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1. Introduction

Comparative statics is important in economics because it describes the main properties of agents behaviour and the main properties of equilibria resulting from the interactions of economic agents. However, despite a long history of studies of the comparative statics in the economic literature[1], the profund mechanisms intervening in comparative statics are not fully understood yet. A better understanding of complex comparative statics is likely to assist the development, the speci&cation and the estimation of realistic models. Moreover, from a theoretical standpoint it is unclear how to compare the decision elasticities with respect to parameters of interest for models that do not share the same objective and constraint functions or the same decisions and parameters. In this paper, we tackle all these difficulties by conducting a full geometrical analysis of the comparative statics of general models. Thus, we break down the decision changes into integrated geometrical structures, easy to understand, to specify and to compare.

Using general models is important because many economic problems involve complications: several nonlinear inequality constraints and general parameters appearing simultaneously in the objective function and in some constraints. These complications occur for: the New Household Economics[2], agricultural household models[3], models with nonlinear budget constraints arising from quality effects[4], nonlinear taxation[5],

[1] e.g. Slutsky (1915), Samuelson (1947), Kalman (1973), Silberberg (1974), Edlefsen (1981), Hatta (1980), Caputo (1999).

[2] Becker (1965), Lancaster (1966).

[3] Sen (1966), Barnum and Squire (1980), Pitt and Rosenzweig (1985), Singh, Squire and Strauss (1986), Benjamin (1992).

[4] Houthakker (1952), Edlefsen (1981, 1983).

[5] Hausman (1985), Weymark (1987), Balestrino, Cigno and Pettini (2000).

productive consumption[6], nonlinear wage schedule[7], rationing[8], nonlinear pricing by &rms with monopoly power, international trade theory, collective household models[9], bargaining or game theory models.

Although this paper is related to the important articles by Kalman and Intriligator (1973), Silberberg (1974), Chichilnisky and Kalman (1978b), Hatta (1980), Edlefsen (1981), Blomquist (1989), the focus of the analysis and the results obtained are substantially different. Our approach is original in several aspects that are not jointly present in previous articles. Firstly, we consider problems with several nonlinear constraints, which is rare[10]. Secondly, parameters are not restricted to prices and income as in consumer theory. In particular, we allow for parameters common to objective and constraints[11]. Parameters common to objective and constraint functions are important because of exogenous variables that in! uence preferences as well as constraints. For example, exogenous skills may in! uence both the preferences and the productivity of an individual. Also, optima considered in incentive or game theory problems often correspond to objective and constraint functions de&ned from similar utility functions. Then, the same exogenous determinants of these utilities may appear in objective and constraint functions. Thirdly, we study several fundamental structures which have not been exhibited before: (1) the intrinsic metric of the optimisation problem; (2) the fundamental hyperplanes of the problem; (3) a projec-

[6] Suen and Hung Mo (1994).
[7] Weymark (1987), Blomquist (1989).
[8] Madden (1991).
[9] Chiappori (1988, 1992), Browning and Chiappori (1998).
[10] See however Silberberg (1974), Chichilnisky and Kalman (1978a, 1978b), Hatta (1980), Caputo (1999), Drandakis (2000).
[11] as in Kalman (1968), Kalman and Intriligator (1973), Chichilnisky and Kalman (1978a, 1978b).

tion on the tangent subspace to the constraints; (4) the change in relative implicit prices respectively associated with the indifference hypersurfaces and the constraints. Fourthly, we breakdown the comparative statics into elementary building blocks that have simple geometric interpretations. This decomposition is pushed further and is more re&ned than what has been done in the literature.

What is the full geometrical structure of the general comparative statics? How does it help us: to clarify the analysis of decisions, to specify and to estimate general models, to compare heterogeneous models? The aim of this paper is to answer these questions. In Section 2, we present the general optimisation problem of interest. We study the geometry of the comparative statics in Section 3. We provide examples of application in Section 4. Finally, we conclude in Section 5. The proofs are generally presented in the appendix.

2. The Fundamental Structures of the Optimisation Problem

2.1. The problem

We consider the following general optimisation programme.

$$\textit{(PB)} \quad \max_x U(x;\theta) \qquad \text{subject to:} \quad g(x;\theta) \leq \mu, \ \ x \in X \subset R^n, \ \ \theta \in \Theta \subset R^m,$$

where x is the decision vector and θ is the parameter vector. U is the objective function from $X \times \Theta$ to R where X and Θ are open sets and X is convex. U is assumed to be of class C^2 at the optimum on X. Moreover, g is a vector of constraint functions from $X \times \Theta$ to R^q, which is of class C^2 at the optimum on X. μ is a parameter in R^q that can be interpreted as the vector level of implicit resources when g is increasing in x. The parameter vector θ is common to U and g, while some components of θ may appear only in U or in g. μ enables us to conveniently denote the constraint levels that will intervene in the interpretation of the comparative statics. However, this is not necessary and more general programmes can be written with this parameter μ absorbed in parameter θ as well stated in Pauwels (1979) and Caputo (1999). They would essentially yield the same type of results. To avoid absurdities we assume that there are more decisions than constraints, i.e. $n > q$.

To ensure the constraint quali&cation, the gradient vectors $\nabla \mathrm{g}^i$ are assumed to be linearly independent for all binding constraints, $i = 1$ through q. The notations for matrix operations and differential operations are given in the appendix.

We use the primal method to derive results of comparative statics because it enables us to break down the sensitivity equation into its basic contributing elements in terms of functions U and g that are often the functions that researchers specify. However, similar results could be obtained by using a dual approach[12].

The Lagrange function associated to Problem *(PB)* is $L = U - \lambda' g$ where λ is a vector of Kuhn-Tucker multipliers. Because of the linear independence of the constraint gradients, the &rst-order Kuhn-Tucker conditions, denoted KTC, are necessary for an optimum.Next, we give a sufficient condition for the KTC to characterise a unique optimum. In the following of the paper, we operate under this weak assumption so as to focus on a unique solution. However, locally unique optima could be treated similarly.

[12]Dual approaches of general optimising models have been studied for example by Silberberg (1974), Hatta (1980), Caputo (1999).

Proposition 1. *If U is strictly pseudo-concave and g is quasi-convex (i.e. each coordinate of g is quasi-convex, which is equivalent to the convexity of the feasible set provided that X is convex), the KTC are sufficient for a unique global optimum.*

With several regimes (sets of binding constraints), the derivation of optimal solutions by using the implicit function theorem may lead to non-continuous or non-differentiable decison functions. We therefore focus on a single regime so as to study the comparative statics with differentiable decision functions, without mixing them with the question of change in regime. In a speci&c regime, the KTC can be simpli&ed as follows.

$$\begin{cases} U_x - g_x\lambda = 0 \\ g(x, \theta) = \mu \\ \lambda \geq 0 \end{cases} \tag{2.1}$$

When the Jacobian matrix of the KTC with respect to (x, λ) is not singular, the implicit function theorem can be applied to the set of equations in System 2.1. Then, there exist optimal and locally unique solutions $x(\theta,\ \mu)$ and $\lambda(\theta,\ \mu)$ of class C^1. Their differentials are represented by eq. 2.2.

$$\begin{pmatrix} dx \\ d\lambda \end{pmatrix} = \begin{bmatrix} L_{xx} & -g_x \\ -g'_x & 0 \end{bmatrix}^{-1} \begin{bmatrix} -L_{\theta x} & 0 \\ g'_\theta & -Id_q \end{bmatrix} \begin{pmatrix} d\theta \\ d\mu \end{pmatrix} \tag{2.2}$$

We now enter in the geometric interpretation of the comparative statics, by discussing &rst the fundamental hyperplanes associated with the KTC, then the intrinsic metric of the optimisation problem. The geometric structure of the comparative statics is anchored on the motion of these hyperplanes and the consequences of this motion on decision changes is measured in terms of this metric.

2.2. The fundamental hyperplanes

We now need to discuss the fundamental hyperplanes that structure the problem. These hyperplanes characterise (1) the directions in which decisions can change; (2) the optimal decisions themselves as solutions of the KTC; (3) the contribution of the different constraints to the decision changes. For comparative statics, these hyperplanes are sufficient &rst-order approximations of the structures of the problem, and therefore summarize fully its marginal changes. Each hyperplane is completely described by its level and its direction vector, whose sole changes will be needed to describe the impact of parameter changes on the problem. Firstly, from the derivation of the ith binding constraints, one can de&ne a *tangent hyperplane*. Its equation is the j^{th} equation in System 2.3. Secondly, by grouping all these equations we obtain System 2.3 that de&nes the *tangent subspace to the constraints* at a point x^0 on the frontier and where x is the unknown vector. Because of the constraint quali&cation condition, the tangent hyperplanes of two distinct constraints cannot be parallel. From the gradient of these hyperplanes, one can de&ne a multi-dimensional cone in which the gradient vector of the optimal indifference hypersurface at the optimal point is included. When there are only equality constraints, the tangent cone is the tangent subspace. Therefore, the tangent subspace is the intersection of all the tangent hyperplanes to speci&c constraints.

$$\begin{cases} g_x^{1\prime}(x^0, \theta).(x - x^0) = 0 \\ \dots \\ g_x^{q\prime}(x^0, \theta).(x - x^0) = 0 \end{cases} \tag{2.3}$$

Thirdly, the separating hyperplane (*SH)* at the optimum separates the optimal indifference hypersurface and the feasible set de&ned by the constraints. Its equation in x can be characterised by the hyperplane tangency to the indifference curve at the optimum, as in eq. 2.4.

$$U_x'(x^*, \theta).(x - x^*) = 0 \tag{2.4}$$

where x^* is the optimal decision vector and x is an unknown decision vector. Equivalently, by using the KTC it can be written in terms of the hyperplane tangency to the constraint frontier at the optimum, as in eq. 2.5. In that case, the gradients of different constraints are linked by the optimal Kuhn-Tucker multipliers λ_i^*, which exist under constraint quali&cation.

$$\sum_{i=1}^{q} \lambda_i^* g_x^{i\prime}(x^*, \theta).(x - x^*) = 0 \tag{2.5}$$

Note that, even at the optimum, the tangent hyperplane to a given constraint is not necessarily tangent to the optimal indifference surface, and therefore does not generally correspond to the SH. However, the tangent hyperplane to a given constraint includes directions that are tangent at the optimum to the optimal indifference hypersurface. They are the directions included in the tangent subspace.

It is useful not to restrict the attention to the SH only as it is usually done, but to examine also the constraint-speci&c tangent hyperplanes. Indeed, in contrast with the SH, once x^* is known the tangent hyperplanes can be observed if the constraints are observable. Then, the unobservable motion of the SH induced by changes in parameters can be described in terms of observable motions of the constraint-speci&c tangent hyperplanes and of generally unobservable KTC multipliers. Each of these motions can be separately analysed. Finally, the normal vector of each tangent hyperplane is important for this paper because it is related to implicit prices that will intervene in the analysis. To be able to relate measures of the motion of the fundamental hyperplanes with measures of the decision changes, we need to study the intrinsic metric of the optimisation problem. To our knowledge, this is the &rst time that this metric has been considered in the economic literature.

2.3. The intrinsic metric

To describe this metric, we borrow our vocabulary from differential geometry (see Dieudonné, 1972). The system $L_x = 0$, which characterises the decision functions from the KTC, describes an hypersurface that can be parameterised by $x \equiv (x_1, \ldots, x_n)$. This hypersurface is the differential manifold of the stationary points of the Lagrange function, or Lagrangian stationary set (denoted LSS). Then, the Jacobian matrix of L_x with respect to x, which is L_{xx} embodies the information contained in the &rst differential form on this hypersurface.

To clarify this point, let us consider the example of a 3-dimensional space and a surface parameterised as $a(u, v)$, where u and v are the surface parameters. On this surface, the length l of a curve parametrised by t as $(u = \phi_1(t), v = \phi_2(t))$ and limited by $a < t < b$ is given by $l = \int_a^b \sqrt{E.(\phi_1'(t))^2 + 2F.\phi_1'(t)\phi_2'(t) + G.(\phi_2'(t))^2} dt$,

where $E =< \frac{\partial a}{\partial u}, \frac{\partial a}{\partial u} >, F =< \frac{\partial a}{\partial u}, \frac{\partial a}{\partial v} >, G =< \frac{\partial a}{\partial v}, \frac{\partial a}{\partial v} >$ and $<, >$ denotes the canonical scalar product. Here, the derivatives of the function a describing the surface determine the surface metric. In the case of the hypersurface of equation $L_x = 0$ in a n-dimensional space, the analogues of the three differential elements E, F, G in dimension 3 can be directly calculated from L_{xx} by using two independent directions for each element. Therefore, L_{xx} characterises the *first fundamental metric form* (Rheinhart and Soeder, 1997) of the hypersurface LSS. Geometric properties that only depend on the first fundamental form are called "intrinsic properties". We have therefore exhibited the intrinsic metric of the LSS.

What is the origin of this metric? Contrarily to what is generally believed, the effect of parameter changes on decisions does not occur freely in the canonical space of decisions. Because we deal with marginal optimal variations, we need to remember that all these changes occur on the hypersurface defined by the KTC. All variations in parameters and decision functions are ineluctably linked by the fact that they correspond to an optimum and must therefore satisfy $L_x = 0$. Therefore, the geometric structure of the comparative statics is more fundamentally characterised in the LSS and by referring to the shape of the LSS, which is locally described by its intrinsic metric.

However, it is usually to express the decision changes in terms of the canonical spacesof decisions, that is in R^n in our case. Therefore, we must convert the fundamental decision changes occurring in the LSS in terms of these canonical spaces.

These conversions are made by using the metric matrix $(L_{xx})^{-1}$, which is symmetric and positive definiteness and therefore defines a scalar product.

The scalar product arises when one considers the second-order local quadratic approximation of the Lagrange function. At the optimum x^*, $L_x = 0$, then $L(x) \simeq L(x^*) + (x - x^*)'.L_{xx}(x^*).(x - x^*)$ where x is the unknown vector. Therefore, matrix L_{xx} summarises the local shape of the optimisation problem at x^*. $(L_{xx})^{-1}$ is the metric matrix that controls for the shape of the Lagrangian in its role of connection of decisions and parameters.

It is unclear which perspective on $(L_{xx})^{-1}$ is the most appropriate: first differential form of the LSS, approximation of the Lagrange function, matrix of change in vector basis between the LSS and R^n. We encompass these different point of view by denoting $(L_{xx})^{-1}$ the "intrinsic metric of the problem". Equipped with the above geometric notions, we are now ready to analyse the geometry of the comparative statics.

3. The Geometry of the Comparative Statics

We now successively examine the sensitivity equation and its geometric components: the generalised substitution and resource matrices; the implicit prices and the implicit resources; the rotations and the translations. Then, we end this section with a discussion of the application of this geometric structure.

3.1. The sensitivity equation

From the inversion in System 2.2 by blocks by denoting

$$\begin{bmatrix} L_{xx} & -g_x \\ -g_x' & 0 \end{bmatrix}^{-1} = \begin{bmatrix} S & -R \\ -R' & -E \end{bmatrix}$$, we obtain the sensitivity equation:

$$dx = -S\,U_{\theta x}\,d\theta + S\sum_k \lambda_k\, g^k_{\theta x}\,d\theta + R\,d\mu - R\,g'_\theta d\theta \tag{3.1}$$

where $R = (L_{xx})^{-1} g_x\, [g_x'\, (L_{xx})^{-1} g_x]^{-1} = \frac{\partial x}{\partial \mu'}$

and $S = (L_{xx})^{-1}[I_n - g_x\,[g_x'\,(L_{xx})^{-1}g_x]^{-1}\,g_x'\,(L_{xx})^{-1}]$.

Eq. 3.1 can also be synthetically rewritten: $dx = -S\,L_{\theta x}\,d\theta + R\,d\mu - R\,g'_\theta d\theta$. Based on this characterisation, Proposition 1 exhibits the fundamental components of the comparative statics.

Proposition 1. *The matrix of the marginal variations of decisions can be written as*

$$\frac{\partial x}{\partial \theta'} = -S\frac{\partial \tilde{P}}{\partial \theta'} + S\frac{\partial \hat{P}}{\partial \theta'} - R\frac{\partial \hat{D}}{\partial \theta'} \tag{3.2}$$

Its general term is

$$\frac{\partial x_i}{\partial \theta_j} = -\sum_{k=1}^{n} S_{ik}\frac{\partial \tilde{P}_k}{\partial \theta_j} + \sum_{p=1}^{q}\sum_{k=1}^{n} S_{ik}\frac{\partial \hat{P}^p_k}{\partial \theta_j} - \sum_{p=1}^{q} R_{ip}\frac{\partial \hat{D}_p}{\partial \theta_j} \tag{3.3}$$

where $\hat{P}^p \equiv \lambda_p g^p_x(x;\theta)$ is denoted the vector of the ***implicit price functions associated with the $\mathbf{p^{th}}$ constraint****, and*

$\hat{P} = \sum_{p=1}^{q} \hat{P}^p$ *is denoted the vector of the* ***implicit price functions associated with the frontier*** *(of the constraints).*

$\frac{\partial \hat{P}}{\partial \theta'}$ *is the tridimensional tensor composed of the* $\frac{\partial \hat{P}^p}{\partial \theta'}$, *p = 1, …, q.*

$\tilde{P} \equiv U_x(x, \theta)$ *is denoted the vector of the* ***implicit price functions associated with the preferences.***

$\hat{D} \equiv g(x, \theta)$ *is denoted the vector of the* ***implicit resource functions associated with the constraints.*** *Moreover,*

$$\frac{\partial x}{\partial \mu'} = R. \tag{3.4}$$

As we shall show eqs. 3.2 and 3.4 embody the geometric content of the comparative statics. Eqs. 3.2 and 3.3 generalise important results for the basic consumer model (Hicks, 1939) and consumer models with one nonlinear constraint (Edlefsen, 1981, Blomquist, 1989, Suen and Hung Mo, 1994). Hicks introduces the diagramme of substitution and income effects for a consumer with two goods. In this diagramme, the motion of the budget line is decomposed in a rotation and a translation that yield convenient illustrations of the substitution and income effects. Edlefsen generalises it by considering the motion of the implicit budget line in the case of one nonlinear budget constraint and with &xed preferences. Our contribution here is to generalise such geometrical approach to the case of several constraints and general parameters allowing for preferences changes. One particularity of the case of several constraints is that several tangent hyperplanes to the constraints can be distinguished.

Also, the change in preferences can be accounted for by the motion of the SH. This leads to multiple terms with implicit prices and implicit resources. The following paragraph clari&es the geometrical nature of the components of the comparative statics, starting with matrices S and R.

3.2. The components

(a) The generalised substitution and resource matrices: eq. 3.4 shows that matrix R describes the effect of the implicit resources, μ, associated with the levels of the constraints. We denoted it the "generalised resource matrix". Matrix S is denoted the "generalised substitution matrix". This is justified by the fact that the term $Rg'_\theta \, d\theta$ in eq. 3.1 describes the resource changes necessary to compensate the translation of constraints corresponding to vector $g'_\theta \, d\theta$. Indeed, the compensation has a form akin to that of the "resources effects", $R \, d\mu$. Thus, matrix S converts $L_{\theta x}$ into the global substitution effect (or "compensated substitution effect").

Various substitution matrices have been well studied in the literature, often for special cases. Matrices similar to S are in Slutsky (1915), Hicks (1939), Samuelson (1947) and Eldlefsen (1981) with one constraint, and in McKay and Withney (1980) and Drandakis (2000) with several constraints; matrices similar to $SL_{\theta x}$ are in Kalman (1968), Kalman and Intriligator (1973), Blomquist (1988) with one constraint and in Chichilnisky and Kalman (1978), Hatta (1980), Drandakis (2000) with several constraints; matrices similar to $L'_{\theta x} SL_{\theta x}$ are in Silberberg (1974), Hatta (1980), Caputo (1999) with several constraints. In the appendix, we further discuss the contribution of these authors who do not deal with the same geometry of the comparative statics and have different focus from ours.

The structural formula of the generalised substitution matrix, shown below eq. 3.1, has been derived in the literature only in the one constraint case (see Kalman and Intriligator) with a scalar division instead of the matrix inversion with several constraints, and we extend it to the multi-constraint case. Symmetry, negative semi-de&niteness, rank and orthogonality properties of S have been studied (Pauwels, 1979) for generalised substitution matrices or related ones and we simply recall them now: S is symmetric; $\dim(KerS) = q$ (the number of linearly independent constraints); S is orthogonal to the gradient vectors of the constraints; S is negative de&nite in the tangent space to the constraints. The symmetry of S is related to Frobenius conditions of local integrability of the Lagrange function from the KTC. The negative semi-de&niteness of matrix S at the optimum corresponds to the negative response of compensated decisions to marginal variations in the direct implicit prices associated to the constraint frontier. It is related to the local stability of the equilibrium that is ensured by the negative de&nitiveness of matrix L_{xx} in the tangent space to the constraints. By contrast, the decomposition of S into a projector matrix and a metric matrix has not been investigated. We &ll this gap in Proposition 2.

Proposition 2. *Inside each regime, the generalised substitution matrix S is as below eq. 3.1 and is equal to* $S \equiv (L_{xx})^{-1}[I_n - P_1]$,

where $P_1 \equiv g_x\,[g_x'\,(L_{xx})^{-1}g_x]^{-1}\,g_x'\,(L_{xx})^{-1}$. *Thus, S satis&es:*

(a) $L_{xx}\ S = I_n - P_1$ *is the matrix of the oblique projector onto the tangent subspace to the constraints, with* $(L_{xx})^{-1}$ *the matrix of the scalar product characterising the obliqueness of the projection.*

P_1 *is the oblique projector onto the subspace spanned by* ∇g *with* $(L_{xx})^{-1}$ *the matrix of the scalar product describing the obliqueness of the projection.*

(b) $S = M[I_n - P_2]M$ *where* $M \equiv (L_{xx})^{-1/2}$ *and* $P_2 \equiv N[N'N]^{-1}N'$ *is the orthogonal projection onto the subspace spanned by the columns of* $N \equiv (L_{xx})^{-1/2}\nabla g$*. Moreover,* $P_2 = MP_1M^{-1}$.

To our knowledge, the results in Proposition 2 have never been stated before. They allow us to break down the formula of S, making it easier to interpret. Given the matrix of the scalar product, the projectors are uniquely determined by the subspace onto which the projection takes place. Result *(a)* and the formula of S indicate that the generalised substitution matrix can be described as a first-stage projection onto the tangent subspace to the constraint followed by a second-stage normalisation based on the scalar product of the intrinsic metric. Result *(b)* exhibits the projection involved in the substitution matrix from the perspective of the LSS. The normalisation with respect to the intrinsic metric can therefore be applied after the projection, or in two steps, before and after the projection. These different perspectives depend on if one considers the projection as in the manifold defined by the KTC or in the canonical affine space.

The orthogonality of the projection explicitly appears in P_2 and not in P_1. This orthogonality is with respect to $(L_{xx})^{-1}g_x$, not with respect to g_x. However, P_1 and P_2 are akin. Indeed, $I - P_2$ represents the projection on the constraints in the LSS, while $I - P_1$ represents it in the canonical space. As *(b)* shows, M is the matrix of the change in basis between these two spaces. The "sphericised" matrix $N = (L_{xx})^{-1/2}\nabla g$ is the transformation of the matrix of the gradient vectors of the constraints, ∇g, as it appears in the canonical space, into the formula in the LSS. Therefore, P_2 is the matrix of the orthogonal projector (in the LSS) onto the hypersurface orthogonal to

N, which is the tangent space to the constraints in the LSS.

The rank and orthogonality properties of S result from the decisions x remaining on the constraints, via the projection operator incorporated in S. We now turn to the generalised resource matrix, R.

Proposition 3. *The Generalised Resource Matrix is defined by*

$$R = (L_{xx})^{-1} g_x \, [g_x' \, (L_{xx})^{-1} g_x]^{-1}.$$

R is a right-inverse of g_x'.

This structural formula for R has not been explicited for the multi-constraint case. It is a generalisation of the matrix of income effects with one nonlinear budget constraint, for the which matrix inversion can be written as a scalar division. R is a right-inverse of g_x' rather than an inverse because there are more decisions than constraints. In a sense, g_x' converts the decision variations into the constraint variations and its right-inverse R plays the opposite role.

It is now easy to exploit the general formulae of matrices S and R to derive a matrical characterisation of the compensation mechanism. This characterisation shows that the compensation has two natural alternative interpretations as a translation or as a dilatation of the inverse of the Hessian of the Lagrange function. Indeed, on the hand $S = L^{-1} - RG'L^{-1}$, where $L = L_{xx}$ and $G = g_x$. The second term in the above right-hand side term is a clearly linear combination of the resource effects described by matrix R. This corresponds to the compensation by a translation of the fundamental hyperplane. In this optic, the decomposition in substitution and resource effects is revealed as a decomposition property of the intrinsic metric represented by $L^{-1} = S + RG'L^{-1}$.

On the other hand, $S = (I_n - RG')L^{-1}$, where matrix $I_n - RG'$ can be seen to dilate the symmetric and negative de&nitive L^{-1}. Here, the substitution effects are seen as resulting from a dilatation of the curvature of Lagrangian, i.e. of the intrinsic metric. These two alternative interpretations are more fundamental than usual substitution/resource effects, since they do not depend on the implicit prices and the implicit resources. Let us now turn to the changes in the relative implicit prices that are combined with matrix S to yield the complete substitution effect.

(b) The relative implicit prices: $\tilde{P}$ is composed of elements similar to marginal utilities in consumer theory, and $\hat{P}$ to marginal productivities (g^i_x if g^i is a production function) in producer theory. Therefore, their interpretation in terms of implicit prices can be naturally related to usual prices in consumer theory and producer theory. Naturally, other normalisations are possible, e.g. by dividing $\hat{P}$ and $\tilde{P}$ by the Lagrange multiplier of a given constraint. The projection in S selects what components of the changes in the implicit prices actually matter for decisions. Eqs. 3.2 and 3.4 correspond to the rigid motion of the general SH, decomposed into the rigid motions of the constraint-speci&c tangent hyperplanes and the rigid motion coming from preferences change. The absolute direction of the SH is determined by vector $\tilde{P}$ (equivalently by vector $\hat{P}$ because of the KTC). Now, that is the change in the vector of *relative implicit prices*, de&ned as $B = \tilde{P} - \hat{P}$, that is crucial for describing the relevant motion of the SH. Indeed, $B_\theta \equiv L_{\theta x}$, drives the relative marginal rotation of the SH. What matters for the properties of decisions are the *relative changes* in the

two types of implicit prices, and not their absolute changes. The decision changes are determined by the *relative motion* of the family of indifference hypersurfaces and of the frontier of the constraints. This situation is summarised by the *relative motion* of the SH with respect to constraints and indifference curves. The (virtual) relative rotation of the SH is the combination of two actual rotations with respect to the axes, respectively corresponding to the variations in $\tilde{P}$ and $\hat{P}$.

An interesting feature has been well noted by Edlefsen with one nonlinear budget constraint, which is here valid in a more general setting. As in consumer theory or producer theory where all the commodity prices intervene in demand and supply functions, the implicit prices of all the decisions may intervene in the general comparative statics. However, this number can be here multiplied by the number of constraints plus one if one wants to push the decomposition to its limits. We now describe the relative rotation of the SH.

(c) The relative rotation: The SH and the constraint-speci&c tangent hyperplanes are rigid hypersurfaces. This is why their motions can be described by rigid motions. Every rigid motion is the commutative product of a translation and an affine rotation (in several dimensions). In the sensitivity equation 3.1, the terms $-Rg'_{\theta}d\theta + Rd\mu$ correspond to in&nitesimal translations of the SH, while the other right-hand-side term, $-SL_{\theta x}d\theta$, corresponds to in&nitesimal affine rotations of the SH. These facts may be more obvious in eq. 3.3 that shows the change in a speci&c decision x_j that results from a change in a scalar parameter θ_j. By analogy with consumer theory, $-SL_{\theta x}$ describes the effect of the compensated rotation of the SH about the initial optimum, although here the rotation is relative to the positions of the indifference

hypersurfaces and the constraints rather than absolute. The affine subspace in which the rotation takes place is determined by the projection in matrix S and by the &xed point $x(\theta, \mu)$. Since matrix S is of rank $n - q$, the subspace in which the rotation occurs is of dimension $n - q$. Thus, the substitution effect is a combination of a projection, a metric operator and a relative rotation of the SH.

The multi-dimensional relative rotation of the SH can be seen as the composition of m rotations in dimension two (into a plane), each one corresponding to a θ_j $(j = 1, ..., m)$. Each of these elementary relative rotation can be decomposed in two absolute rotations, respectively associated with the change in the two types of implicit prices.

Let us complete the description of the relative rotation by characterising the rotation angle. Note &rst that 2π radiant rotations are possible, therefore also periodic positions of the SH that are equivalent for the changes in decisions. Moreover, what matters is the difference in changes in $\hat{P}$ and $\tilde{P}$, and not only their absolute changes. These two facts imply that the observation of decision changes may be insufficient to recover the changes in the implicit prices.

$(\tilde{P}, \tilde{P} + d\tilde{P})$ and $(\hat{P}, \hat{P} + d\hat{P})$ respectively de&ne the angles β_j and γ_j of the two absolute rotations of the SH that are caused by a variation $d\theta_j$ of the j^{th} coordinate of θ. The angles β_j and γ_j are easy to characterise because they can be de&ned respectively from the two absolute rotations of the normal vector of SH. The angle formulae are valid in any dimension since the angle of two non-colinear vectors can be de&ned in the plan de&ned by these two vectors. The angle of the relative rotation can then be de&ned as $\alpha_j = \beta_j - \gamma_j \ \ [2\pi]$. The angles can be calculated as shown in the following proposition.

Proposition 4. *The angles β_j and γ_j of the two absolute rotations of the SH that correspond to a variation $d\theta_j$ of the j^{th} coordinate of the parameter are:*

$$\beta_j = (||U_x \wedge (U_x + U_{\theta_j x})||/||U_x||^2)\, d\theta_j \qquad [2\pi]$$

$$\text{and } \gamma_j = (||\textstyle\sum_k \sum_p \lambda_k \lambda_p \left\{g_x^k \wedge (g_x^p + g_{\theta_j x}^p)\right\}||/||\sum_k \lambda_k g_x^k||^2)\, d\theta_j \qquad [2\pi],$$

where $\wedge$ is the vectorial product and $||.||$ is the euclidean norm.

Let us now turn to the other rigid motion of the SH, its translation.

(d) The translation: A change in θ_j also induces a translation of the SH, associated with decision changes $Rg'_{\theta_j} d\theta_j$, along the directions orthogonal to the tangent space to the constraints. Indeed, $g_x R g'_{\theta_j} d\theta_j = g'_{\theta_j} d\theta_j$ and besides $g_x R \frac{\partial x}{\partial \mu'} = \frac{\partial x}{\partial \mu'}$. The vector of resource effects on decision x_i of a per-unit translation associated with the k^{th} constraint, with translation vector $g^k_{\theta_j} d\theta_j$, is measured by R_{ik}. The global translation effect can be broken down into several terms, each one corresponding to a change in a speci&c constraint level. As a direct consequence of Proposition 3, matrix R satis&es the following property similar to Engel aggregation in consumer theory.

Proposition 5.

At the optimum: $\hat{P}'R = \tilde{P}'R = \lambda'$.

Proposition 5 shows that the Lagrange multipliers can be interpreted as the implicit values of the implicit resource effects for unitary translation vector. Thus, they can be seen as implicit prices of the changes in constraint levels. We now present a graphic illustration that shows how changes in decisions, in constraints and in indifference hypersurfaces are related to the motions of the SH and of the optimum.

(e) Graphic illustration: Figure 1 shows the substitution effect corresponding to a parameter variation ("the shock") that implies changes in the objective indifference curves and in the two constraint curves. Marginal variations are metaphorically represented by finite variations. Owing to the shock, the optimal decision goes from points A to D. Points B and C are optimal points associated with the successively envisaged changes in the constraint curves g1 and g2 (respectively denoted g1i and g2i before the shock; g1f and g2f after the shock), without change in the indifference curves. Finally, the shift of the optimal decision from C to D corresponds to the effect of the change in the family of indifference curves (represented by Ui before the shock and by Uf after the shock). The generalised substitution effect corresponds to the move of the optimal decision from A to A$''$, where A and A$''$ are on the same SH, and where the SH moves from BCi (the initial implicit "budget constraint") to BCf$''$ (the final implicit budget constraint without resource effects) due to the simultaneous changes in the constraints and in the indifferences curves. BCf$''$ has the same direction as BCf (the final implicit budget constraint).

The motion from BCi to BCf represents the absolute motion of the SH, while the motion from BCi to BCf$''$ only represents the absolute rotation of the SH. The relative motion of the SH is the difference of two absolute motions associated with the motion from the solution from point A to point C, which corresponds to the change in $\hat{P}$; and the motion from C to D, which corresponds to the change in $\tilde{P}$. We now wrap up the discussion by envisaging the application of the geometrical structure.

3.3. Application of the geometrical structure

The geometrical framework of the comparative statics that we have exhibited can be used for several purposes: understanding, specifying, testing, estimating and comparing models.

*(a) Understanding the model mechanisms*d by decomposing the decision changes in elementary elements that are geometrically inter-related: the decomposition of the comparative statics is very detailed, since beyond resource and substitution effects, it is carried out across all the parameters, all the decisions, all the constraint and objective functions. Each combination of these elements in eq. 3.3 corresponds to an elementary comparative statics effect. Not only this but also the structure of matrices S and R provide considerable re&nement of the simple use of the implicit function theorem for the KTC. Thhis new perspective on the projection and metric structures is likely to guide the theoretical debate on the subject. In particular, special attention must be devoted to matrices L_{xx} and g_x.

Importantly, the list of the relevant implicit prices and implicit resources is now straightforward to establish and facilitates model analysis. We have shown that what matters is the difference in the changes of the two types of implicit price vectors. This suggests to simultaneously analyse and specify the objective and constraint changes.

One can choose the aggregation level of the comparative statics equations to understand it better. At the highest aggregation level, the term $(-SL_{\theta x})$ in eq. 3.1 is the sum of all substitution effects and has the simple geometric interpretation of representing the (relative) rotation of SH as for for consumer theory. The component S_{ik}

of S measures the per-unit substitution effect on x_i of the SH relative pivoting associated with the marginal variation in the k^{th} relative implicit price ($\frac{\partial L}{\partial x^k \partial \theta_j} = \frac{\partial \tilde{P}_k}{\partial \theta_j} - \frac{\partial \hat{P}_k}{\partial \theta_j}$ for any j). At a lower aggregation level, $-Sg^p_{\theta x}$ describes the effect of the absolute rotation of the tangent hyperplane of the p^{th} constraint. The rotations of the tangent hyperplanes to the various constraints can be linearly aggregated by using Lagrange multipliers (λ_p) that express the impact of a change in level of this p^{th} constraint on the objective level. Similarly, the resource effects caused by the change in θ, $-R\frac{\partial \hat{D}}{\partial \theta'}$ in eq. 3.2, can be decomposed in q constraint-speci&c terms. By cancelling the appropriate terms in eq. 3.3, we can derive contra-factual comparative statics to study cases where some constraints or the indifference curves are &xed. The comparative statics of different optimisation regimes can thus be easily related.

Finally, the projection reveals that what matters for the substitution effects are only the features that are conserved by this projection. For example, Muller (2001) exploits this property for deriving weak concavity conditions for general models. All this clari&cation of the structure of the comparative statics can be used in the model speci&cation as we discuss now.

(b) Specifying models for objective, constraint and decision functions: we mean by "specifying", de&ning the functional form of the functions and of the associated error terms, but also selecting the decisions, parameters and constraints to consider. The analysis shows that it may be appropriate to directly start the speci&cation with implicit prices and implicit resources (or with rotation angles and translation vectors) rather than with constraint and objective functions. In particular, a preliminary re! exion on the variations of the implicit prices and implicit resources may yield

models with better properties. For general models, the speci&cation of the interaction of the constraint functions and of the objective function should be carefully dealt with. This simultaneous speci&cation is possible because the list of the implicit prices and implicit resources is easy to read in the comparative statics equations. Moreover, each of these implicit prices and implicit resources can be the object of specialised speci&cation work if they are of particular interest for the study. Another usual approach is to start from the functional form of the constraint and utility functions. These forms can be examined to characterise the properties of the various decision-speci&c implicit prices. This should yield a better understanding of the mechanisms involved in the comparative statics and guide the choice of these functional forms.

The comparative statics equations can also be used to de&ne reduced-form equations where only the terms for implicit resources and implicit prices need be speci&ed. The coefficients coming from matrices R and S are here considered as unknown or are to be estimated from the data. In that case, the differentied KTC need not be solved explicitely. Alternatively, explanatory variables of the implicit prices and implicit resources can be used to de&ne the reduced-forms by replacing these implicit elements with, for example, linear combinations of the explanatory variables. In particular, the aggregative hierarchisation of the implicit prices and implicit resources could guide the introduction of these explanatory variables. Then, theoretical restrictions on the coefficients of the reduced-form equations could be imposed or tested by using the properties of matrices R and S. We discuss now how some data to test or to estimate the model.

(c) Helping model test and estimation for applied work: Our framework highlights

the geometrical meaning of the theoretical restrictions on matrices S and R. Symmetry and negative semi-de&niteness restrictions are ultimately originated in L_{xx} that is inversed and truncated by a projection to produce matrix S. The decomposition of the comparative statics may help to express these restrictions in terms of observable elements because it facilitates the manipulation of equations. This will be illustrated in the example section.

Because elementary implicit prices and elementary implicit resources are clearly identi&ed and characterised, one can estimate the comparative statics equations in two stages by plugging for the second stage the predicted implicit prices and predicted implicit resources obtained from a preliminary estimation. This method has been used by Jacoby (1993) for models with one technological constraint for the analysis of peasant family labour supply in Peru. It can be generalised to complex models when sufficient data is available.

(d) Comparing heterogeneous model: &nally, our geometric framework evinces the fundamental elements to compare across heterogeneous models that may incorporate different constraints, different functional forms, different decisions and different parameters. Because the comparative statics is summarised by $S, R, \hat{P}_\theta, \tilde{P}_\theta$ and $\hat{D}_\theta$, these elements can be compared across models sharing a decision and a parameter of interest. Also, the formulae of the angles and translation vectors can be compared across models. Therefore, the response of the decision to a change in the parameter can be compared from a theoretical standpoint in the different models. More fundamentally, one could directly compare the properties of matrices L_{xx} and g_x for different models, since they are the basic components of S and R.

A more restrictive way of comparing heterogeneous models is to compare the substitution effects embodied in S across models sharing the same set of decisions since the dimensions of S depends only on the number of decisions and not on the constraints or on the parameters. This approach was &rst proposed by Edlefsen with one constraint. It can be extended to several constraints and varying preferences. Similarly, the resource effects can be compared via matrix R across models sharing the same set of decisions and constraints, but not necessarily the same parameters and the same objective. However, what we have shown above is that the range of comparisons can be much broader and does not even require that the models to compare necessarily have the same set of decisions or the same set of constraints. To illustrate our geometric framework, we now examine a few examples, starting with well-known economic models.

4. Examples

(a) Consumer model: the optimisation programme is: $\max_{x} U(x)$ subject to $p'x = \mu$, where U is a strongly quasi-concave utility function of type C^2 and increasing in its arguments, x is the vector of consumption demands, p is the price vector and is the parameter of interest. μ is the consumer income. To simplify the exposition, the positivity constraints are omitted for all the examples. λ denotes the Lagrange multiplier in all the examples with one constraint. The fundamental elements of the model are:

$\hat{P} = \lambda p; \tilde{P} = U_x; \hat{D} = p'x; \frac{\partial \hat{P}}{\partial p'} = \lambda . I_n; \frac{\partial \tilde{P}}{\partial p'} = 0; \frac{\partial \hat{D}}{\partial p'} = x'.$

$S = (U_{xx})^{-1}\left[I - \frac{pp'(U_{xx})^{-1}}{p'(U_{xx})^{-1}p}\right]$ and $R = \frac{(U_{xx})^{-1}p}{p'(U_{xx})^{-1}p}$.

Under this form S is sometimes called the "fundamental matrix of consumer theory". These well-known formulae of S and R have not been found easy to interpret in the literature. In our framework, the projection subspace and the metric associated with the optimisation problem provide a natural interpretation of these matrices. Then, all that need to be known for a clear understanding of S and R are the matrices $(L_{xx})^{-1}$ and g_x. In the consumer case, matrix S embodies: (1) an oblique projection on the budget constraint of direction determined by p, and (2) a metric $(U_{xx})^{-1}$ determined by the convexity of preferences.

It is now easy to understand why the economists have not noticed this geometrical structure of S in consumer theory. Indeed, firstly, in the above formula of S the division by a scalar compound hides the matrical shape that would help to recognize a projection operator. Secondly, the linear constraint disappears in $L_{xx} = U_{xx}$, which hides the presence of the intrinsic metric of the optimisation problem. Finally, the price vector p appears instead of the explicit constraint gradient that would have revealed the role of the tangent space to the constraint.

This suggests that for each model of interest, it is useful to systematically examine the tangent subspace as projection set and the intrinsic metric. This is particularly interesting when the metric has special characterics. For example, in the consumer model the metric does not depend on the Lagrange multiplier nor on the constraints.

For models where there is only one constraint, we denote from now R_{i1} as R_i.The comparative statics equations of the consumer model are: $\frac{\partial x_i}{\partial p_j} = S_{ij}\lambda - R_i x_j$ and

$\frac{\partial x_i}{\partial \mu} = R_i$ $(i, j = 1, \ldots, n)$. The fact that only one elementary substitution effect $(S_{ij}\lambda)$ intervenes in the &rst equation results from two simpli&cations: (1) there is no change in preferences; (2) there is a unique constraint that is bilinear in parameters and decisions. Let us now see what happens if preferences can change.

(b) Consumer with prices in the preferences (Kalman, 1968, Kalman and Intriligator, 1973): the optimisation programme is: $\max_x U(x, p)$ subject to $p'x = \mu$, with the same notations than for Example (a). This gives: $\hat{P} = \lambda p; \tilde{P} = U_x; \hat{D} = p'x; \frac{\partial \hat{P}}{\partial p'} = \lambda.I_n; \frac{\partial \tilde{P}}{\partial p'} = U_{px}; \frac{\partial \hat{D}}{\partial p'} = x'$.

The formulae for S and R are the same than for the basic consumer model. By contrast, the equations of the comparative statics are now more complex:

$\frac{\partial x_i}{\partial p_j} = \left(-\sum_k S_{ik} U_{p_j x_k} \right) + S_{ij}\lambda - R_i x_j$ and $\frac{\partial x_i}{\partial \mu} = R_i$.

The comparative statics of Examples (a) and (b) only differ by the presence of terms from $U_{\theta x}$, while they admit the same structural formulae for S and R. This corresponds to the fact that the decision changes are driven by the derivatives of the implicit prices ($\tilde{P}_\theta$ and $\hat{P}_\theta$) and of the implicit resources ($\hat{D}_\theta$) rather by the levels of these implicit prices and implicit resources. The term in parentheses in the right-hand-side term of the &rst equation was not decomposed by past authors. However, the above equation clearly shows that this term can be analysed to see how its sign may change depending on the type of in! uence of prices on the various arguments of preferences and thereby to guide the model speci&cation. Note that even the traditional direct substitution effect (in the sense of $\frac{\partial x_i}{\partial p_i} + x_i \frac{\partial x_i}{\partial \mu}$) is not necessary negative in that case. Therefore, the usual negativity test of consumer theory would fail. We now examine an extension of this model with a nonlinear production function in the budget constraint.

(c) General mono-producer model (Kalman and Intriligator, 1973): the optimisation programme is: $\max_x U(x;p,w)$ subject to $\pi = pf(x) - w'x$, where U is the objective function, x is the vector of input demands, p is the output price, w is the vector of input prices, f is the production function and π is the profit that is considered as fixed (e.g. by stockholders). From now, the generalised concavity and differentiability properties of the functions are omitted to avoid a too tedious presentation. The parameters are: $\theta = (p, w')'; \mu = \pi$. The implicit prices, implicit resources and their variations are: $\hat{P} = \lambda[pf_x - w]; \tilde{P} = U_x; \hat{D} = pf(x) - w'x; \frac{\partial \hat{P}}{\partial p} = \lambda f_x; \frac{\partial \tilde{P}}{\partial p} = U_{px}; \frac{\partial \hat{D}}{\partial w'} = -x'$.

$\frac{\partial \hat{P}}{\partial w'} = -\lambda I_n; \frac{\partial \tilde{P}}{\partial w'} = U_{wx}; \frac{\partial \hat{D}}{\partial p} = f(x)$. Matrices S et R can be deduced from $L_{xx} = U_{xx} - \lambda p f_{xx}$ and $g_x = pf_x - w$. This shows that this model is of different nature from those of the previous examples that are isomorphic as far as $S, R, \tilde{P}$ and $\hat{P}$ are concerned. The comparative statics equations are:

$\frac{\partial x_i}{\partial p} = -\sum_k S_{ik} U_{px_k} + \sum_k S_{ik} \lambda f_{x_k} - R_i f(x);$

$\frac{\partial x_i}{\partial w_j} = -\sum_k S_{ik} U_{w_j x_k} + S_{ij}\lambda - R_i f(x)$ and $\frac{\partial x_i}{\partial \mu} = R_i$.

Again, the global substitution effect can be of either sign. Note the new occurrence of the production function in these equations as compared with Example (b). We now turn to two special cases of the general mono-producer model.

(d) Profit maximising mono-producer: it is Example (c) with $U(x, p, w) = pf(x) - w'x$ with elimination of the constraint. We obtain: $\frac{\partial x_i}{\partial p} = -\sum_k S_{ik} f_{x_k}$ and $\frac{\partial x_i}{\partial w_j} = -S_{ij}$. There is no income effect. The well-known symmetry and positivity restrictions

of input demands are apparent.

$S = (f_{xx})^{-1}/p$. The formula of S is simple because the constraint has been eliminated and the objective function is simple. In this case the projection incorporated in S is the identity transformation. Then, S is the intrinsic metric matrix itself. Note that specifying the problem differently as a pro&t maximisation subject to a production function leads to L_{xx} singular. This difficulty disappears by substitution of the production in the objective function.

- *(e) Baumol producer:* it is Example (c) with $U(x,p,w) = pf(x)$. Here, the constraint is not substituted, which generates implicit prices associated with the objective, $\tilde{P} = pf_x$, and implicit prices associated with the constraint, $\hat{P} = \lambda(pf_x - w)$. The metric matrix is $\frac{(f_{xx})^{-1}}{(1-\lambda)p}$. The equations of the comparative statics are:

$\frac{\partial x_i}{\partial p} = \sum\limits_k S_{ik} f_{x_k}.(\lambda - 1) - R_i f(x)$; $\frac{\partial x_i}{\partial w_j} = -S_{ij}\lambda + R_i f(x)$ and $\frac{\partial x_i}{\partial \mu} = R_i$.

There is a resource effect that is proportional to the production level. The next example includes consumption demands and two production constraints, but no markets.

(f) Autarkic household with two productions: the optimisation programme is: $\max\limits_{x_1,x_2,l} U(x_1,x_2,l,E)$ subject to: $x_1 = F(L,E) - \mu_1; x_2 = G(N,E) - \mu_2; L + N + l = T$, where U is the utility function, x_1 and x_2 are the consumption demands of the two productions, l is the leisure, L and N are the labour inputs allocated to the production of each good, T is the total available time. F and G are the two production functions, μ_1 and μ_2 are &xed costs for the respective production processes. E is an exogenous

skill parameter, which may in! uence preferences and technologies. The last two equations can be regrouped by substitution of N. This yields: $x_2 = G(T - L - l; E) - \mu_2$. The two obtained constraints are binding because the marginal utilities of the two productions are assumed in&nite at zero or above (e.g. if the utility function is of the Cobb-Douglas type). The decision vector is: $x = (x_1, x_2, l, L)'$. The parameter vector of interest is: $\theta = (E, T)'$.

The implicit resource vector is $\hat{D} = (x_1 - F(L, E), x_2 - G(T - L - l, E))'$.

The implicit price vectors are: $\tilde{P} = (U_{x_1}, U_{x_2}, U_l, 0)$; $\hat{P}^1 = (\lambda_1, 0, 0, -\lambda_1 F_L)$ and $\hat{P}^2 = (0, \lambda_2, \lambda_2 G_N, \lambda_2 G_N)$, where λ_1 and λ_2 are the Lagrange multipliers respectively associated with the two constraints. Moreover,

$$L_{xx} = \begin{bmatrix} U_{x_1x_1} & U_{x_1x_2} & U_{x_1l} & 0 \\ U_{x_2x_1} & U_{x_2x_2} & U_{x_2l} & 0 \\ U_{lx_1} & U_{lx_2} & U_{ll} + \lambda_2 G_{NN} & \lambda_2 G_{NN} \\ 0 & 0 & \lambda_2 G_{NN} & \lambda_1 F_{LL} + \lambda_2 G_{NN} \end{bmatrix}.$$

In general L_{xx} is not block-diagonal because of the presence of l in the system. In the case $G_{NN} = 0$ (i.e. G affine in N), a block invertion of L_{xx} is possible to yield a block-diagonal metric matrix. The resource effects are $\frac{\partial x_i}{\partial \mu^1} = R_{i1}$ and $\frac{\partial x_i}{\partial \mu^2} = R_{i2}$ for $x_i = x_1, x_2, l, L$ and $i = 1, 2, 3, 4$.

The effects on the i^{th} decision of a change in the skill level and in the total available time are: $\frac{\partial x_i}{\partial E} = -S_{i1} U_{x_1E} - S_{i2} U_{x_2E} - S_{i3}(U_{lE} - \lambda_2 G_{NE}) + S_{i4}(-\lambda_1 F_{LE} + \lambda_2 G_{NE}) + R_{i1} F_E + R_{i2} G_E$, for $x_i = x_1, x_2, l, L$ and $i = 1, 2, 3, 4$.

$\frac{\partial x_i}{\partial T} = (S_{i3} + S_{i4})\lambda_2 G_{NN} + R_{i2} G_N$, for $x_i = x_1, x_2, l, L$ and $i = 1, 2, 3, 4$.

These equations show the variety of the implicit prices and implicit resources that may intervene even in relatively simple models. There are two resource effects to compensate in order to isolate the global substitution effect of a change in skill, but only one resource effect for the decision changes caused by a change in total time because of the initial substitution of the variable of the labour input in F by the residual time from other time uses. In general the global substitution effects are of ambiguous sign.

The difference in implicit prices, $U_{lE} - \lambda_2 G_{NE}$, describes the impact of skills on the difference between the value of leisure from the preferences and the value of working time for the second technology. Similarly, $\lambda_2 G_{NE} - \lambda_1 F_{LE}$ describes the impact of skills on the difference between the values of working time for the two technologies. These two differences of implicit prices could be the object of speci&c modelling since they express a central feature of the model. That may be a convenient alternative to the separate speci&cation of U, G and F. The household allocates the three time uses according to their relative marginal valuations. The presence of $S_{i3} + S_{i4}$ in the comparative statics equations is related to the existence of two channels, each associated to one production, for the impact of a change in the implicit price of leisure. When this price changes, there is substitution of leisure with the two labour inputs.

The effect on decisions of a change in the available time can be conveniently estimated in two stages, with a preliminary estimation of production function G that would yield predictions for G_N and G_{NN} that could be plugged in the equation for

$\frac{\partial x_i}{\partial T}$. This approach is more delicate for the effect of the skill parameter because of the large number of implicit prices and implicit resources intervening and the difficulty of observing preferences changes.

What happens when the skill parameter does not intervene in the preferences or in one of the technologies? In that case the general formulae for $S, R, \tilde{P}, \hat{P}, \hat{D}$ stay the same, but the variations in $\tilde{P}, \hat{P}$ and $\hat{D}$ can be simplified. One obtains comparative statics equations that can be deduced from the initial equation by cancelling appropriate terms according to modifications in the definitions of $\frac{\partial \tilde{P}}{\partial \theta'}$, $\frac{\partial \hat{P}}{\partial \theta'}$ and $\frac{\partial \hat{D}}{\partial \theta'}$. We now conclude this section with a simple game theory model.

(g) Collective household model: Browning and Chiappori (1998), denoted B&C, propose the following model of collective household.

$$\max_{q^A, q^B, Q} \left\{ \begin{array}{c} \omega(p,I).u^A(q^A, q^B, Q) + (1-\omega(p,I)).u^B(q^A, q^B, Q) \\ \text{subject to } p'(q^A + q^B + Q) - I = 0 \end{array} \right\},$$

where function $\omega(p, I)$ that represents the relative decision power of member A is homogeneous of degree 0. u^A (u^B) is the utility of member A (B). q^A (q^B) is the vector of private consumption of member A (B). Q is the vector of the household public consumption. p is the price vector and I is the household exogenous income. Here, $x = (q^{A\prime}, q^{B\prime}, Q')'$ and $\theta = (p', I)'$. There is no parameter uniquely describing the level of the budget constraint, although I could partly play this role. Naturally, implicit resource effects still exist and could be explicited by introducing virtual parameters. Although, B&C do not comment the comparative statics since their main interest is the derivation of theoretical restrictions for household global demands, it is easy to do so. The metric matrix and the constraint gradients are formally identical to what they are in consumer theory. The vectors of implicit prices, the implicit resource and the comparative statics equations are: $\tilde{P} = \omega u_x^A + (1-\omega)u_x^B, \hat{P} = \lambda p, \hat{D} = p'(q^A + q^B + Q) - I$.

$\frac{\partial x_i}{\partial I} = -\sum\limits_k S_{ik}(u_x^A - u_x^B)\omega_I - R_i$.

$\frac{\partial x_i}{\partial p_j} = -\sum\limits_k S_{ik}(u_x^A - u_x^B)\omega_{p_j} + \sum\limits_k S_{ik}\lambda + R_i(q^A + q^B + Q)$.

This model is a special case of the consumer "with prices (and income) in the preferences" and is largely isomorphic to the models of Examples (a) and (b) with the same type of matrices S and R. The difference of the comparative statics with the pure consumer case essentially comes from the presence of terms with changes in implicit prices associated with the preferences. Here, these terms involve the quantity $u^A - u^B$, revealing the importance of the cardinal specification of preferences.

We now show that it is easy to derive B&C theoretical restrictions from our comparative statics equations without having to use their dual approach. Indeed, by combining the above equations we obtain:

$\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial I} = \left(-\sum\limits_k S_{ik}\right)(u_x^A - u_x^B)(\omega_I x_j + \omega_{p_j}) + \lambda \sum\limits_k S_{ik}$.

The last term in the right-hand-side term is similar to the Slutsky matrix term in the pure consumer model. The first term can be written as taken from a matrix $a.b'$, which is the main result of B&C for whom the general term of b is $\omega_I x_j + \omega_{p_j}$ and the general term of a is $\frac{\partial x_i}{\partial \omega}$. Then, the traditional Slutsky matrix is the sum of a symmetric matrix with a matrix of rank 1. We just need to show that $\left(-\sum\limits_k S_{ik}\right)(u_x^A - u_x^B) = \frac{\partial x_i}{\partial \omega}$. This is easy to obtain by returning to the initial programme and considering ω as a parameter itself instead of a function of p and I. Then, applying again the geometrical derivation of the comparative statics for a change in parameter ω delivers the result.

5. Conclusion

We provide in this paper a full geometrical elucidation of the comparative statics of general optimising models with several nonlinear constraints and with general parameters. The results can be used: to clarify the fundamental mechanisms of the decision changes, to help model speci&cation and estimation, and to compare heterogenous models.

The comparative statics of a complex model can be entirely decomposed into elementary geometric elements across parameters, decisions, constraint and objective functions. Our approach improves on the previous literature by several features: very general problems of comparative statics are studied with several constraints, varying preferences and general parameters; more fundamental mathematical structures are exhibited, including the intrinsic metric of the problem and a fundamental projection; the importance of simultaneously considering the changes in the constraints and in the preferences is stressed and explicitely treated; general implicit prices and implicit resources are introduced, a full characterisation of the rigid motions involved in the comparative statics is achieved, notably by identifying fundamental hyperplanes (i.e. the constraint-speci&c tangent hyperplanes and the separating hyperplane) and by expliciting the formulae for rotation angles and translation vectors.

A generalised substitution matrix and a generalised resource matrix help the convertion of the decision changes from the stationary manifold of the Lagrange function into the canonical space. Parameter variations are converted into variations in implicit prices of the decisions and in implicit resources, which can all be decomposed across constraint and objective functions. Ultimately, what matters for the comparative statics is the change in the *difference* in several types of implicit prices, respectively associated with the preferences and with the constraints. The changes in implicit prices can be interpreted in terms of rotation of the fundamental hyperplanes, while the change in implicit resources correspond to translations of these hyperplanes.

The analysis clari&es the mechanism intervening in the general comparative statics. Notably, the geometrical framework allows us to easily derive generalised Slutsky equations for general models. Also, elementary geometrical elements can be easily studied and handled. Such clari&cation can assist speci&cation, test and estimation of general models, by exhibiting what are the structural characteristics of the decision functions, but also by allowing estimation procedures with preliminary estimations of implicit prices and implicit resources. Finally, the comparison of heterogeneous models can be based on the comparison of the basic elements of the geometrical structure of the comparative statics.

The changes in optimisation regime can be partly dealt with by substituting terms in the general comparative statics equation. However, the possibility of discontinuity or non-differentiability of decisions at the point of regime change must be studied with other methods. Note &nally that a similar geometric analysis can be carried out for the dual approach of the problem.

Abstract

How to analyse decision changes in general optimising models? How to guide the speci&cation and the estimation of these models? How to compare heterogeneous models sharing some decisions and parameters? To explore these questions, we derive the fundamental geometrical structure of the comparative statics for models with several nonlinear constraints, varying preferences and general parameters.

We clarify the comparative statics, &rstly by identifying the intrisic metric of the optimisation problem and secondly by decomposing the decision changes by using a projection onto the tangent subspace to the constraints, and the rigid motions of the fundamental hyperplanes of the problem. Changes in implicit prices and implicit resources associated with the objective and with the constraints describe the rotations and translations of the fundamental hyperplanes on which the decomposition of the comparative statics is based. Finally, we illustrate our theoretical results with several examples.

Résumé

Comment analyser les décisions de modèles d’optimisation complexes? Comment guider la spéci&cation et l’estimation de ces modèles? Comment comparer des modèles hétérogènes partageant certaines décisions et certains paramètres? Pour répondre à ces questions, nous mettons en évidence la structure géométrique fondamentale de la statique comparative pour les modèles représentés par des programmes d’optimisation impliquant plusieurs contraintes non linéaires, des préférences évolutives et paramètres généraux.

Nous clari&ons les méchanismes de la statique comparative d’abord en identi&ant la métrique intrinsèque du problème d’optimisation et les hyperplans fondamentaux; ensuite en décomposant les variations des décisions en utilisant une projection sur le sous-espace tangent aux contraintes, et les déplacements affines des hyperplans fondamentaux.

Les variations des prix implicites et des ressources implicites associés aux fonctions objectif et contraintes décrivant les caractéristiques des rotations et translations des hyperplans fondamentauc. Finalement, nous illustrons nos résultats théoriques par plusieurs examples.

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Appendix

Notations:

We denote the transpose matrix of M by M'. All derivatives are written in vector or matrix form. They are denoted with a subscript corresponding to the vector with respect to which the differentiation is made. We also denote the Jacobian matrix of a vector function f with respect to vector x, by $\frac{\partial f}{\partial x'}$ or f'_x. Its general term is $\frac{\partial f^i}{\partial x_j}$, where i denotes the row number and j the column number. The Hessian matrix of a function g with respect to vector x is given by g_{xx}. Its general term is $\frac{\partial^2 g}{\partial x_j \partial x_i}$, with i the row index and j the column index.

For a vector function g, $\nabla g \equiv g_x$ denotes the matrix $\frac{\partial g^j}{\partial x_i}$ where i is the row index and j is the column index.

General substitution matrices in the literature

Silberberg (1974) proposes a systematic primal-dual approach of problems with several constraints in which the Hessian of the Lagrangian for a “primal-dual” problem is symmetric and negative semi-de&nite, and can therefore pass for a substitution matrix if we accepts that these properties are characteristic of a possible substitution matrix. Pauwels (1979) recovers Silberberg’s results by using the primal approach. For a similar problem, Chichilnisky and Kalman (1976) discuss a “Slutsky-Hicks-Samuelson operator”. In another dual approach, Hatta (1980) uses a “gain function” with several constraints, to derive envelop theorems, while Caputo (1999) explicits the link between Silberberg’s and Hatta’s approaches. Silberberg, Hatta and Caputo do not explicitly mention any generalised substitution matrix, although they use matrices whose product is the opposite of the Hessian of the gain function, and is symmetric and negative semi-de&nite. Thus, past articles exhibit Hessian matrices that could qualify for being considered as generalised substitution matrices. Despite the differences in the formulae, our matrix $SL_{\theta x}$ is proportional to the Hessians of Silberberg’s and Hatta’s dual objective functions.

Proof of Propositions:

Proof of Proposition 1:

Let x be a decision vector chosen in the feasible set and (x^*, λ^*) satisfying the KTC such that $x \neq x^*$. Then, $g^i(x) \leq g^i(x^*)$, for all i and $\lambda^* \geq 0$.

Since all g^i are quasi-convex, we have $\nabla g(x^*)(x - x^*) \leq 0$, for all i.

Therefore, using the KTC and since $\lambda^{i*} \geq 0$, for all i, we can deduce:

$\nabla U(x^*)(x - x^*) \geq 0$, which implies since U is strictly pseudo-concave and $x \neq x^*$, that $U(x^*) > U(x)$. QED.

Proof of Proposition 2:

(a) $L_{xx}S = I - g_x[g'_x(L_{xx})^{-1}g_x]^{-1}g'_x(L_{xx})^{-1}$ is of the form $I - G[G'F^{-1}G]^{-1}G'F^{-1}$. This shows that $L_{xx}S$ is the oblique projector onto the tangent subspace to the

constraints, with $F \equiv (L_{xx})^{-1}$ the matrix of the scalar product describing the obliqueness of the projection. See Magnus and Neudecker (1988) for a description of oblique projectors. Therefore, $Ker(L_{xx}.S)$ is the subspace spanned by $G \equiv g_x$ and $\dim(Ker(L_{xx}S)) = q$, which implies $\dim(KerS) = q$ since L_{xx} is invertible. Then, if $L_{\theta x}$ is full column rank, $Rank(S) = n - q$.

(b) is obtained by simple algebra.

Proof of Proposition 3:

$R = (L_{xx})^{-1} g_x [g_x' (L_{xx})^{-1} g_x]^{-1}$. Let us denote $G = g_x'$, $F = L_{xx}$, then $R = F^{-1} G' [G F^{-1} G']^{-1}$. Then, $g_x' R = G F^{-1} G' [G F^{-1} G']^{-1} = I_q$.

Proof of Proposition 4:

We have $\sin \beta_j = \frac{\left\| \tilde{P} \wedge (\tilde{P} + d\tilde{P}) \right\|}{\left\| \tilde{P} \right\| \left\| \tilde{P} + d\tilde{P} \right\|}$ and $sin \gamma_j = \frac{\left\| \hat{P} \wedge (\hat{P} + d\hat{P}) \right\|}{\left\| \hat{P} \right\| \left\| \hat{P} + d\hat{P} \right\|}$, where $\tilde{P} = U_x$, $\hat{P} = \sum_k \lambda_k \, g_x^k$, $d\tilde{P} = \frac{\partial^2 U}{\partial \theta_j \partial x} d\theta_j$ *and* $d\hat{P} = \sum_k \lambda_k \frac{\partial^2 \, g^k}{\partial \theta_j \partial x} d\theta_j$. Therefore, because the rotation angle is in&nitesimal a &rst-order Taylor expansion yields the angle formulae shown in the Proposition. Note that a similar formula of sine cannot be given for α_j because a relative direction described by a vector $B = \tilde{P} - \hat{P}$ has no meaning at the optimum for which $\tilde{P} - \hat{P} = 0$. QED.

Proof of Proposition 5:

We have $\tilde{P} \equiv U_x$; $\hat{P}^p \equiv \lambda_p g_x^p$ and $\hat{P} = \sum_{p=1}^{q} \hat{P}^p$. At the optimum $\tilde{P} = \hat{P}$. Therefore, $\tilde{P}' R = \hat{P}' R = \sum_{p=1}^{q} \lambda_k g_x^{p\prime} R = \begin{bmatrix} \lambda_1 & \cdots & \lambda_q \end{bmatrix} \begin{bmatrix} g_x^{1\prime} \\ \vdots \\ g_x^{q\prime} \end{bmatrix} R =$

$\begin{bmatrix} \lambda_1 & \cdots & \lambda_q \end{bmatrix} g_x' R = \begin{bmatrix} \lambda_1 & \cdots & \lambda_q \end{bmatrix} I_q = \lambda'$. QED.