UNIVERSITY OF NOTTINGHAM

Discussion Papers in Economics

Discussion Paper
No. 04/01

'REGULAR' CHOICE AND THE WEAK AXIOM OF STOCHASTIC REVEALED PREFERENCE

by Indraneel Dasgupta and Prasanta K. Pattanaik

January 2004 DP 04/01
ISSN 1360-2438

UNIVERSITY OF NOTTINGHAM

Discussion Papers in Economics

Discussion Paper
No. 04/01

'REGULAR' CHOICE AND THE WEAK AXIOM OF STOCHASTIC REVEALED PREFERENCE

by Indraneel Dasgupta and Prasanta K. Pattanaik

Indraneel Dasgupta is Lecturer, School of Economics, University of Nottingham and Prasanta K. Pattanaik is Professor, Department of Economics, University of California

January 2004

‘Regular’ Choice and the Weak Axiom of Stochastic Revealed Preference[•]

Indraneel Dasgupta
School of Economics, University of Nottingham, UK.

and

Prasanta K. Pattanaik
Department of Economics, University of California, USA.

This version: 26 January 2004

Abstract: We explore the relation between two 'rationality conditions' for stochastic choice behaviour: regularity and the weak axiom of stochastic revealed preference (WASRP). We show that WASRP implies regularity, but the converse is not true. We identify a restriction on the domain of the stochastic choice function, which suffices for regularity to imply WASRP. When the universal set of alternatives is finite, this restriction is also necessary for regularity to imply WASRP. Furthermore, we identify necessary and sufficient domain restrictions for regularity to imply WASRP, when the universal set of alternatives is finite and stochastic choice functions are all degenerate. Results in the traditional deterministic framework regarding the relation between Chernoff's Condition and the Weak Axiom of Revealed Preference follow as special cases.

KEYWORDS: Stochastic Choice, Regularity, Chernoff Condition, Weak Axiom of Revealed Preference, Weak Axiom of Stochastic Revealed Preference, Complete Domain, Incomplete Domain.

JEL Classification No. D11

<u>Address for Correspondence</u>:

Prasanta K. Pattanaik
Department of Economics, University of California,
Riverside, CA 92521, U.S.A.
Telephone: (909) 827 1592; Fax: (909) 787-5685.
E-mail: prasanta.pattanaik@ucr.edu

[•] We are grateful to Kunal Sengupta and seminar participants at ISI, Calcutta, for helpful conversations.

1. Introduction

A large literature exists on stochastic choice behaviour of an agent.[1] This literature has advanced a number of rationality, or consistency, restrictions for such choice behaviour. Perhaps the best known and intuitively most compelling of these restrictions is the so-called 'regularity' (RG) condition. This simply postulates that the probability of choosing from any subset of a feasible set of options cannot rise if the feasible set is expanded. Recently, in a series of papers, Bandyopadhyay, Dasgupta and Pattanaik (2003, 2002, 1999) have introduced a different rationality postulate for stochastic choice, the weak axiom of stochastic revealed preference (WASRP), and analyzed its implications in the context of consumers' behaviour.[2]

A priori, RG and WASRP appear to have very different focus. While RG specifies consistency restrictions for choice behaviour when the feasible set is contracted or expanded, it does not explicitly spell out any such restriction across two feasible sets, neither of which includes the other. WASRP, however, is formulated explicitly to cover such cases. A natural question to ask, therefore, is: what is the relationship between these two rationality postulates, which, at first sight, appear to have very different focus? Since both RG and WASRP have independent intuitive appeal, if it can be shown that one in fact implies the other, then the case for taking the latter seriously would be further strengthened. In particular, since RG is arguably the weakest and intuitively most plausible rationality condition suggested in the literature, if it can be shown that even RG suffices to imply WASRP, at least for a large class of cases, then the justification for WASRP would be significantly enhanced.

We address this issue in this paper. We first show that WASRP necessarily implies RG, regardless of the domain of the stochastic choice function. However, if the stochastic choice function is not defined for some subsets of the universal set of alternatives, then it can violate WASRP while satisfying RG. Thus, in this sense, WASRP is stronger than RG. We then identify a restriction on the domain of the stochastic choice function, under which RG turns out to be equivalent to WASRP. This restriction allows the possibility that the domain of the stochastic choice function is 'incomplete', i.e., not defined for some subsets of the universal set of alternatives. A corollary of our results is that, for every stochastic choice function with a complete domain, WASRP and regularity are equivalent properties. We proceed to show that our domain restriction is also necessary for RG to imply WASRP, when the universal set is finite. Lastly, we provide a necessary and sufficient domain restriction under which RG and WASRP are equivalent, when the universal set is finite and stochastic choice functions are constrained to be degenerate. Results in the traditional, deterministic, framework

[1] See Barbera and Pattanaik (1986), Block and Marschak (1960), Cohen (1980), Corbin and Marley (1974), Falmagne (1978), Fishburn (1973, 1977, 1978), Halldin (1974), Luce (1958, 1959, 1977), Luce and Suppes (1965), Manski (1977), Marschak (1960), Quandt (1956), Sattath and Tversky (1976) and Yellott (1977).

[2] We define RG and WASRP formally in Section 2 below.

regarding the relation between Chernoff's Condition (or Sen's Condition α) and the Weak Axiom of Revealed Preference follow as special cases of our general analysis.

Section 2 sets out the basic notation and definitions. We present our main results in Sections 3 and 4. Section 3 deals with the general case where stochastic choice functions may be degenerate, but are not constrained to be so. Section 4 deals with the special case where the universal set of alternatives is assumed to be finite and stochastic choice functions are constrained to be degenerate. Section 5 concludes. Proofs are relegated to the Appendix.

2. The notation and definitions

Let X denote the (non-empty) universal set of alternatives, and let χ denote the class of all non-empty subsets of X. Let $\Im$ be the class of all non-empty subsets of χ. Thus, an element of $\Im$ is a non-empty class of non-empty subsets of X.

Definition 2.1. Let $Z \in \Im$. A *stochastic choice function* (SCF) over Z is a function which, for every $A \in Z$, specifies exactly one finitely additive probability measure over the class of all subsets of A.

If F is an SCF over $Z \in \Im$ and $A \in Z$, then the probability measure specified by F for A will be denoted by p_{FA}. When there is no ambiguity about the SCF, F, we shall write simply p_A rather than p_{FA}. Given an SCF F over $Z \in \Im$ and given $A \in Z$, for every subset B of A, $p_A(B)$ is to be interpreted as the probability that the agent's choice from the set A will lie in B. When B contains exactly one element, say x, we write $p_A(x)$ rather than $p_A(\{x\})$.

Definition 2.2. An SCF F over $Z \in \Im$ is *degenerate* iff, for all $A \in Z$, there exists $x \in A$ such that $p_A(x) = 1$.

Definition 2.3. Let $Z \in \Im$. A *deterministic choice function* (DCF) over Z is a function which, for every $A \in Z$, specifies exactly one alternative in A.

Remark 2.4. Let $Z \in \Im$. Let F be a degenerate SCF over Z and f be a DCF over Z. We say that F induces f iff, for all $A \in Z$, $f(A) = x$, where $p_A(x) = 1$; and we say that f induces F iff, for all $A \in Z$, $p_A(x) = 1$, where $x = f(A)$. It is clear that every degenerate SCF induces a DCF, which, in turn, induces the degenerate SCF under consideration. Similarly, every DCF induces a degenerate SCF, which, in turn, induces the DCF under consideration.

Definition 2.5. Let $Z \in \Im$.

(i) An SCF F over Z satisfies *regularity* (RG) iff, for all $A, B, C \in Z$ such that $C \subseteq B \subseteq A$, $p_B(C) \geq p_A(C)$.

(ii) A DCF f over Z satisfies *Chernoff's Condition* (CC) iff, for all $A, B \in Z$ such that $B \subseteq A$, $(B - \{f(B)\}) \subseteq (A - \{f(A)\})$.

RG is arguably the weakest rationality property of stochastic choices discussed in the literature. It stipulates that, if we start with $B \in Z$, and if, by adding some alternatives to B, we arrive at a new set $A \in Z$, then the probability that the agent's choice will lie in a subset C of B cannot increase when we pass from the feasible set B to the feasible set A. CC, first introduced by Chernoff (1954), and later discussed by Sen (1969), who called it α, requires that an alternative that is rejected in a set B cannot be chosen, when the set B is expanded by adding new alternatives.

Remark 2.6. It is clear that a degenerate SCF F satisfies RG iff the DCF induced by F satisfies CC, and, conversely, a DCF f satisfies CC iff the degenerate SCF induced by f satisfies RG.

Definition 2.7. Let $Z \in \Im$.

(i) An SCF F over Z satisfies the *weak axiom of stochastic revealed preference* (WASRP) iff, for all $A, B \in Z$,

$$p_B(C) - p_A(C) \leq p_A(A - B) \text{ for all } C \subseteq A \cap B. \tag{2.1}$$

(ii) A DCF f over Z satisfies the *weak axiom of revealed preference* (WARP) iff, for all $A, B \in Z$, and, for all distinct $x, y \in X$, if $x, y \in A \cap B$ and $f(A) = x$, then $f(B) \neq y$.

WARP, introduced by Samuelson in the context of competitive consumers' choices,[3] and reformulated by Houthakker (1950) for our general choice context, is a very familiar property of DCFs and hardly needs any explanation. WASRP was introduced by Bandyopadhyay, Dasgupta and Pattanaik (1999). The intuition behind WASRP is as follows. Suppose, initially, A is the set of all available alternatives, and $C \subseteq A$. Then $p_A(C)$ is the probability that the agent's choice from A lies in C. Now suppose that the set of available alternatives changes to B, but $C \subseteq B$. If, at all, the new choice probability, $p_B(C)$, for C is greater than $p_A(C)$, then this increase must be due to the fact that the alternatives in C face 'less competition' to the extent that the alternatives in A - B are no longer available. Hence $p_B(C) - p_A(C)$ must not be greater than the choice probability for $A - B$, $p_A(A - B)$, in the initial situation.

Remark 2.8. A degenerate SCF F satisfies WASRP iff the DCF induced by F satisfies WARP. Conversely, a DCF f satisfies WARP iff the degenerate SCF induced by f satisfies WASRP.

Definition 2.9. An SCF is said to have a *complete domain* iff its domain is χ. An SCF has an *incomplete domain* iff its domain is a proper subset of χ. Similarly, we have notions of complete

[3] See, for example, Samuelson (1948).

and incomplete domains for DCFs.

3. <u>The relation between RG and WASRP: the general case</u>

We are now ready to explore the relation between RG and WASRP. We first show that WASRP implies RG.

Proposition 3.1. *An SCF satisfying WASRP must satisfy RG.*

Proof: See the Appendix.

In light of Remarks 2.6 and 2.8, it is clear that Proposition 3.1 yields the familiar result that WARP implies CC. The following example shows that an SCF satisfying RG does not necessarily satisfy WASRP. Since WASRP was first formulated for stochastic choices of a competitive consumer, we have constructed our counter-example in terms of a competitive consumer's choice.

Example 3.2. Consider a competitive consumer in an economy with exactly two goods. We specify a DCF for the consumer that satisfies CC but violates WARP (by Remarks 2.6 and 2.8, the SCF induced by this DCF will satisfy RG but violate WASRP). Assume that the domain of the DCF is the set of all 'budget triangles' in $\Re^2_+$. Let x_1 and x_2 denote the quantities of the two goods, 1 and 2, and let r_1 and r_2 be their respective prices. Consider Figure 1. The DCF of the consumer is specified as follows. If $r_1 \leq r_2$, then the chosen consumption bundle is given by the intersection of the budget frontier and the ray *og* through the origin. If $r_1 > r_2$, then the chosen consumption bundle is given by the intersection of the budget frontier and the ray *og'*. It can be checked that this DCF satisfies CC. However, it violates WARP: for example, x is chosen from the budget triangle *oab*, while x' is chosen from the budget triangle *oa'b'*.

Insert Figure 1

Since, in general, RG does not imply WASRP, the question now arises as to whether one can formulate necessary and sufficient conditions, in terms of restrictions on the domains of SCFs, for RG to imply WASRP. Our notions of sufficiency and necessity in this context are as follows.

(1) A *sufficient condition* (in terms of domain restrictions) for RG to imply WASRP specifies a non-empty subclass $\Im'$ of $\Im$, such that, for all $Z \in \Im'$, if an SCF has Z for its domain and satisfies RG, then it must satisfy WASRP.

(2) A *necessary condition* (in terms of domain restrictions) for RG to imply WASRP specifies a proper subclass $\Im''$ of $\Im$, such that, for every $Z \in \Im - \Im''$, there exists an SCF with Z for its domain, which satisfies RG but violates WASRP.

Proposition 3.4 below establishes a sufficient condition for RG to imply WASRP.

Notation 3.3. Let $\Im^*$ be the set of all $Z \in \Im$ such that, for all $A, B \in Z$, at least one of the following two conditions holds:

$$|A \cap B| \leq 1; \tag{3.1}$$

there exist $A', B' \in Z$ such that $A' \subseteq A$, $B' \subseteq B$, $A' \cap B' = A \cap B$, and

$$[(A' \cap B') \in Z \text{ or } (A' \cup B') \in Z]. \tag{3.2}$$

We now show that $\Im^*$ constitutes a sufficient condition for RG to imply WASRP.

Proposition 3.4. *Let $Z \in \Im^*$. Then every SCF with domain Z that satisfies RG must satisfy WASRP.*

Proof: See the Appendix.

Propositions 3.1 and 3.4 immediately yield the following.

Corollary 3.5. *Let $Z \in \Im^*$, and let F be any SCF with domain Z. F satisfies RG if, and only if it satisfies WASRP.*

Since $\chi \in \Im^*$, Corollary 3.5 implies the following.

Corollary 3.6. *An SCF with a complete domain satisfies RG if and only if it satisfies WASRP.*

Remark 3.7. Note that $\Im^*$ contains many elements of $\Im$, other than χ. Therefore, by Corollary 3.5, it follows that RG can be equivalent to WASRP even for SCFs defined over incomplete domains.

Remark 3.8. By Corollary 3.6 and Remarks 2.6 and 2.8, it follows that a DCF with a complete domain satisfies WARP iff it satisfies CC, a fact known in the literature on DCFs.[4]

Remark 3.9. Proposition 3.4 naturally raises the question whether $\Im^*$ also constitutes a necessary condition for RG to imply WASRP, i.e., whether for every $Z \in \Im - \Im^*$, there exists an SCF with domain Z, which satisfies RG but violates WASRP. We have not been able to resolve this problem for the case where the universal set of alternatives is infinite. However, $\Im^*$ does turn out to be a necessary condition for RG to imply WASRP when the universal set of alternatives is finite.

Proposition 3.10. *Let X be a finite set. Then, for all $Z \in \Im - \Im^*$, there exists an SCF with domain Z, which satisfies RG but violates WASRP.*

Proof: See the Appendix.

Remark 3.11. By Propositions 3.4 and 3.10, it follows that, if X is a finite set, then $\Im^*$ constitutes a necessary and sufficient condition for RG to imply WASRP.

---

[4] See, for example, Sen (1969) and Kreps (1988, pp. 13-14).

4. Finite X and degenerate SCFs

We now consider the special case where, not only is X assumed to be finite, but SCFs are also constrained to be degenerate. Since this case corresponds to the problem of the relationship between CC and WARP in the traditional, deterministic, framework, it is of considerable interest.

Note that Proposition 3.4 allows SCFs to be degenerate as well as non-degenerate. Therefore, by Proposition 3.4, for every $Z \in \Im^*$, every degenerate SCF with domain Z, which satisfies RG, must satisfy WASRP. However, as the following example shows, it is not true that, for every $Z \in \Im - \Im^*$, one can construct a degenerate SCF with domain Z, which satisfies RG but violates WASRP.

Example 4.1. Let $X = \{a,b,c,d,e\}, A = \{a,c,d,e\}, B = \{b,c,d,e\}$, $C_1 = \{c,d\}$, $C_2 = \{d,e\}$, $C_3 = \{e,c\}$, and $Z = \{A,B,C_1,C_2,C_3\}$. The reader can easily check that $Z \in \Im - \Im^*$, yet it is not possible to construct a degenerate SCF with domain Z, which satisfies RG but violates WASRP.

We now introduce a restriction on the domain that turns out to be both necessary and sufficient for RG to imply WASRP, when X is finite and F is assumed to be degenerate.

Notation 4.2. Let $\overline{\overline{\Im}}$ be the set of all $Z \in \Im$, such that, for all $A, B \in Z$, we have (3.1) or (4.1) or (4.2) below:

$$A \cup B \in Z; \tag{4.1}$$

for all distinct $x, y \in A \cap B$, [for some $A', B' \in Z, \{x,y\} \subseteq A' \subseteq A, \{x,y\} \subseteq B' \subseteq B$, and $(A' \cup B') \subset (A \cup B)$]. (4.2)

Remark 4.3. It is evident that $\Im^* \subseteq \overline{\overline{\Im}}$. However, it is not true in general that $\overline{\overline{\Im}} \subseteq \Im^*$: Z, as specified in Example 4.1, belongs to $\overline{\overline{\Im}}$, but it does not belong to $\Im^*$.

Proposition 4.4. *Let X be a finite set. Then*:

(i) *for every $Z \in \overline{\overline{\Im}}$, and, for every degenerate SCF F with domain Z, if F satisfies RG, then F satisfies WASRP;*

(ii) *for all $Z \in \Im - \overline{\overline{\Im}}$, there exists a degenerate SCF with domain Z, which satisfies RG but violates WASRP.*

Proof: See the Appendix.

By Remark 2.8, the following result follows immediately from Proposition 4.4.

Corollary 4.5. *Let X be a finite set. Then:*

(i) *for every $Z \in \overline{\overline{\Im}}$, and, for every DCF f with domain Z, if f satisfies CC, then F satisfies WARP;*

(ii) *for all $Z \in \Im - \overline{\overline{\Im}}$, there exists a DCF with domain Z, which satisfies CC but violates WARP.*

5. Conclusion

In this paper, we have clarified the relationship between two rationality restrictions on stochastic choice behaviour, viz., regularity and the weak axiom of stochastic revealed preference. We have shown that WASRP implies RG, though the converse is not necessarily true. Our next result specified a restriction on the domain of SCFs, under which RG implies WASRP, so that RG and WASRP turn out to be equivalent. A corollary of this result is that, for every SCF with a complete domain, RG and WASRP are equivalent. For the special case where the universal set of alternatives is finite, we have shown that this domain restriction constitutes a necessary, as well as sufficient, condition for RG to imply WASRP. We have also identified another domain restriction as both necessary and sufficient for RG to imply WASRP when one constrains SCFs to be degenerate, in addition to assuming the universal set to be finite. This result also provides a necessary and sufficient condition, in the standard deterministic framework, for Chernoff's Condition to imply WARP.

There remains one unresolved problem: we have not been able to formulate a condition (in terms of domain restrictions) that is necessary as well as sufficient for RG to imply WASRP in the general case where the universal set of alternatives is permitted to be infinite. In particular, in this general case, we do not know whether for every $Z \in [\Im - \Im^*]$, there exists an SCF with domain Z, which satisfies RG but violates WASRP.

Appendix

Proof of Proposition 3.1. Let F be an SCF and let Z be its domain. Suppose F violates RG. Then there must exist $A, B, C \in Z$ such that $C \subseteq B \subseteq A$ and $p_B(C) < p_A(C)$. In that case, $[p_A(C) - p_B(C) > 0]$ and $[p_B(B - A) = p_B(\varnothing) = 0]$. Then F violates WASRP. ◊

We prove Proposition 3.4 via the following lemmas.

Lemma N.1. Let $Z \in \Im$, and let F be an SCF with domain Z. For all $A, B \in Z$, if $A \cap B = \varnothing$ or $A \cap B$ is a singleton, then (2.1) holds.

Proof of Lemma N.1. Let $A, B \in Z$ and suppose either $A \cap B = \varnothing$ or $A \cap B$ is a singleton. Let $C \subseteq A \cap B$. Then either $C = \varnothing$ or $C = A \cap B$. If $C = \varnothing$, then $p_A(A - B) \geq p_B(C) - p_A(C) = 0$. If $C = A \cap B$, then $p_A(C) + p_A(A - B) = 1 \geq p_B(C)$. Thus, if $C = \varnothing$ or $C = A \cap B$, then (2.1) holds. ◊

Lemma N.2. Let $Z \in \Im$, and let F be an SCF with domain Z that satisfies RG. Let $A, B \in Z$ be such that $(A \cap B) \in Z$ or $(A \cup B) \in Z$. Then, (2.1) holds.

Proof of Lemma N.2. Suppose, for some $C \subseteq A \cap B$, $p_B(C) - p_A(C) > p_A(A-B)$. Then

$$p_B(C) > p_A(C) + p_A(A-B). \tag{A1}$$

We shall show that (A1) leads to a contradiction, given our assumptions. First, consider the case where $(A \cap B) \in Z$. In this case, by RG,

$$p_{A\cap B}(C) \geq p_B(C), \tag{A2}$$

and

$$p_{A\cap B}((A \cap B) - C) \geq p_A((A \cap B) - C). \tag{A3}$$

By (A1-A3), we get $\left[p_{A\cap B}(C) + p_{A\cap B}((A \cap B) - C) > 1 \right]$, a contradiction.

Next, consider the case where $(A \cup B) \in Z$. In this case, by RG,

$$p_B(B - A) \geq p_{A\cup B}(B - A), \tag{A4}$$

$$p_B((A \cap B) - C) \geq p_{A\cup B}((A \cap B) - C), \tag{A5}$$

and

$$p_A(C) + p_A(A - B) \geq p_{A\cup B}(C) + p_{A\cup B}(A - B). \tag{A6}$$

By (A1) and (A6),

$$p_B(C) > p_{A\cup B}(C) + p_{A\cup B}(A - B). \tag{A7}$$

By (A4), (A5) and (A7), $p_B(B) > p_{A\cup B}(A \cup B) = 1$, which is a contradiction. ◊

Lemma N.3. Let $Z \in \Im$ and let F be an SCF with domain Z that satisfies RG. Let $A, B, A', B' \in Z$ be such that $A' \subseteq A$, $B' \subseteq B$, and $A \cap B = A' \cap B'$. Then, for all $C \subseteq A' \cap B' = A \cap B$, $[p_{A'}(C) - p_{B'}(C) \leq p_{A'}(A' - B')]$ implies $[p_B(C) - p_A(C) \leq p_A(A - B)]$.

Proof of Lemma N.3. Let $A, B, A', B' \in Z$ be such that $A' \subseteq A$, $B' \subseteq B$ and $A \cap B = A' \cap B'$. Let $C \subseteq A \cap B = A' \cap B'$ and let $[p_{A'}(C) - p_{B'}(C) \leq p_{A'}(A' - B')]$.

By RG, noting $A' \subseteq A$,

$$p_{A'}((A \cap B) - C) - p_A((A \cap B) - C) \geq 0. \tag{A8}$$

Note that:

$p_A(C) + p_A((A \cap B) - C) + p_A(A - B) = 1 = p_{A'}(C) + p_{A'}((A \cap B) - C) + p_{A'}(A' - B')$. Hence,

$$p_A(C) + p_A(A - B) \geq p_{A'}((A \cap B) - C) - p_A((A \cap B) - C) + p_{A'}(C) + p_{A'}(A' - B'). \tag{A9}$$

By (A8) and (A9),

$$p_A(C) + p_A(A - B) \geq p_{A'}(C) + p_{A'}(A' - B'). \tag{A10}$$

By assumption,

$$p_{B'}(C) - p_{A'}(C) \leq p_{A'}(A' - B'). \tag{A.11}$$

From (A10) and (A11),

$$p_A(C) + p_A(A-B) \geq p_{B'}(C).$$

Noting that, by RG, $p_{B'}(C) \geq p_B(C)$, it follows that $p_A(A-B) \geq p_B(C) - p_A(C)$. ◊

Proof of Proposition 3.4. Let F be an SCF with domain $Z \in \mathfrak{I}^*$, and let F satisfy RG. Let $A, B \in Z$ and let $C \subseteq A \cap B$. We show that: $p_B(C) - p_A(C) \leq p_A(A-B)$. Since $Z \in \mathfrak{I}^*$, either (3.1) holds or (3.2) holds. If (3.1) holds, the claim follows immediately from Lemma N.1. If (3.2) holds, then consider $A', B' \in Z$ as specified by (3.2). By Lemma N.2, [for all $D \subseteq A' \cap B'$, $p_{B'}(D) - p_{A'}(D) \leq p_{A'}(A'-B')$]. Then the claim follows by Lemma N.3. ◊

To prove Proposition 3.10, we first introduce a new notation and prove a lemma.

Notation N.4. Let $\hat{\mathfrak{I}}$ be the set of all $Z \in \mathfrak{I}$ such that, for all $A, B \in Z$, at least one of (3.1) and the following two conditions holds:

$A \cup B \in Z$; (A.12)

there exist $A', B' \in Z$, such that $(A' \cap B') = (A \cap B)$, $A' \subseteq A$, $B' \subseteq B$, and $(A' \cup B') \subset (A \cup B)$. (A.13)

Lemma N.5. *Let X be a finite set. Then,* $\mathfrak{I}^* = \hat{\mathfrak{I}}$.

Proof of Lemma N.5. Given the definitions of $\mathfrak{I}^*$ and $\hat{\mathfrak{I}}$ (see Notations 3.3 and N.4), it is evident that $\mathfrak{I}^* \subseteq \hat{\mathfrak{I}}$. Hence, to establish Lemma N.5, we need to show that $\hat{\mathfrak{I}} \subseteq \mathfrak{I}^*$ for finite *X*.

Let $Z \in \hat{\mathfrak{I}}$. Suppose $Z \notin \mathfrak{I}^*$. Then there exist $A, B \in Z$, such that neither (3.1) nor (3.2) is satisfied. Consider such A and B. Let $A \cap B = I$.

Since (3.2) is violated, (A.12) cannot hold. Given that neither (3.1) nor (A.12) holds, as $Z \in \hat{\mathfrak{I}}$ by assumption, (A.13) must hold. Then

there exist $A_1, B_1 \in Z$, such that $A_1 \cap B_1 = I$, $A_1 \subseteq A$, $B_1 \subseteq B$, and $(A_1 \cup B_1) \subset (A \cup B)$. (A.14)

Since $(A' \cap B') = I$, noting the violation of (3.1), $|A_1 \cap B_1| \geq 2$. Further, since (3.2) is violated by assumption, $A_1 \cup B_1 \notin Z$. Hence, as $Z \in \hat{\mathfrak{I}}$, there exist $A_2, B_2 \in Z$, such that $A_2 \cap B_2 = A_1 \cap B_1 = I$, $A_2 \subseteq A_1$, $B_2 \subseteq B_1$, and $(A_2 \cup B_2) \subset (A_1 \cup B_1)$.

Proceeding in this fashion, we get the following:

there exists an infinite sequence of ordered pairs $\langle A_i, B_i \rangle$, $i \in \{0,1,2,...\}$, such that $\langle A_0, B_0 \rangle = \langle A, B \rangle$, and, for all $i \in \{0,1,2,...\}$, [$A_i, B_i \in Z$, and $(A_{i+1} \cup B_{i+1}) \subset (A_i \cup B_i)$]. (A.15)

(A.15) contradicts the assumption that X is finite. ◊

Proof of Proposition 3.10.

Suppose $Z \in \Im - \hat{\Im}$. We shall construct an SCF defined over Z, which satisfies RG but violates WASRP. In light of Lemma N.5, this will suffice to establish Proposition 3.10.

Since $Z \in \Im - \tilde{\Im}$, there exist $A, B \in Z$ such that each of (3.1), (A.12) and (A.13) is violated. Consider such A and B. Consider the following subsets of Z.

$$Z_1 = \{E \in Z : E \cap (X - A - B) \neq \phi\};$$

$$Z_2 = \{E \in Z : E \subseteq (A - B) \cup (B - A)\}$$

$$Z_3 = \{E \in Z : (A \cap B) \subset E \subseteq A\};$$

$$Z_4 = \{E \in Z : (A \cap B) \subset E \subseteq B\};$$

$$Z_5 = \{E \in Z : (A \cap B) \subset E \subset (A \cup B); E - A \neq \phi; \& E - B \neq \phi\};$$

$$Z_6 = \{E \in Z : \phi \neq (E \cap A \cap B) \subset (A \cap B); E \subset A \cup B\}.$$

It can be checked that these six subsets of Z are pairwise disjoint. Furthermore, since violations of (A.12) and (A.13), imply, respectively, that $(A \cup B) \notin Z$ and $(A \cap B) \notin Z$, it can be seen that the union of all of them is Z. Also, since (A.13) is violated,

$$Z_3 = \{A\} \text{ and } Z_4 = \{B\}. \tag{A.16}$$

Further, by the specification of Z_5,

for all $E \in Z_5$, [if $A \subseteq E$, then not ($B \subseteq E$)] and [if $B \subseteq E$, then not ($A \subseteq E$)]. (A.17)

Since neither (3.1) nor (A.12) holds, $|A \cap B| \geq 2$, $A - B \neq \varnothing$, and $B - A \neq \varnothing$. Let $A = \{a_1, ..., a_u\} \cup \{c_1, ..., c_v\}$ and $B = \{b_1, ..., b_w\} \cup \{c_1, ..., c_v\}$, where $a_1, ..., a_u, b_1, ..., b_w, c_1, ..., c_v$ are all distinct, $u, w \geq 1$, and $v \geq 2$. Let G be a linear ordering over $X - (A \cap B)$.

Specify an SCF with domain Z as follows. For all $E \in Z_1$, $p_E(x) = 1$, where x is the *G*-greatest element in $E \cap [X - (A \cup B)]$. For all $E \in Z_2$, $p_E(x) = 1$, where x is the *G*-greatest element in E. $p_A(c_j) = \frac{1}{v-1}$ for all $j \in \{1, 3, ..., v\}$; and $p_B(c_k) = \frac{1}{v-1}$ for all $k \in \{2, ..., v\}$ (recall (A.16)). For every $E \in Z_5$, [if $A \subseteq E$, then we have $p_E(c_k) = \frac{1}{v-1}$ for all $k \in \{1, 3, ..., v\}$] and [if not($A \subseteq E$), then $p_E(c_k) = \frac{1}{v-1}$ for all $k \in \{2, ..., v\}$] (recall (A.17)). Finally, for all $E \in Z_6$, $p_E(y) = \frac{1}{m}$ for all $y \in (E \cap A \cap B)$, where $m = |E \cap A \cap B|$.

Given that none of (3.1), (A.12) and (A.13) hold, and noting (A.16) and (A.17), it can be checked that the SCF as specified above satisfies RG. However, the SCF violates WASRP since $p_A(v_2)=0, p_B(v_2)>0,$ and $p_A(A-B)=0.$ ◊

Proof of Proposition 4.4.

(i) Let X be a finite set and let $Z \in \overline{\mathfrak{I}}$. Let F be a degenerate SCF with domain Z, which satisfies RG and violates WASRP. We first establish the following claim:

for all distinct $x, y \in X$ and all $D, E \in Z$, if $[\{x,y\} \subseteq D \cap E$ and $p_D(x)=1$ and $p_E(y)=1]$, then there exist $D', E' \in Z$, such that $[D' \subseteq D, E' \subseteq E, \{x,y\} \subseteq D' \cap E', D' \cup E' \subset D \cup E$, and $(p_{D'}(x)=1$ and $p_{E'}(y)=1)]$. (A.18)

Suppose there exist $D, E \in Z$ and distinct $x, y \in X$ such that $[\{x,y\} \subseteq D \cap E$ and $p_D(x)=1$ and $p_E(y)=1]$. Then, since F satisfies RG, by Lemma N.2, we have $(D \cup E) \notin Z$. Since $Z \in \overline{\mathfrak{I}}$, it follows that there exist $D', E' \in Z$ such that $\{x,y\} \subseteq D' \subseteq D$, $\{x,y\} \subseteq E' \subseteq E$, and $(D' \cup E') \subset (D \cup E)$. (A.18) follows by RG.

Notice now that, since the degenerate SCF F violates WASRP,

there exist $A, B \in Z$ and distinct $x, y \in A \cap B$, such that $p_A(x)=1$ and $p_B(y)=1$. (A.19)

By (A.18) and (A.19), we have an infinite sequence of ordered pairs, $(A_1,B_1),(A_2,B_2),\ldots$, such that $(A_1,B_1)=(A,B)$ and, for every positive integer $i \in \{1,2,\ldots\}$, $(A_{i+1} \cup B_{i+1}) \subset (A_i \cup B_i)$. However, this contradicts our assumption that X is a finite set. This completes the proof of Proposition 4.4 (i).

(ii) Let $Z \in \mathfrak{I} - \overline{\mathfrak{I}}$. We shall construct a degenerate SCF F with domain Z such that F satisfies RG but violates WASRP. Since $Z \in \mathfrak{I} - \overline{\mathfrak{I}}$, there exist $A, B \in Z$ such that each of (3.1), (4.1), and (4.2) are violated, so that

$|A \cap B| \geq 2$ and $A \cup B \notin Z$ (A.20)

and

for some distinct $x, y \in A \cap B$, and for all $A', B' \in Z$, if $\{x,y\} \subseteq A' \subseteq A$ and $\{x,y\} \subseteq B' \subseteq B$, then $A' = A$ and $B' = B$. (A.21)

Consider the following subsets of Z:

$Z_1 = \{E \in Z : E \cap (X - A - B) \neq \phi\};$

$Z_2 = \{E \in Z : E \subseteq (A - \{x,y\}) \cup (B - \{x,y\})\}$

$Z_3 = \{E \in Z : \{x,y\} \subset E \subseteq A\};$

$Z_4 = \{E \in Z : \{x,y\} \subset E \subseteq B\};$

$Z_5 = \{E \in Z : \{x,y\} \subset E \subset A \cup B; E - A \neq \phi; \& E - B \neq \phi\};$

$Z_6 = \{E \in Z : \phi \neq (E \cap \{x, y\}) \subset \{x, y\}; E \subset A \cup B\}$.

It follows from (A21) that these six subsets of Z are pairwise disjoint. Since (A.20) and (A.21) yield $A \cup B \notin Z$ and $\{x, y\} \notin Z$, the union of all of them is Z. Further, by (A.21),

$$Z_3 = \{A\} \text{ and } Z_4 = \{B\}. \quad \text{(A.22)}$$

Let G be a linear ordering over $X - \{x, y\}$. Construct a degenerate SCF F, with domain Z, as follows. For all $E \in Z_1$, $p_E(z) = 1$, where z is the *G*-greatest element in $E \cap [X - (A \cup B)]$. For all $E \in Z_2$, $p_E(z) = 1$, where z is the *G*-greatest element in E. $p_A(x) = 1$; and $p_B(y) = 1$ (recall (A.22)). For every $E \in Z_5$, [if $A \subseteq E$, then we have $p_E(x) = 1$] and [if not ($A \subseteq E$), then $p_E(y) = 1$]. Finally, for all $E \in Z_6$, [if $E \cap \{x, y\} = \{x\}$, then $p_E(x) = 1$], and [if $E \cap \{x, y\} = \{y\}$, then $p_E(y) = 1$].

It can be checked that F satisfies RG but violates WASRP. ◊

References

Bandyopadhyay, T., I. Dasgupta and P.K. Pattanaik (2003): "A General Revealed Preference Theorem for Stochastic Demand Behaviour", forthcoming in *Economic Theory*.

Bandyopadhyay, T., I. Dasgupta and P.K. Pattanaik (2002): "Demand Aggregation and the Weak Axiom of Stochastic Revealed Preference", *Journal of Economic Theory*, 107: 483-489.

Bandyopadhyay, T., I. Dasgupta and P.K. Pattanaik (1999): "Stochastic Revealed Preference and the Theory of Demand", *Journal of Economic Theory,* 84: 95-110.

Barbera, S. and P. K. Pattanaik (1986): "Falmagne and the Rationalizability of Stochastic Choices in terms of Random Orderings", *Econometrica* (54): 707-715.

Block, H. D. and J. Marschak (1960): "Random Orderings and Stochastic Theories of Response", in I. Olkin, S. Ghurye, W. Hoeffding, W. Madow and H. Mann (eds) *Contributions to Probability and Statistics* (Stanford: Stanford University Press).

Chernoff, H. (1954): "Rational Selection of Decision Functions", *Econometrica*, 22: 422-443.

Cohen, M. A. (1980): "Random Utility Systems: The Infinite Case", *Journal of Mathematical Psychology* (22): 1-23.

Corbin, R. and Marley, A. A. J. (1974): "Random Utility Models with Equality: An Apparent, But Not Actual, Generalization of Random Utility Models", *Journal of Mathematical Psychology* (11): 274-293.

Falmagne, J. C. (1978): "A Representation Theorem for Finite Random Scale Systems", *Journal of Mathematical Psychology* (18): 52-72.

Fishburn, P. (1978): "Choice Probabilities and Choice Functions", *Journal of Mathematical Psychology* (18): 205-219.

Fishburn, P. (1977): “Models of Individual Preference and Choice”, *Synthese* (36): 287-314.

Fishburn, P. (1973): “Binary Choice Probabilities: On the Varieties of Stochastic Transitivity”, *Journal of Mathematical Psycho*logy (10): 327-352.

Halldin, C. (1974): “The Choice Axiom, Revealed Preference and the Theory of Demand”, *Theory and Decision* (5): 139-160.

Houthakker, H.S. (1950): “Revealed Preference and the Utility Function”, *Economica* (N.S.) 17: 159-174.

Kreps, D.M. (1988): *Notes on the Theory of Choice*, Westview Press, Boulder.

Luce, R. D. (1977): “The Choice Axiom after Twenty Years”, *Journal of Mathematical Psychology* (15): 215-233.

Luce, R. D. (1959): *Individual Choice Behaviour* (New York: Wiley).

Luce, R. D. (1958): “A Probabilistic Theory of Utility”, *Econometrica* (26): 193-224.

Luce, R. D. and Suppes, P. (1965): “Preference, Utility and Subjective Probability”, in R. D. Luce, R. R. Bush and E. Galanter (eds) *Handbook of Mathematical Psychology*, Vol III (New York: John Wiley and Sons).

Manski, C. F. (1977): “The Structure of Random Utility Models”, in G. L. Eberlein, W. Kroeber-Riel, W. Leinfellner and F. Schick (eds) *Theory and Decision* (Dordrecht: Reidel).

Marschak, J. (1960): “Binary Choice Constraints and Random Utility Indicators”, in K. J. Arrow, S. Karlin and P. Suppes (eds) *Proceedings of a Symposium on Mathematical Methods in the Social Sciences* (Stanford: Stanford University Press).

Quandt, R. (1956): “A Probabilistic Theory of Consumer Behaviour”, *Quarterly Journal of Economics* (70): 507-536.

Samuelson, P.A. (1948): “Consumption Theory in Terms of Revealed Preference”, *Economica* 15: 243-253.

Sattath, S. and Tversky, A. (1976): “Unite and Conquer: A Multiplicative Inequality for Choice Probability”, *Econometrica* (44): 79-89.

Sen, A.K. (1969): “Quasi-transitivity, Rational Choice and Collective Decisions”, *Review of Economic Studies* (36): 381-393.

Yellott, J. I., Jr. (1977): “The Relationship between Luce’s Choice Axiom, Thurstone’s Theory of Comparative Judgment, and the Double Exponential Distribution”, *Journal of Mathematical Psychology* (15): 109-144.

Figure 1