of the implied share function, shown in Panel (c). Draw the ray from the origin that passes through a given point, say $(Y^0; y^0)$, on the graph of the replacement function. At the point where this ray where $Y = 1$, its height measures the share value implied by $(Y^0; y^0)$. The associated point on the graph of the share function is therefore $(Y^0; \sigma^0)$. Although this purely geometric argument contributes no additional economic insights, it serves to stress that the three functions are no more than alternative ways of presenting precisely the same information. The choice between them may be guided by analytical convenience. Moreover, the simple piecewise-linear form of the share function suggests that this is the most convenient to use.

Important properties of the share function for subsequent analysis are that it is continuous (indeed piece-wise linear), its value approaches 1 as $Y \to 0$, it falls (linearly) as Y increases until $Y = a_i R$, and remains at zero thereafter.

3.2 Nash equilibria

Figure 2 shows the share functions for four individuals with different unit costs. It also shows the graph of the aggregate share function, obtained simply by adding the graphs of the individual share functions vertically. Nash equilibrium values of Y occur where the aggregate share function equals unity. In our example, this gives Y^* satisfying

$$\sum_{i=1} s_i(Y^*) = \sum_{i=1} \max\left\{1 - \frac{Y^*}{a_i R}; 0\right\} = 1 \qquad (8)$$

The thick line in Figure 2 is the graph of the vertical sum of the individual share functions, and the Nash equilibrium value, Y^*, is the unique value at which this sum is unity. Given Y^*, the corresponding equilibrium strategy profile is found by multiplying Y^* by each contestant's share evaluated at Y^*: $y_i^* = Y^* s_i(Y^*)$.

The properties of the individual share functions imply the following properties of the aggregate share function: (i) it is continuous, (ii) for sufficiently small values of Y its value exceeds 1, while for sufficiently large values of Y its value is zero, and (iii) it is strictly decreasing in Y whenever it is positive. From these observations we can immediately infer existence and uniqueness.

Theorem 3.1 *If $f_i(x) = a_i x$ for $i = 1, \dots, n$, the contest has a unique Nash equilibrium.*

3.3 Large contests

Students of rentseeking contests have been particularly concerned with the question of whether, and under what conditions, the resources expended in the contest fully dissipate the rent. In particular, when all contestants are identical, technology is linear and there are no barriers to entry, the rent is completely dissipated. Rather than model entry directly, we address the issue by considering the limit as the number of players tends to infinity. In the same spirit, we can regard results on rent dissipation with a finite number of players as a reflection of positive barriers to entry.

The extent of rent dissipation is easily addressed using share functions. Recall, however, that the value of resources expended is measured not by Y. Rather, it is the value of $X = \sum_i x_i$ that we need to examine. The next result puts an upper bound on X/R.

Theorem 3.2 *Suppose that $f_i(x) = a_i x$ for $i = 1, \dots, n$ and m contestants are active in equilibrium. The proportion of rent dissipated in equilibrium will not exceed $(m - 1)/m$ and will equal this if and only if all active contestants are identical.*

Note particularly that, if each contestant has the same technology, then each of the n contestants will be an active contestant and the proportion of rent dissipated is $(n - 1)/n$ and in a large contest the rent is almost fully dissipated.

However, the theorem allows us to analyse large asymmetric contests. A simple way of generating such a contest is as follows. Imagine a population consisting of a discrete number of types of individual, distinguished by their ability parameters. Figure 2 may be interpreted as showing the share function of each of four ability types: $s_1(\cdot)$ is the share function of an individual with the highest ability and $a_1 > a_2 > a_3 > a_4$. Call an individual whose efficiency factor is a_i a 'type i'. The use of share functions shows in a very simple and direct manner the implications of increasing the number of contestants by repeated draws from the population of types. If two type 1's are drawn, the aggregate share function of these two individuals alone is the sum of their two individual share functions. It passes through the point H and its slope is twice that of $s_1(\cdot)$. If a third is added, the graph of the aggregate share function of the three individuals again passes through H, and is yet steeper - three times the slope of $s_1(\cdot)$ - and so on. Clearly there is a finite number of type 1's at which the equilibrium level of Y implied by their aggregate share function exceeds the point D at which the next highest type drops out of the contest. If the number of type 1's equals or exceeds this value, then only this type will actively contest the prize at equilibrium. More generally, let there be T ability types, and adopt the convention that $a_1 > a_2 > \cdots > a_T$. Denote the number of type i's drawn from the population by n_i. Consider the sum of the share functions of the highest ability type, whose ability parameter is a_1. As their number grows, so too does the value of Y that would represent a Nash equilibrium if only type 1's were contesting the rent. For large enough n_1 this value exceeds the maximum value of Y at which type 2 individuals would want to contest the rent. This situation arises if $Y^* > a_2 R$, where

$$n_1 \left(1 - \frac{Y^*}{a_1 R} \right) = 1.$$

which simplifies to $n_1 > a_1/(a_1 - a_2)$ and yields the following result:

Theorem 3.3 Let there be contestants of T ability types, with efficiency factors $a_1 > a_2 > \cdots > a_T$. Then, if $n_1 > a_1/(a_1 - a_2)$, only type 1 contestants will devote a positive level of effort to contesting the rent. As n_1 tends to infinity, the rent is wholly dissipated.

In the next section, we examine a more general convex technology for converting effort into input. Our objective is to test the robustness of the conclusions of the present section and to establish analogous results in a more general setting.

4 Contests with convex technologies

Throughout this section, we maintain the following assumption:

A.1 For contestant $i (= 1, \ldots, n)$ the production function f_i satisfies the following conditions:

$$f_i(0) = 0, \text{ and } f_i'(x) > 0,\ f_i''(x) \leq 0 \text{ for all } x > 0.$$

Note that the implied cost function, $g_i = f_i^{-1}$, has the following properties:

$$g_i(0) = 0, \text{ and } g_i'(y) > 0,\ g_i''(y) \geq 0 \text{ for all } y > 0.$$

4.1 Share functions

As in the previous section, the key to the analysis of Nash equilibria and large contests is the share function of individual contestants. In contrast to the case of constant returns to scale, typically we can no longer write down an explicit functional form for the share function. However, this will not prevent us from using the first order conditions for best responses to obtain an implicit equation for shares from which appropriate qualitative properties can be deduced. This is facilitated by the recognition that contestant i's payoff function, given by (2) is a strictly concave function of y_i under assumption A.1. As a result, the first-order conditions are necessary and sufficient for best responses and can be written in the form (recalling that the share value $\sigma_i = y_i/Y$):

$$g_i'(\sigma_i Y)Y \geq (1 - \sigma_i)R \tag{9}$$

with equality if $\sigma_i > 0$. This condition leads directly to the next proposition, proved in Appendix 2.

Proposition 4.1 If A.1 holds for contestant i there a share function: $s_i(Y)$. $s_i(Y)$ satisfies $s_i(Y) = 0$ if and only if $f_i'(0) < \infty$ and $Y \geq Rf_i'(0)$. Otherwise, $s_i(Y) = \sigma$, where σ is the unique solution of:

$$(1 - \sigma)R = g_i'(\sigma Y)Y \tag{10}$$

We may use this proposition to infer that the crucial qualitative properties of the share function derived under the assumption of constant returns to scale continue to hold for decreasing returns. The full details are set out in the following proposition, proved in Appendix 2.

Proposition 4.2 If A.1 holds for contestant i, the share function $s_i(Y)$ has the following properties:

1. $s_i(Y)$ is continuous,
2. $\lim_{Y \to 0} s_i(Y) = 1$,
3. $s_i(Y)$ is strictly decreasing where positive,
4. if i's marginal product is bounded, $s_i(Y) > 0$ for $0 < Y < Rf_i'(0)$ and $s_i(Y) = 0$ if $Y \geq Rf_i'(0)$,
5. if i's marginal product is unbounded, $s_i(Y) > 0$ for all $Y > 0$ and $s_i(Y) \to 0$ as $Y \to \infty$.

Diminishing marginal product means that whether the marginal product is bounded or not is determined by behaviour at the origin. In the former case $f_i'(0)$ is finite and the share function decreases continuously from one to zero over the interval $(0, Rf_i'(0))$ beyond which it takes the value zero. We refer to $Rf_i'(0)$ as player i's **dropout point**. If equilibrium Y exceeds a player's dropout point, that player will not be an active participant. Unbounded marginal product means that $f_i'(0)$ is infinite and there is no dropout point. The share function decreases from one to zero over all positive Y.

These observations shed light on a significant implication of the form of $f_i(\cdot)$ suggested by Tullock and adopted by others. Assume, following Tullock's suggestion, that $f_i(x) = x^{r_i}$ where $0 < r_i \leq 1$. In the case $r_i = 1$, dealt with in the preceding section, marginal product is bounded, indeed constant, and therefore the share function takes the value zero for large enough Y. One consequence, explored in that section, is that a contestant with such a production function may be driven out of active participation by more efficient opponents. Another consequence, as we shall see later, is the full rent dissipation discovered in that section which turns on the boundedness or otherwise of marginal product. The case $r < 1$ captures in a simple way the idea of decreasing returns. However, it also implies that the marginal product of contestant i's effort is unbounded above: $f'(0)$ is infinite. Then, Proposition 4.2 asserts that contestant i makes a positive effort (though one approaching zero for large Y) in any equilibrium. There is no value of Y at which the contestant will drop out of the contest. She will always want to apply a strictly positive level of effort.

Unbounded marginal product appears to deny the economic fact of scarcity[2]. It seems more plausible to suppose that the marginal product is bounded above, so that there is a strictly positive lower bound to the marginal cost. This can be arranged by a slight modification of the production function. For example, the form $f_i(x_i) = (x_i + k_i)^{r_i} - k_i^{r_i}$ also exhibits decreasing returns to scale but has bounded marginal product provided $k_i > 0$. This significantly changes the nature of equilibria in large games.

[2] The Inada conditions, widely used in applications such as growth theory include a similar assumption. However, these conditions are typically employed to avoid awkward boundary problems by pushing solution paths away from the boundaries and their infinite marginal products. By contrast, in large contests, participants supply vanishingly small levels of input, placing them right in the economically implausible region where the marginal product becomes arbitrarily large.

4.2 Nash equilibria

Properties of share functions enable us to study Nash equilibria. Recall that Y^* is an equilibrium value of aggregate input if and only if the sum of the share functions over all contestants takes the value one at Y^*. It follows from Proposition 4.2 that the aggregate share function exceeds one for small Y and is less than one for large Y. Since it is also continuous, there is an equilibrium value of aggregate input. Proposition 4.2 also implies that the aggregate share function is strictly decreasing, so the equilibrium value is unique. Finally, recall that a unique Y^* implies a unique strategy profile and we have the following result:

Theorem 4.1 If A.1 holds for all participants, the contest has a unique Nash equilibrium.

This result was first proved by Szidarowszky and Okuguchi[27], using a more involved application of replacement functions rather than direct use of share functions.

4.3 Large games

Whether or not the marginal product is bounded has profound implications for the extent to which rent is dissipated in the limit. In particular, when the marginal product is bounded in a large symmetric game, almost the whole rent is dissipated. The intuition behind this conclusion is straightforward. Bounded marginal product implies that share functions eventually become zero (Part 4 of Proposition 4.2). This dropout point places an upper bound on the equilibrium value of aggregate input for any number of players. Since there is a unique equilibrium (Theorem 4.1), each player's input becomes arbitrarily small. Then a bounded marginal product allows us to use a linear approximation and hence exploit the results for linear production functions obtained in the preceding section. In particular, rent is fully dissipated in the limit. The following theorem states this result and a formal proof is given in Appendix 2.

Theorem 4.2 Suppose every contestant has the same technology which has a bounded marginal product and satisfies A.1. As the number of contestants tends to infinity, the proportion of rent dissipated tends to unity.

The assumption that all contestants are identical is not a prerequisite for full rent dissipation in the limit. An argument similar to that used to justify Proposition 3.3 can be applied here. Suppose there are finitely many types of contestant. If all contestants have a bounded marginal product, there is a dropout point for each type and we refer to the type with the highest dropout point as the most efficient. If there are enough contestants of the most efficient type, the aggregate share function will exceed one at the dropout points of all other types. This means that only contestants of the most efficient type participate and the preceding theorem can be applied to show that almost all the rent will be dissipated in large contests.

When marginal product is unbounded, these results must be modified. To address these issues, we focus on the case where all production functions are proportional to x^r, where $0 < r < 1$. Such production functions, of course, have unbounded marginal product and the next proposition, proved in Appendix 2, gives an upper bound to equilibrium rent dissipation.

Theorem 4.3 Suppose $f_i(x) = a_i x^r$ for $i = 1, \ldots, n$, where $0 < r < 1$. The proportion of rent dissipated in equilibrium will not exceed $(n - 1)r/n$ and will equal this if and only if all contestants are identical.

As in the case of linear production functions discussed in the preceding section, strategic effects in contests with identical participants lead to equilibrium aggregate effort being reduced by a factor inversely proportional to the number of players. By contrast, however, even when there are many contestants rent dissipation is incomplete by a factor r. We reiterate that what leads to incomplete dissipation is not decreasing returns to scale per se but rather the unbounded marginal product implicit in the specific functional form. If this economically problematic assumption is avoided by modifying the production function to a form which has decreasing returns but bounded marginal product, Theorem 4.2 shows that the whole rent is dissipated in the limit.

5 Contests with nonconvex technologies

In this section, we characterise Nash equilibria for contests in which some or all contestants have production functions exhibiting increasing returns. Increasing returns introduce complications not found when returns are constant or decreasing. First, a contestant's payoff is no longer a concave or even pseudo-concave function of her own strategy. This means that the first-order conditions are not sufficient to characterise optima and non-unique best responses may arise. Second, and consequently, there may be more than one share value consistent with a given level of aggregate input. Hence, share functions may be multi-valued and we need to resort to correspondences to examine equilibria.

Under increasing returns details of the analysis depend on the precise form of the production function. We follow the literature by modelling increasing returns with the production function $f_i(x) = a_i x^{r_i}$ where $r_i > 1$ (and a_i is an efficiency factor as above). However, our method is much more general; qualitatively similar results obtain for a wide range of production functions. In the next subsection, we define and elucidate the properties of share correspondences for contestants with such production functions. In the following subsections we apply these results to characterise Nash equilibria and to investigate large games.

5.1 Share correspondences

We define contestant i's **share correspondence** $S_i(Y)$ for $Y > 0$ to be the set of share values σ such that $y_i = \sigma Y$ is a best response to $Y_{-i} = Y - y_i$. In order to derive the properties of share correspondences we first need to derive necessary and sufficient conditions for best responses. When $r_i > 1$, the payoff function $\pi_i(y_i; Y_{-i})$ is not concave but neither is it without structure. When $Y_{-i} > 0$, it takes the value 0 at $y = 0$, is initially decreasing, has at most two stationary points and is eventually negative. Figure 3 graphs contestant i's payoff against y_i for several values of Y_{-i}. Depending on the value of Y_{-i}, the maximum occurs either at $y_i = 0$ or at the rightmost stationary point or possibly both. There are two maxima if

$$Y_{-i} = \underline{Y}_i \stackrel{\text{def}}{=} a_i R^{r_i} \frac{(r_i - 1)^{r_i - 1}}{r_i^{r_i}}.$$

For positive $Y_{-i} < \underline{Y}_i$, the best response is at the unique level of input at which marginal payoff is zero and the payoff itself is positive. For $Y_{-i} > \underline{Y}_i$, the best response is zero. For example, suppose $a_i = 1$, $r_i = 3/2$ and $R = 2$. If $Y_{-i} = \underline{Y}_i = \sqrt{32/27}$, there are two local maxima: $y = 0$ and $y = \hat{y}_i = \sqrt{8/27}$. For $Y_{-i} < \sqrt{32/27}$, the interior maximum is indeed the true global maximum, whereas for $Y_{-i} > \sqrt{32/27}$, the maximum payoff of zero is achieved at $y = 0$. A detailed justification of these claims may be found in Appendix 3 as part of the proof of the next proposition.

Proposition 5.1 *Suppose $f_i(x) = a_i x^{r_i}$. Then*

1. *if $Y_{-i} = 0$, player i has no best response,*
2. *if $0 < Y_{-i} < \underline{Y}_i$, player i has a unique best response: the solution in y of*

$$r_i a_i^{1/r_i} R Y_{-i} y^{(r_i-1)/r_i} = \left(y + Y_{-i}\right)^2 \text{ and } a_i^{1/r_i} R y^{1-\frac{1}{r}} > \left(y + Y_{-i}\right), \quad (11)$$

3. player i has two best responses to $\hat{Y}_i = \underline{Y}_i$: 0 and

$$\hat{y}_i = a_i R^{r_i} \left(\frac{r_i - 1}{r_i} \right)^{r_i},$$

4. if $\hat{Y}_i > \underline{Y}_i$, player i's unique best response is 0.

Proposition (5.1) can be used to characterise share correspondences. First, it follows immediately that $S_i(Y)$ contains zero if and only if $Y \geqslant \underline{Y}_i$. There is also a range of values of Y for which a positive share lies in $S_i(Y)$. Specifically, we have a function $s_i^+(Y)$ defined for $0 < Y \leqslant \overline{Y}_i$ where

$$\overline{Y}_i = a_i R^{r_i} \left(\frac{r_i - 1}{r_i} \right)^{r_i - 1} > \underline{Y}_i$$

and $s_i^+(Y)$ is a member of $S_i(Y)$ if and only if $0 < Y \leqslant \overline{Y}_i$. Furthermore, $S_i(Y)$ contains no other positive shares. This is illustrated in Figure 4. Observe that, in the interval $\left[\underline{Y}_i, \overline{Y}_i\right]$, the share correspondence has one positive and one zero member. Outside this interval, $S_i(Y)$ is a singleton. Furthermore, $s_i^+(Y)$ approaches unity for small Y and decreases strictly to $(r_i - 1)/r_i$ at $\overline{Y}_i$. Typically, we cannot obtain an explicit expression for s_i^+ but it is the partial inverse of an explicit function given in the next proposition, which also summarises the results above. The proof is in Appendix 3.

Proposition 5.2 If $f_i(x) = a_i x^{r_i}$, where $r_i > 1$, contestant i's share correspondence satisfies

$$S_i(Y) = \begin{cases} \left\{ s_i^+(Y) \right\} & \text{if } 0 < Y < \underline{Y}_i, \\ \left\{ 0, s_i^+(Y) \right\} & \text{if } \underline{Y}_i \leqslant Y \leqslant \overline{Y}_i, \\ \{0\} & \text{if } Y > \overline{Y}_i, \end{cases}$$

where s_i^+ is the inverse of the function

$$\phi_i(\sigma) = aR^{r_i} r_i^{r_i} [1 - \sigma]^{r_i} \sigma^{r_i - 1}. \tag{12}$$

restricted to the interval $(r_i - 1)/r_i \leqslant \sigma < 1$. Furthermore, s_i^+ is continuous, strictly decreasing and satisfies

$$s_i^+(Y) \to 1 \text{ as } Y \to 0, \text{ and}$$
$$s_i^+\left(\overline{Y}_i\right) = \frac{r_i - 1}{r_i}.$$

5.2 Nash equilibria

As with share functions, the properties of the share correspondences set out in Proposition 5.2 allow us to characterise Nash equilibria in contests where all contestants have production functions with increasing returns. This is achieved by studying the aggregate share correspondence of a game which is obtained by combining the individual share correspondences using addition of sets. Formally,

$$S^{\star}(Y) = \sum_{i=1} S_i(Y) \stackrel{\text{def}}{=} \left\{ \sigma : \sigma = \sum_{i=1} \sigma_i,\ \sigma_i \in S_i(Y)\ \forall i = 1; \dots; n \right\}.$$

A strategy profile $(y_1^*; \dots; y_n^*)$ is a Nash equilibrium if and only if $S^{\star}(Y^*)$ contains unity and $y_i^*/Y^* \in S_i(Y^*)$ for $i = 1; \dots; n$. This is readily verified by chasing definitions.

Figure 5 shows the individual and aggregate share correspondences of a contest involving two players with different technologies. The share correspondences of contestants A and B are labelled, respectively, aa/aa and bb/bb. The zero branch of the aggregate share correspondence coincides with the zero branch of player B. Elsewhere, it is represented by the thicker lines in the diagram which are obtained by set addition. In the contest depicted by the figure, there is a single equilibrium level of Y, at which both contestants are active.

Note that, even if there is a unique value of Y^* such that $S^{\star}(Y^*)$ contains unity, we may not be able to conclude that there is a unique Nash equilibrium (though this is not the case in Figure 5). This is because there may be more than one set of shares contained in the individual correspondences and summing to one. In particular, consider an asymmetric equilibrium of a symmetric contest. Then any permutation of the active players will yield another equilibrium with the same aggregate input. This suggests that there are two ways in which equilibria may fail to be unique. Firstly, if $S^{\star}(Y^*)$ contains unity for only one value of Y^*, we may still have multiple equilibria. But, if we have reason to believe that the players can coordinate on an equilibrium, we can predict the value of Y^* even if we cannot predict individual strategies. On the other hand, $S^{\star}(Y)$ contains one for several values of Y, the problem posed by multiple equilibria is more severe: without a full resolution of the coordination problem, we cannot even predict equilibrium Y.

In contests in which some participants have increasing returns to scale and others have decreasing returns, the former will have share correspondences whilst the latter will have share functions. However, we can apply the same technique in the natural way. By regarding share functions as single-valued correspondences, they can be added to the multi-valued correspondences of participants with decreasing returns.

Now consider a contest in which all players have production functions of the form $f_i(x) = a_i x^{r_i}$ where all $r_i > 1$. It will prove convenient to refer to a subset of contestants I_A as an **active set** if it is the set of contestants supplying positive effort in some Nash equilibrium. For any active set, there is a unique equilibrium. This follows by a similar argument to that used to establish Proposition 4.1 and rests on the fact that s_i^+ is strictly decreasing. However, an arbitrarily chosen subset I_A of contestants may not be an active set, for, if i is to be an active participant, equilibrium Y cannot exceed the maximum value for which s_i^+ is defined: $\overline{Y}_i$. Hence, (i) the sum of s_i^+ over I_A evaluated at the least $\overline{Y}_i$ amongst the contestants in I_A should not exceed unity. Similarly, for contestant j to be inactive, equilibrium Y must be at least $\underline{Y}_j$. This requires that (ii) the sum of s_i^+ over I_A evaluated at the greatest $\overline{Y}_j$ amongst the contestants not in I_A should be at least unity. Conversely, if (i) and (ii) both hold for some set I_A, there is a value of Y at which the sum of s_i^+ over I_A is equal to one. The set of active participants at the corresponding equilibrium is precisely I_A.

The following theorem summarises these observations and extends them to include contestants with both increasing and decreasing returns. In this case, active sets refer only to subsets of the contestants with increasing returns: I_A is an active set if there is a Nash equilibrium in which contestant i with $r_i > 1$ supplies positive effort if and only if $i \in I_A$.

Theorem 5.1 Suppose that $f_i(x) = a_i x^{r_i}$, with $r_i > 1$ for all $i \in I$, where I is a non-empty subset of contestants. If A.1. is satisfied for all other contestants, then $I_A \subseteq I$ is an active set if and only if

$$s_A\left(\min_{i \in I_A} \overline{Y}_i\right) \leq 1 \leq s_A\left(\max_{j \in I - I_A} \underline{Y}_j\right), \qquad (13)$$

where

$$s_A(Y) = \sum_{i \in I_A} s_i^+(Y) + \sum_{k \notin I} s_k(Y).$$

In this sum, s_k is the share function for each contestant $k \notin I$ and $s_i^+(Y)$ is defined as in Proposition 4.1. Furthermore, for each active set I_A, there is a unique Nash equilibrium.

With a little care in its interpretation, the theorem holds even if the active set I_A is empty or equal to the set I itself, provided we interpret the maximum of an empty set as 0 and the minimum of an empty set as ∞. (As usual, sums over an empty set are taken to be 0.) We also need to interpret the value of s_A as a limit when its argument is 0 or ∞. But s_A approaches zero for large Y, so the left-hand inequality in (13) is automatically satisfied when I_A is empty and the condition reduces to the right-hand inequality. Similarly, since all s_i^+ and s_k approach unity for small Y, the right-hand side of (13) is automatically satisfied if all players are active: $I_A = I$; the condition reduces to the left-hand inequality.

Theorem 5.1 has several consequences. For I_A to be an active set, we may use the fact that each s_i^+ is decreasing (by Proposition 5.1) to deduce that

$$1 \geq s_A\left(\min_{i\in I_A}\overline{Y}_i\right) \geq \sum_{i\in I_A} s_i^+\left(\min_{i\in I_A}\overline{Y}_i\right) \geq \sum_{i\in I_A} s_i^+\left(\overline{Y}_i\right) = \sum_{i\in I_A}\frac{r_i-1}{r_i}$$

by the first inequality of the previous proposition. This conclusion implies the following corollary.

Corollary 5.2 Under the hypotheses of the preceding theorem, $I_A \subseteq I$ cannot be an active set if

$$\sum_{i\in I_A}\frac{1}{r_i} < n_A - 1, \tag{14}$$

where I_A has n_A members.

This result may be applied to the case when all contestants have increasing returns. Under the hypothesis of Corollary 5.2, if the contestants are labelled so that $1 < r_1 \leq r_2 \leq \cdots \leq r_n$ and

$$\frac{1}{r_1} + \frac{1}{r_2} < 1, \tag{15}$$

the game has no equilibria. This follows since there must be at least two active participants and (15) implies that (14) holds for all I_A with at least two members.

If $r_i = r > 1$ for all i, stronger conclusions can be drawn even if the efficiency factors a_i vary between contestants. For example, it follows immediately from (15) that a contest with $r > 2$ can have no equilibria. This conclusion is valid no matter what values are taken by the efficiency factors. However, non-existence is a more extensive problem. Indeed, if $r > 1$, the efficiency factors can be chosen so that the resulting contest has no equilibria for any $r > 1$. To see this, set $n = 2$ and $a_2 > a_1/(r-1)$ where $r \leq 2$. Since $\overline{Y}_i$ is proportional to a_i, $\min\{\overline{Y}_1, \overline{Y}_2\} = \overline{Y}_1$ and, if we can show that

$$s_1^+\left(\overline{Y}_1\right) + s_2^+\left(\overline{Y}_1\right) > 1, \tag{16}$$

it follows from Theorem 5.1 that both contestants cannot be active so there is no equilibrium. To justify (16), first recall that $s_1^+\left(\overline{Y}_1\right) = (r-1)/r$. Second, since $a_2 > a_1/(r-1)$, we have

$$\psi_2\left(\frac{1}{r}\right) = a_2 R^r \frac{[r-1]^r}{r^{r-1}} > a_1 R^r \left(\frac{r-1}{r}\right)^{r-1} = \overline{Y}_1,$$

where ψ_2 is defined as in (12). Hence, $s_2^+\left(\overline{Y}_1\right) > 1/r$ and (16) follows.

The strongest results are obtained when all contestants are identical. Our conclusions are set out in the following theorem, which extends the results of Perez-Castrillo and Verdier for symmetric contests. The proof is given in Appendix 3 and is a direct application of the characterisation of equilibria in Theorem 5.1.

Theorem 5.3 Suppose $f_i(x) = ax^r$ for $i = 1, \dots, n$ where $r > 1$ and (dropping subscripts) write ς for $[s^+(\underline{Y})]^{-1}$. The set of all equilibria for $1 < r \leqslant 2$ may be characterised as follows.

1. If $n < \varsigma$, there is a unique equilibrium in which all participants are active.
2. If $n \geqslant \varsigma$, there is an equilibrium with m active participants if and only

$$\varsigma \leqslant m \leqslant \min\{r/(r-1), n\}.$$

3. In an equilibrium with m active participants, each supplies effort $R\,r\,(m-1)/m^2$.

Note that both ψ and $\underline{Y}$ are proportional to a which means that ς in this theorem depends on r but not on a. The following lemma, proved in Appendix 3, gives more information on ς.

Lemma 5.1 If ς is defined as in Theorem 5.3, the interval $[\varsigma; r/(r-1)]$ contains an integer.

The following existence result is an immediate consequence..

Corollary 5.4 If $f_i(x) = ax^r$ for $i = 1, \dots, n$, the contest has an equilibrium if and only if $r \leqslant 2$.

5.3 Large games and rent dissipation

Extending the asymptotic results in the previous sections encounters difficulties when there are contestants with increasing returns, since equilibria of large contests may involve few active participants. Share correspondences help to illustrate what underpins these results. Suppose all contestants are identical and their number grows large. If each contestant had a share function with a finite dropout point, the aggregate share function would pivot clockwise about this point pushing the equilibrium level towards the dropout point and leading to full dissipation in the limit. When the share function is replaced by a correspondence, the maximum value of the correspondence at any Y has a finite downwards jump at $\overline{Y}$ and as the number of contestants increases by one, as well as rotating clockwise the aggregate value of this maximum moves up [by $r/(r-1)$]. Eventually this exceeds unity so that, although the rightmost equilibrium moves towards $\overline{Y}$, it may stop short of $\overline{Y}$ even when there are many contestants. Indeed, the 'limit' is $\overline{Y}$, implying full rent dissipation if and only if $r/(r-1)$ is an integer. Even when contestants differ, an upper bound on rent dissipation can be derived provided that they all have the same r_i. Details are given in the next proposition, proved in Appendix 3.

Proposition 5.3 Suppose $f_i(x) = a_i x^r$ for $i = 1, \ldots, n$, where $r > 1$.

1. The number of active participants in equilibrium cannot exceed $r/(r-1)$.
2. The proportion of the rent dissipated at an equilibrium with m active participants will not exceed $(1-1/m)\,r\,(\leq 1)$ and will equal this if and only if all active players are identical.

Part 2 implies that rent dissipation will be maximised when contestants are identical. In this respect, results are the same for $r > 1$ as for $r \leq 1$. However, the maximum rent dissipation depends in a non-monotonic way on r. Theorem 5.3 and Lemma 5.1 show that there is an equilibrium with m active participants where m is the greatest integer not exceeding $r/(r-1)$. If $r/(r-1)$ is an integer (m), Part 2 of the theorem implies that the rent will be fully dissipated. Even where $r/(r-1)$ is not an integer, m must satisfy

$$m > \frac{r}{r-1} - 1 = \frac{1}{r-1}$$

and therefore the proportion of rent dissipated in this equilibrium is

$$\left(1 - \frac{1}{m}\right) r > 1 - \frac{1}{m^2}. \tag{17}$$

Since two or more contestants must be active, at least 75% of the rent is dissipated in any such equilibrium and, for r close to one, m is large and almost all the rent is dissipated. Note that it is not necessary to have a large number of active contestants for all or nearly all the rent to be dissipated. All these conclusions make very strong assumptions about the ability of the contestants to coordinate on an equilibrium, particularly when the interval $[\&; r/(r-1)]$ contains more than one integer. In the latter case, there will be several equilibrium values Y some entailing rent dissipation below the lower bound in (17).

When returns are non-increasing, we saw that more efficient contestants, provided they are sufficiently numerous, will drive their less efficient opponents out of the game. No such conclusion can be drawn when there are increasing returns. Suppose that all contestants share the same value of r but there are several types with different efficiency factors. If $r/(r-1)$ is equal to an integer m, there are equilibria in which m participants of the same type are active and no other contestants are active. This applies for any type and so includes equilibria in which only the most inefficient contestants actively participate. Even for more general r there can be equilibria with active contestants who are less efficient than inactive contestants. Note that this will have an effect on rent dissipation. However, if the contestants can coordinate so that all active participants are of the same type, the lower bound on rent dissipation in (17) still applies. If $r/(r-1)$ is an integer, all the rent is dissipated in such an equilibrium.

Figures 6 and 7 illustrate the possibilities implied by our formal analysis and are drawn on the assumption that $a = R = 1$. In Figure 6 $r = 9/7$, which implies that $r/(r-1) = 4.5$. The share correspondence of a typical contestant consists of two branches. These are the segments marked aa. Observe that for all Y such that $\underline{Y}_i < Y < \overline{Y}_i$, there are two values of the individual's share - one zero and one strictly positive - consistent with the value of the aggregate. The aggregate share correspondence consists of the thick segments. Between the values $\underline{Y}_i$ and $\overline{Y}_i$ the segments of the aggregate function are obtained by vertically adding the positive branches of the individual share correspondence. The figure shows that there are two equilibria. The point N^3 represents an equilibrium at which there are 3 active contestants. There is a second equilibrium at N^4 at which there are four active contestants.

Figure 7 shows the individual and aggregate share correspondences implied by $r = 2$. With three potential contestants, the aggregate share correspondence is represented by the thick segments. In this case, there is a unique Nash equilibrium value of Y, at which any two of the three contestants are active. It is marked N^2 in the figure. Note that, if r increases above 2, the positive branch of the individual share correspondence shifts upwards. Therefore, so too does the segment of the aggregate correspondence associated with two active contestants. In this case, an equilibrium does not exist.

6 Conclusions

A number of themes have emerged in our analysis of contests.

1. Contests have a unique Nash equilibrium when all production functions have constant or decreasing returns.
2. Contestants with increasing returns typically imply multiple equilibria and consequent problems of coordination.
3. If the elasticities of production functions are too large, contests may have no equilibria.
4. The rent is almost fully dissipated in large symmetrical contests with constant or decreasing returns.
5. Under increasing returns, the number of active participants is bounded even in large games but all the rent may still be dissipated.
6. Rent dissipation is reduced by limiting the number of active participants (strategic e¤ects) and by asymmetry.

We have also sought to illustrate the usefulness of share functions and share correspondences as tools for analysing contests. They contain the same information as best response functions and correspondences but are usually easier to handle (by simple addition) often permitting a complete analysis of Nash equilibria, especially existence, uniqueness and large-game limits. As we have seen they can take simpler forms than the corresponding best response functions. More is possible. For reasons of space, we have not given results for comparative statics but the share function approach provides a useful tool (cf. Nti[19], for a traditional approach). It can also be used when contestants are risk averse (another possible explanation for partial rent dissipation): Cornes and Hartley[4] apply the approach to the case of constant absolute risk aversion, and examine situations in which contestants di¤er in their attitudes to risk.

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APPENDICES

Appendix 1

In this appendix we give the proof of Proposition 3.2:

Proof. Let S_C denote the set of active contestants and $Y^*[X^*]$ the equilibrium values of aggregate input [effort]. If $\sigma_i = s_i(Y^*)$,

$$X^* = \sum_{i \in S_C} x_i^* = \sum_{i \in S_C} \frac{\sigma_i Y^*}{a_i} = R \sum_{i \in S_C} \sigma_i (1 - \sigma_i),$$

since $\sigma_i = 1 - (Y^*/a_i R)$ for $i \in S_C$ by Theorem 3.1. In equilibrium, $\sum_{i \in S_C} \sigma_i = 1$, so

$$\frac{X^*}{R} = \sum_{i \in S_C} \sigma_i (1 - \sigma_i) = \frac{m - 1}{m} - \sum_{i \in S_C} \left(\frac{1}{m} - \sigma_i \right)^2$$

which establishes the proposition. ■

Appendix 2

In this appendix, we give the proofs of propositions and theorems in Section 4. In the proofs of Propositions 4.1 and 4.2, we can drop the subscript i without confusion.

Proof of Proposition 4.1. The left-hand side of (9) exceeds the right at $\sigma = 1$ and is non-decreasing in σ, whereas the right-hand side is strictly decreasing. We may conclude that there is a unique share value for any $Y > 0$ which is zero if and only if $g'(0)Y \geq R$. The proof is completed by observing that $g'(0) = [f'(0)]^{-1}$. ■

Proof of Proposition 4.2. First, note that shares are continuous (indeed differentiable where positive) by the implicit function theorem, establishing Part 1. Second, since $g'(0)$ is finite, letting $Y \to 0$ in both sides of (10) shows that the share must approach one as Y approaches zero, giving Part 2. Third, observe that the left-hand side of (10) is strictly decreasing in σ and the right-hand side is non-decreasing in σ and strictly increasing in Y for positive σ. We may deduce that positive shares are strictly decreasing in Y. Part 3. The fourth part is an immediate consequence of Proposition 4.1.

Finally, suppose that the marginal product $f'(0)$ is unbounded, which implies $g'(0) = 0$. Then (10) can hold as $Y \longrightarrow \infty$ only if $s(Y)Y$ approaches zero in this limit and hence if s itself also approaches zero. ■

Proof of Theorem 4.2. Let $Y^{(n)}[X^{(n)}]$ denote the equilibrium level of $Y[X]$ when there are n identical contestants. At the Nash equilibrium, we know that $\sum_{i=1}^{n} s_i\left(Y^{(n)}\right) = 1$. Since the contest is symmetric and has a unique equilibrium, $s_i\left(Y^{(n)}\right) = 1/n$ for all i. Therefore, as $n \longrightarrow \infty$, Proposition 4.2 implies that $Y^{(n)} \longrightarrow Rf_i'(0)$. Hence, as $n \to \infty$, we have

$$X^{(n)} = \sum_{i=1} x_i^{(n)} = ng_i\left[Y^{(n)}/n\right] \to [Rf_i'(0)]g_i'(0) = R,$$

where $x_i^{(n)}$ is player i's equilibrium effort in the n-player contest. ■

Proof of Proposition 4.3. Let $Y^*[X^*]$ denotes the equilibrium value of aggregate input [effort]. By hypothesis, $g_i(y) = (y/a_i)^{1/r}$ so that $g_i'(y) = y^{(1-r)/r}/ra_i^{1/r}$ for each i. Applying Proposition 4.1, contestant i's share function satisfies $s_i(Y^*) = \sigma_i$, where

$$(1-\sigma_i)\, rRa_i^{\frac{1}{r}} = Y^*\sigma_i^{\frac{1-r}{r}}(Y^*)^{\frac{1-r}{r}}.$$

Hence,

$$X^* = \sum_{i=1} x_i^* = \sum_{i=1} \left(\frac{\sigma_i Y^*}{a_i}\right)^{\frac{1}{r}} = rR\sum_{i=1} \sigma_i(1-\sigma_i).$$

The proof is completed as in the proof of Proposition 3.2 in Appendix 1. ■

Appendix 3

In this section, we give proofs of propositions and theorems in Section 5. In the proofs of Propositions 5.1 and 5.2 as well as Lemma 5.1, we drop the subscript i for ease of exposition.

Proof of Proposition 5.1. With production functions as specified in the proposition, a contestant's payoff may be expressed in the form:

$$\pi\left(y;\psi\right) = \frac{y}{y+\psi}R - \left(\frac{y}{a}\right)^{\frac{1}{r}},$$

except when $y = \psi = 0$, in which case $\pi(0;0) = 0$.

Part 1 is established by observing that the least upper bound of $\pi(y;0)$ is R but, because of the discontinity at $y = 0$, no non-negative y achieves this. Hence, there is no best response to $\psi = 0$.

Turning to best responses to $\hat{Y} > 0$, we aim to relate them to the first-order conditions. Note first that $\pi\left(0; \hat{Y}\right) = 0$ and $\pi\left(y; \hat{Y}\right) \to -\infty$ as $y \to \infty$. For $y > 0$,

$$\frac{\partial \pi}{\partial y} = \frac{\hat{Y}}{\left(y + \hat{Y}\right)^2} R - \frac{1}{ry}\left(\frac{y}{a}\right)^{\frac{1}{r}}.$$

The marginal payoff is negative for all small enough y and has at most two zeroes. To justify the latter claim observe that

$$\frac{\partial \pi_i}{\partial y} = 0 \iff y^{\frac{r-1}{2r}} \sqrt{a^{\frac{1}{r}} r R \hat{Y}} = y + \hat{Y}.$$

This condition sets a strictly concave function of y equal to a linear function of y and such an equation can have at most two solutions.

To summarise our conclusions so far, as y increases from 0, the payoff $\pi\left(y; \hat{Y}\right)$ starts at 0, initially decreases, has at most two stationary points and is eventually negative. Hence the payoff is maximised either at 0 or at the right-most stationary point. The latter holds if the payoff evaluated at this stationary point is zero or greater. (The payoff at the other stationary point is always negative.) Hence, $\hat{y} > 0$, is a best response to $\hat{Y} > 0$ if and only if it is a stationary point with non-negative payoff. When the payoff at $\hat{y}$ is exactly 0, there are two maxima: 0 and $\hat{y}$. This happens if and only if

$$\frac{\partial \pi}{\partial y}\left(\hat{y}; \hat{Y}\right) = 0 \text{ and } \pi\left(\hat{y}; \hat{Y}\right) = 0.$$

These equations have a unique solution: $\hat{Y} = \underline{Y}$,

$$\hat{y} = aR^r \left(\frac{r-1}{r}\right)^r.$$

This establishes Part 3.

To obtain Part 2, note that $\pi(\hat{y}; \underline{Y}) = 0$ implies that $\pi\left(\hat{y}; \hat{Y}\right) > 0$ for $0 < \hat{Y} < \underline{Y}$ since $\pi\left(\hat{y}; \hat{Y}\right)$ is a strictly decreasing function of $\hat{Y}$. Because the payoff can take positive values for $\hat{Y}$ in this interval, our previous remarks show that the best response is positive and therefore must have positive payoff and zero marginal payoff. Rearranging these conditions in a more convenient form gives Part 2.

To prove Part 4, we start from the fact that $y = 0$ maximises $\pi(y; \underline{Y})$, from the definition of $\underline{Y}$. This shows that $\pi(y; \underline{Y}) \leq 0$ for all $y > 0$. Using the fact that $\pi\left(y; \hat{Y}\right)$ is a strictly decreasing function of $\hat{Y}$, we deduce that, if $0 < \hat{Y} < \underline{Y}$, then $\pi\left(y; \hat{Y}\right) < 0 = \pi\left(0; \hat{Y}\right)$ for all $y > 0$. The best response to such a $\hat{Y}$ must be 0, completing the proof. ■

Proof of Proposition 5.2. Zero shares were dealt with in the preamble to the proposition. By definition, $\sigma \in S(Y)$ if and only if σY is a best response to $(1 - \sigma) Y$ and Proposition 5.1 implies that this is true for positive σ if and only if

$$Y = \phi(\sigma) \text{ and } Y \leq aR^r \sigma^{r-1}. \tag{18}$$

These conditions are obtained by substituting σY for y and $(1 - \sigma) Y$ for $\hat{Y}$ in (11) and rearranging the results. For (18) to be satisfied, we must have $r^r (1 - \sigma)^r \leq 1$ or $\sigma \geq (r - 1)/r$. Now ϕ is non-negative, has a unique stationary point at $\sigma = (r - 1)/(2r - 1) < (r - 1)/r$ and satisfies $\phi(0) = \phi(1) = 0$. It follows that the restriction of ϕ to the set $[(r - 1)/r; 1]$ is strictly decreasing. Since $\phi([r - 1]/r) = \overline{Y}$, we can define a function

$$s^+ : \left(0; \overline{Y}\right] \to \left(1; \frac{r-1}{r}\right]$$

by $\sigma = s^+(Y)$ if and only if $Y = \phi(\sigma)$. Clearly, s^+ is continuous, strictly decreasing and approaches 1 as $Y \to 0$. ■

Proof of Proposition 5.3. The theorem follows readily from Proposition 5.1 which states that there is an equilibrium with $m < n$ players if and only if

$$ms^+\left(\overline{Y}\right) = m\frac{r-1}{r} \leq 1 \leq ms^+(\underline{Y}) = \frac{m}{\rho}.$$

For $m = n$, this condition is modified by dropping the right hand inequality. Part 1 and 2 follow immediately. (For Part 1, we use the fact that $1/\rho > 1 - 1/r$ since s^+ is decreasing.) Part 3 is an immediate corollary of Part 2 of Proposition 5.3 proved below. ■

We can prove Lemma 5.1 by establishing an appropriate upper bound on ρ. In doing this, we find it convenient to divide the interval $(1; 2]$ into two subintervals.

Lemma .1 If $r \geq \sqrt{2}$, then $\rho \leq 2$.

Proof. The inequality $(r - 2)^2 \geq 0$, satisfied for all r, can be rewritten

$$\frac{r^2}{2} \geq 2(r - 1).$$

Since, by assumption, $r^2 \geq 2$,

$$\left(\frac{r^2}{2}\right)^r \geq \left(\frac{r^2}{2}\right)^{r-1} \geq 2^{r-1}(r-1)^{r-1}.$$

It follows that, if ϕ is defined as in (12),

$$\phi\left(\frac{1}{2}\right) = \frac{a r^r R^r}{2^{2r-1}} = aR^r \left(\frac{r^2}{2}\right)^r \frac{1}{r^r 2^{r-1}} \geq aR^r \frac{[r-1]^{r-1}}{r^r} = \underline{Y}.$$

Since s^+ is the inverse function of ϕ in the relevant range, $s^+(\underline{Y}) \geq 1/2$ so $\sigma = [s^+(\underline{Y})]^{-1} \leq 2$. ∎

Lemma .2 If $1 < r \leq \sqrt{2}$, then $\sigma < (r-1)^{-1}$.

Proof. Define $\psi(r) = r^2(2-r)$. Then $\psi''(r) < 0$ so that ψ is strictly concave. Furthermore, $\psi(1) = 1 < \psi(\sqrt{2})$ and we may conclude that $\psi(r) > 1$ for $1 < r \leq \sqrt{2}$. It follows that, if ϕ is defined as in (12),

$$\phi(r-1) = aR^r \frac{[\psi(r)]^r [r-1]^{r-1}}{r^r} > aR^r \frac{[r-1]^{r-1}}{r^r} = \underline{Y}.$$

As in the preceding lemma, $s^+(\underline{Y}) > r-1$ so $\sigma < (r-1)^{-1}$: ∎

These lemmas allow us to complete the proof of Lemma 5.1.

Proof of Lemma 5.1. When $\sqrt{2} \leq r \leq 2$, we have $2 < r/(r-1)$ and 2 lies in the interval $[\sigma; r/(r-1)]$. When $1 < r \leq \sqrt{2}$ we have

$$\frac{r}{r-1} - 1 = \frac{1}{r-1} \geq \sigma$$

and the greatest integer less than $r/(r-1)$ lies in the interval $[\sigma; r/(r-1)]$. ∎

Proof of Proposition 5.3. Let I_A be a subset of I with n_A members. The first part is a consequence of Corollary 5.2 since inequality (14) becomes $n_A/r < n_A - 1$. Hence, if $n_A > r/(r-1)$, I_A cannot be an active set.

Let $\hat{Y}$ denote the value of Y at an equilibrium with m active participants and write $\sigma_i = s_i^+\left(\hat{Y}\right)$ for participant i. If i is active, Proposition 5.1 implies

$$\hat{Y} = \phi_i(\sigma_i) = a_i R^r r^r (1-\sigma_i)^r \sigma_i^{r-1}, \qquad (19)$$

where ϕ_i is defined as in (12) and using the fact that s_i^+ is the partial inverse of ϕ_i. Writing $\hat{x}_i$ for participant i's effort, the production function implies that $a_i \hat{x}_i^r = \sigma_i \hat{Y}$. Summing over the active participants:

$$\hat{X} = \sum_i \hat{x}_i = \sum_i \left(\frac{\sigma_i}{a_i}\right)^{\frac{1}{r}} \hat{Y}^{\frac{1}{r}} = rR \sum_i \sigma_i (1-\sigma_i),$$

using (19). The proof of Part 2 is completed as in the proof of Proposition 3.2 in Appendix 1. (The inequality follows from Part 1.) ■

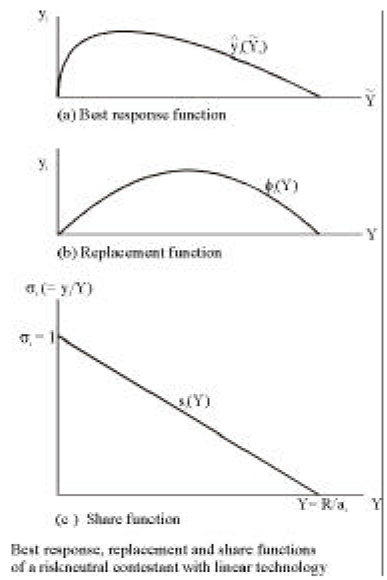


Figure 1:

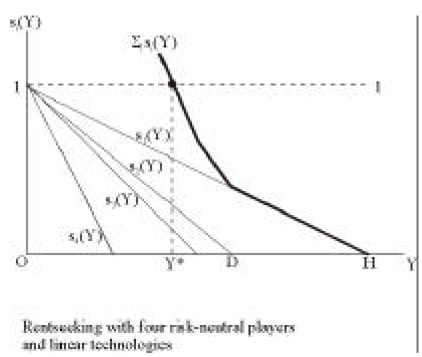


Figure 2:

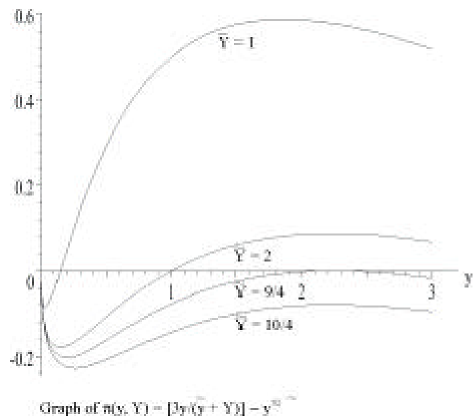
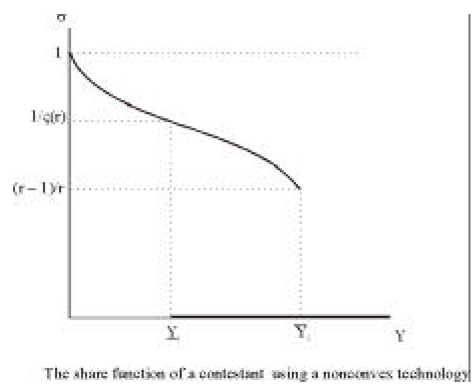


Figure 3:



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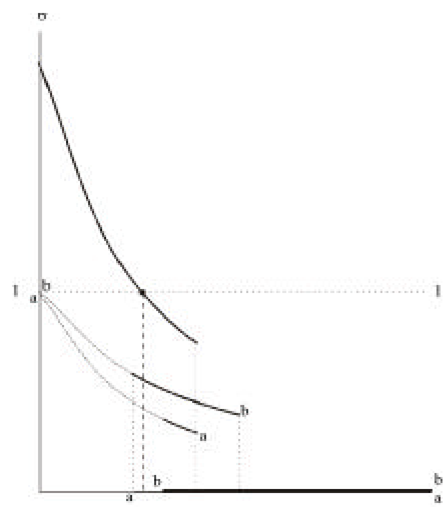
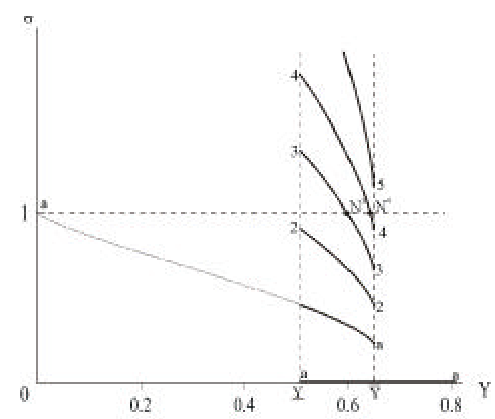


A contest with two heterogeneous contestants

Figure 5:



An aggregate share correspondence:
$r = 9/7$ [$\Rightarrow r/(r-1) = 4.5$]

Figure 6:

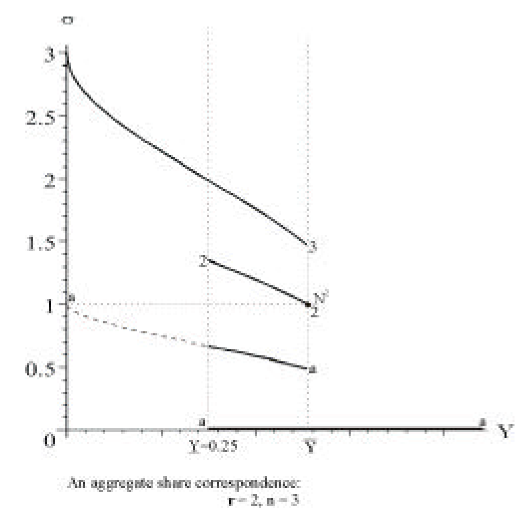

σ
3
2.5
2
1.5
1
0.5
0
3
2
N
2
Y=0.25
Y
Y
An aggregate share correspondence:
r = 2, n = 3

Figure 7: