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Evolutionary Stability of Constant Consistent Conjectures

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Abstract

The paper shows that there is a close connection between (constant) consistent conjectures in a given game and evolutionary stability of conjectures. Evolutionarily stable conjectures are consistent and consistent conjectures are the only interior candidates for evolutionary stability.

1 Introduction

Recently, in the context of a particular linear-quadratic duopoly model, Muller and Normann (2003) showed that consistent conjectures are evolutionarily stable, while Dixon and Somma (2003) demonstrated that an explicit evolutionary process converges to consistent conjectures. The purpose of this note is to show that (constant) consistent conjectures and evolutionary stability are closely connected in a general setting of two-player games.

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2 The Model

2.1 The Game

Consider a two-player game $G = (\{1,2\}, \{X_1, X_2\}, \{u_1, u_2\})$, where $X_1, X_2 \subset \mathbb{R}$ are convex strategy spaces and u_1, u_2 are payoff functions of the two players. In what follows, i refers to either Player 1 or Player 2, and j to the other player ($j \neq i$). The payoff functions are assumed to be twice continuously differentiable.

The players have (constant) conjectures about the marginal reaction of the opponent to a marginal change in strategy. Let $r_{ij} \in Y_i \subset \mathbb{R}$, where Y_i is a convex set, be this conjecture of Player i about Player j, that is, Player i believes that $\frac{dx_j}{dx_i}(x_i, x_j) = r_{ij}\ \forall x_i, x_j$. I work with constant conjectures because they allow some selection; if conjectures depend on x_i, x_j many strategy profiles can be supported by (weakly) consistent conjectures (Laitner, 1980; Boyer and Moreaux, 1983).

Since Player i believes that x_j depends on x_i, Player i's maximization problem is $\max_{X_i} u_i(x_i, x_j(x_i))$. At an interior solution x_i^* of this problem $\frac{\partial u_i}{\partial x_i}(x_i, x_j(x_i)) + \frac{\partial u_i}{\partial x_j}(x_i, x_j(x_i))\frac{dx_j}{dx_i} = 0$. Since the player does not attempt to conjecture the whole reaction function $x_j(x_i)$, but only its slope $\frac{dx_j}{dx_i} = r_{ij}$, x_j is an independent variable, so at x_i^* $\frac{\partial u_i}{\partial x_i}(x_i, x_j) + \frac{\partial u_i}{\partial x_j}(x_i, x_j) r_{ij} = 0$. It follows that for given x_j and corresponding x_i^* it holds that $-\frac{\partial u_i/\partial x_i(x_i,x_j)}{\partial u_i/\partial x_j(x_i,x_j)} = r_{ij}$, when $\frac{\partial u_i}{\partial x_j}(x_i, x_j) \neq 0$ at x_i^*.

Claim 1 *For given r_{ij} and x_j, at an interior best response x_i^* of Player i it holds that $-\frac{\partial u_i/\partial x_i(x_i,x_j)}{\partial u_i/\partial x_j(x_i,x_j)} = r_{ij}$, when $\frac{\partial u_i}{\partial x_j}(x_i^*, x_j) \neq 0$.*

Condition 1 $\frac{\partial u_i}{\partial x_j}(x_i, x_j) \neq 0$ *at* x_i^*.

Let $F_i(x_i, x_j; r_{ij}) = \frac{\partial u_i}{\partial x_i}(x_i, x_j) + \frac{\partial u_i}{\partial x_j}(x_i, x_j) r_{ij}$. At an interior solution x_i^* of Player i's maximization problem $F_i(x_i, x_j; r_{ij}) = 0$. If the solution is unique and interior for each x_j, $F_i(x_i, x_j; r_{ij}) = 0$ implicitly defines the reaction function $x_i^*(x_j; r_{ij})$ of Player i. To be able to use this reaction function I require

Condition 2 *For all $r_{ij} \in Y$, all $x_j \in X_j$, problem $\max_{X_i} u_i(x_i, x_j(x_i))$ has unique interior solution x_i^*.*

2.2 Consistent Conjectures

To distinguish the consistency notion I use from that in (some of) the literature (e.g. Bresnahan, 1981, where consistent conjectures are functions and are required to coincide with all derivatives of the actual reaction function in the neighborhood of equilibrium) I call a conjecture of Player i *weakly consistent* if the conjectured reaction of Player j equals the actual slope of the reaction function of Player j *at best response*, i.e. at x_i^* $r_{ij} = \frac{dx_j^*}{dx_i}(x_i; r_{ji})$.

Since the reaction function is determined implicitly by $F_i(x_i, x_j; r_{ij}) = 0$, the slope of it is $\frac{dx_i^*}{dx_j} = -\frac{\partial F_i/\partial x_j(x_i,x_j)}{\partial F_i/\partial x_i(x_i,x_j)}$, when $\frac{\partial F_i}{\partial x_i}(x_i, x_j) \neq 0$. Therefore, a conjecture r_{ij}^C of Player i is weakly consistent when at the best response $x_i^*(r_{ij}^C)$ of Player i

$$r_{ij}^C = \frac{dx_j^*}{dx_i}(x_i^*; r_{ji}) = -\frac{\partial F_j/\partial x_i(x_i^*, x_j^*; r_{ji})}{\partial F_j/\partial x_j(x_i^*, x_j^*; r_{ji})} \tag{1}$$

where x_j^* is the interior best response of Player j when Player j's conjecture is r_{ji} and Player i's strategy is x_i^*.

Claim 2 *For given r_{ij}, r_{ji}, at best responses (x_i^*, x_j^*), conjecture r_{ij} is weakly consistent iff $r_{ij} = -\frac{\partial F_j/\partial x_i(x_j^*, x_i^*, r_{ji})}{\partial F_j/\partial x_j(x_j^*, x_i^*; r_{ji})}$, when $\frac{\partial F_j}{\partial x_j}(x_i^*, x_j^*; r_{ji}) \neq 0$.*

Condition 3 $\frac{\partial F_j}{\partial x_j}(x_i^*, x_j^*) \neq 0$ *for any $r_{ij} \in Y_i, r_{ji} \in Y_j$.*

Conjectures r_{ij}^C, r_{ji}^C are mutually consistent if $r_{ij}^C = \frac{dx_j^*}{dx_i}(x_i^*; r_{ji}^C)$ and $r_{ji}^C = \frac{dx_i^*}{dx_j}(x_j^*; r_{ij}^C)$. When the game is symmetric, the reaction functions are symmetric. Then a symmetric conjecture $r^C = r_{ij}^C = r_{ji}^C$ is consistent when $r^C = \frac{dx_i^*}{dx_j}(x_j^*; r^C)$.

2.3 Evolutionary Stability of Conjectures

Suppose that conjectures is something a player is born with (one can interpret them as optimism/pessimism attitudes). Consider a large population of players who are repeatedly randomly matched. In a match, players observe the conjectures of each other and behave according to equilibrium of the game with these conjectures. The (evolutionary) success of a given conjecture is determined by averaging of equilibrium payoffs of players with this conjecture over all matches. The proportions of players with given conjecture change according to their evolutionary success.

For given conjecture r_{ji} of Player j, *evolutionarily stable* (ES) conjecture of Player i is such conjecture r_{ij}^{ES} that no other conjecture r_{ij} perform better or equally well in a population of Players i almost exclusively composed of players with conjecture r_{ij}^{ES} (and the rest of the population have conjecture r_{ij}). If in a monomorphic population of players with conjecture r_{ij}^{ES} a small proportion of mutants with some other conjecture r_{ij} appears, evolutionary forces will eliminate the mutants. This approach is a generalization to asymmetric games of the indirect evolution approach of Guth and Yaari (1992).

Given conjectures r_{ij}, r_{ji}, let $u_i(r_{ij}, r_{ji}) = u_i(x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji}))$ be the payoff of Player i when equilibrium $x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$ is played. For given conjecture r_{ji}^* of Player j, conjecture r_{ij}^{ES} of Player i is evolutionarily stable if $u_i(r_{ij}^{ES}, r_{ji}^*) > u_i(r_{ij}, r_{ji}^*)$ for any $r_{ij} \neq r_{ij}^*$ (asymmetric games ESS, Selten, 1980). A conjecture of Player i is evolutionarily stable against a given conjecture of Player j if it is unique best response to that conjecture of Player j in the game with payoffs $u_i(r_{ij}, r_{ji})$.

Given conjectures r_{ij}, r_{ji} of the players, if the solutions of optimization problems are interior, x_i, x_j satisfy $F_i(x_i, x_j; r_{ij}) = 0, F_j(x_i, x_j; r_{ji}) = 0$. Consider the problem

$$\begin{aligned} \max_{x_i, x_j, r_{ij}} u_i(x_i, x_j) & \\ \text{s.t. } F_i(x_i, x_j; r_{ij}) &= 0 \\ F_j(x_i, x_j; r_{ji}) &= 0 \end{aligned} \tag{2}$$

By the implicit function theorem, the system of equations $F_i(x_i, x_j; r_{ij}) = 0, F_j(x_i, x_j; r_{ji}) = 0$ determines functions $x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$, when $\frac{\partial F_i}{\partial x_i}\frac{\partial F_j}{\partial x_j} - \frac{\partial F_i}{\partial x_j}\frac{\partial F_j}{\partial x_i} \neq 0$ at $r_{ij}, r_{ji}, x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$. Substituting the implicit functions, problem (2) is equivalent to

$$\max_{r_{ij}} u_i(x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})) \tag{3}$$

Problem (3) is exactly the problem to find a best response conjecture for Player i, given conjecture r_{ji} of Player j.

Since problems (2) and (3) are equivalent, they have the same solution. At an interior solution of problem (2) the following first order conditions hold:

$$\frac{\partial u_i}{\partial x_i} - \lambda \frac{\partial F_i}{\partial x_i} - \mu \frac{\partial F_j}{\partial x_i} = 0 \quad (4a)$$

$$\frac{\partial u_i}{\partial x_j} - \lambda \frac{\partial F_i}{\partial x_j} - \mu \frac{\partial F_j}{\partial x_j} = 0 \quad (4b)$$

$$\frac{\partial u_i}{\partial r_{ij}} - \lambda \frac{\partial F_i}{\partial r_{ij}} - \mu \frac{\partial F_j}{\partial r_{ij}} = 0 \quad (4c)$$

where λ, μ are Lagrangean multipliers. Since u_i does not depend directly on r_{ij}, $\frac{\partial u_i}{\partial r_{ij}} = 0$. Since F_j does not depend directly on r_{ij}, $\frac{\partial F_j}{\partial r_{ij}} = 0$. Furthermore, since $F_i = \frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_j} r_{ij}$, $\frac{\partial F_i}{\partial r_{ij}} = \frac{\partial u_i}{\partial x_j}$. By Condition 1 $\frac{\partial u_i}{\partial x_j} \neq 0$, thus $\frac{\partial F_i}{\partial r_{ij}} \neq 0$. Then from (4c) $\lambda = 0$, and from (4a) and (4b) it follows that $\frac{\partial u_i/\partial x_i}{\partial u_i/\partial x_j} = \frac{\partial F_j/\partial x_i}{\partial F_j/\partial x_j}$ (since $\frac{\partial F_j}{\partial x_j} \neq 0$ by Condition 3).

Claim 3 *At an interior solution of* $\max_{r_{ij}} u_i(x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji}))$ *it holds that* $\frac{\partial u_i/\partial x_i}{\partial u_i/\partial x_j} = \frac{\partial F_j/\partial x_i}{\partial F_j/\partial x_j}$.

Condition 4 $\frac{\partial F_i}{\partial x_i}\frac{\partial F_j}{\partial x_j} - \frac{\partial F_i}{\partial x_j}\frac{\partial F_j}{\partial x_i} \neq 0$ *at* $r_{ij}, r_{ji}, x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$ *for all* $r_{ij} \in Y_i, r_{ji} \in Y_j$.

By Claim 1 $r_{ij} = -\frac{\partial u_i/\partial x_i}{\partial u_i/\partial x_j}$. Combining with Claim 3, if interior conjecture r_{ij} is evolutionarily stable, then $r_{ij} = -\frac{\partial F_j/\partial x_j}{\partial F_j/\partial x_i}$. But by Claim 2 this means that r_{ij} is weakly consistent. We have

Proposition 1 *Suppose Conditions 1 to 4 are satisfied. If interior conjecture* r_{ij} *is evolutionarily stable against given* r_{ji}*, then it is weakly consistent for this* r_{ji}.

A partial reverse of the proposition may be stated as following:

Proposition 2 *Suppose Conditions 1 to 4 are satisfied. If interior conjecture* r_{ij} *is not consistent for given* r_{ji}*, then it is not evolutionarily stable against* r_{ji}.

Additional conditions are needed for the full reverse. Condition 4 guarantees existence of equilibrium functions $x_i^*(r_{ij}, r_{ji}), x_j^*(r_{ij}, r_{ji})$. If sufficient conditions for global unique optimum of problem (3) are satisfied, then weak consistency of interior conjecture r_{ij}^* implies that r_{ij}^* is an ES conjecture. One such sufficient condition is global concavity of the payoff function.

Proposition 3 *Suppose Conditions 1 to 4 are satisfied. If interior* r^*_{ij} *is weakly consistent for given* r_{ji} *and* $[u_i(x^*_i(r, r_{ji}), x^*_j(r, r_{ji})]''_r < 0$ *for all* r*, then* r^*_{ij} *is evolutionarily stable against* r_{ji}*.*

Another condition, often easier to check, is local concavity together with the uniqueness of the critical point. Thus

Proposition 4 *Suppose Conditions 1 to 4 are satisfied. If interior* r^*_{ij} *is weakly consistent,* $[u_i(x^*_i(r, r_{ji}), x^*_j(r, r_{ji})]'_r = 0$ *has unique solution* r^*_{ij}*, and* $[u_i(x^*_i(r, r_{ji}), x^*_j(r, r_{ji}))]''_r|_{r=r^*_{ij}} < 0$*, then* r^*_{ij} *is evolutionarily stable.*

The analysis above is for Player i and for given conjecture r_{ji} of Player j. Analogous analysis can be performed for Player j, keeping constant conjecture r_{ij} of Player i. If interior conjectures r^*_{ij}, r^*_{ji} are mutually evolutionarily stable, then they are mutually consistent. If interior conjectures are not mutually consistent, then they are not mutually evolutionarily stable. Analogous extensions hold for the other propositions.

When the game is symmetric, it is natural to expect players to hold symmetric conjectures. Although in the symmetric case evolutionary stability is not equivalent to strict best response, the propositions hold for the symmetric case as well. An interior evolutionarily stable conjecture is a best response to itself, so first order conditions have to hold, thus implying Propositions 1 and 2. Since a strict symmetric equilibrium is evolutionarily stable in the symmetric case, sufficient conditions of Propositions 3 and 4 imply evolutionary stability in this case too.

The graphical illustration of the close connection between consistency and evolutionary stability is given in the example in the next section. Intuitively, r_{ji} determines the reaction function of Player j. By varying r_{ij}, Player i can change his own reaction function and so can change its point of intersection with the reaction function of Player j. Player i will choose such a point on the reaction function of Player j where it is tangent to level curves of Player i's payoff function. But since r_{ij} by Claim 1 equals the slope of these level curves, best response r_{ij} has to be equal to the slope of the reaction function of Player j.

If conjecture r_{ij} is consistent, Player i "knows" the reaction of Player j to small changes in x_i. Thus a player with consistent conjecture maximizes the "correct" function $u_i(x_i, x_j(x_i))$, and so has higher payoff than if conjecture is not consistent. Therefore the obtained result may look obvious. Nevertheless, Muller and Normann (2003) state "the result that the evolutionarily stable conjectures coincide with the consistent conjectures is surprising as there is no obvious analogy between the two", and the result was certainly also surprising for me. There was no reason to expect apriori that 'more rationality' (consistency) should lead to the same result as 'less rationality' (evolution); only after interpreting the result did the connection appear obvious.

3 Examples

3.1 Linear-Quadratic Cournot Duopoly

Constant conjectures are justified when the reaction functions are indeed linear. This is the case, for example, when in a duopoly the demand function is linear and cost functions are quadratic. This is the case considered in Muller and Normann (2003) and Dixon and Somma (2003).

Consider a symmetric Cournot duopoly with inverse demand function $P(q_i, q_j) = a - b(q_i + q_j), a > 0, b > 0$ and cost function $c(q_i) = \frac{c}{2}q_i^2, c > 0$. The strategy space is $X = [0, \frac{b}{a})$. Let the conjecture space be $Y = (-1, 1)$. The payoff function is $\pi_i(q_i, q_j) = P(q_i, q_j)q_i - c_i(q_i) = (a - b(q_i + q_j))q_i - \frac{c}{2}q_i^2$.

Since $\frac{\partial \pi_i}{\partial q_j} = -bq_i \neq 0$ in the interior of X, Condition 1 is satisfied. Player i's problem is $\max_{q_i}(a - b(q_i + q_j(q_i)))q_i - \frac{c}{2}q_i^2$, and $\frac{dq_j}{dq_i} = r_{ij}$. The first order condition is $F_i = a - 2bq_i - bq_j - br_{ij}q_i - cq_i = 0$. This implies $q_i^* = \frac{a - bq_j}{b(2+r_{ij})+c}$, which is interior. Since the second order condition $-2b(1 + r_{ij}) - c < 0$ is satisfied for all q_i, Condition 2 is satisfied. Thus the reaction function is given by $F_i = 0$.

Since $\frac{\partial F_i}{\partial q_i} = -b(2 + r_{ij}) - c \neq 0$, Condition 3 is satisfied. Finally, since $\frac{\partial F_i}{\partial q_i}\frac{\partial F_j}{\partial q_j} - \frac{\partial F_i}{\partial q_j}\frac{\partial F_j}{\partial q_i} = (-2b - c - br_{ij})(-2b - c - br_{ji}) - b^2 \neq 0$ when $r_{ij}, r_{ji} \in (-1, 1)$, Condition 4 is also satisfied.

Consistent symmetric conjecture can be found from $r = \frac{dq_i^*}{dq_j} = -\frac{\partial F_j/\partial q_i}{\partial F_j/\partial q_j}$. Then $r = -\frac{b}{2b+c+br}$, or $br^2 + (2b + c)r + b = 0$. Let $H(r) = br^2 + (2b + c)r + b$. Since $H(-1) = -c < 0, H(0) = b > 0$, and $H(1) = 4b + c > 0$, there is one root on $(-1, 1)$ and it is between -1 and 0. Thus there is unique consistent conjecture $r^C \in (-1, 0)$. By Proposition 2 it is the unique interior candidate for an evolutionarily stable conjecture.

The profit function can be written as $\pi_i(q_i, q_j) = \frac{1}{2}(q_i)^2 \left(\frac{2(a-b(q_i+q_j))}{q_i} - c\right)$. From the reaction functions $(a - b(q_i^* + q_j^*)) - (b + br_{ij})q_i^* - cq_i^* = 0$, or $(a - b(q_i^* + q_j^*)) = (b + br_{ij} + c)q_i^*$. Therefore, at equilibrium $\pi_i(q_i^*, q_j^*) = \frac{1}{2}(q_i^*)^2 (2b(1 + r_{ij}) + c)$ and $\frac{\partial \pi_i(q_i^*, q_j^*)}{\partial r_{ij}} = (q_i^*)^2 b + q_i^* \frac{\partial q_i^*}{\partial r_{ij}} (2b(1 + r_{ij}) + c)$.

The equilibrium for given r_{ij}, r_{ji} is $q_i^* = \frac{a(b+c+br_{ji})}{(2b+c+br_{ij})(2b+c+br_{ji})-b^2}$. Then

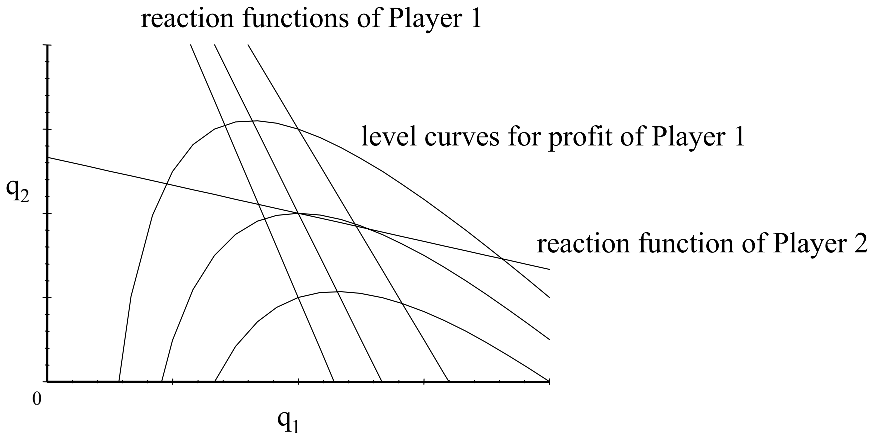

Figure 1: Reaction functions and level curves

$\frac{\partial q_i^*}{\partial r_{ij}} = \frac{-bq_i^*(2b+c+br_{ji})}{(2b+c+br_{ij})(2b+c+br_{ji})-b^2}$, and $\frac{\partial \pi_i(q_i^*,q_j^*)}{\partial r_{ij}} = (q_i^*)^2 \frac{b^2(-(2b+c+br_{ji})r_{ij}-b)}{(2b+c+br_{ij})(2b+c+br_{ji})-b^2}$. The unique solution of the first order condition $\frac{\partial \pi_i(q_i^*,q_j^*)}{\partial r_{ij}} = 0$ is $r_{ij} = -\frac{b}{2b+c+br_{ji}}$. When $r_{ji} = r^C$, the unique solution is $r_{ij} = r^C \in (-1, 0)$.

Furthermore, $\frac{\partial^2 \pi_i(q_i^*,q_j^*)}{\partial r_{ij}^2} = \frac{b^2(q_i^*)^2(2b+c+br_{ji})}{((2b+c+br_{ij})(2b+c+br_{ji})-b^2)^2}((2b+c+br_{ji})(-(2b+c)+2br_{ij})+4b^2)$. The sign of $\frac{\partial^2 \pi_i(q_i^*,q_j^*)}{\partial r_{ij}^2}$ is determined by the sign of $(2b+c+br_{ji})(-(2b+c)+2br_{ij})+4b^2$. When $r_{ij} = r_{ji} = r^C$, this expression becomes $-\frac{b}{r^C}(2br^C-(2b+c))+4b^2 = 2b^2 + \frac{(2b+c)b}{r^C} = \frac{2b^2(r^C+1)+bc}{r^C} < 0$. By Proposition 4

Proposition 5 *In the linear-quadratic Cournot duopoly there exist unique consistent conjecture and it is unique evolutionarily stable one.*

To get the intuition behind the result, also for the general case of the previous section, it is useful to consider a diagram. Conjecture r_{21} determines a reaction function of Player 2, which is linear in this case. Varying r_{12} varies reaction functions of Player 1, three of which are drawn on Figure 1. The equilibrium is on the intersection of the reaction functions, thus varying r_{12} allows Player 1 to move along the given reaction function of Player 2. Some level curves of Player 1 for the linear-quadratic duopoly are also drawn in the figure. Profit is increasing in the south-east direction.

Since Player 1 can vary the equilibrium point by moving along the reaction function of Player 2, the best profit Player 1 can achieve is at the point where a level curve is tangent to the reaction function of Player 2. At this point the slope of the reaction function $-\frac{\partial F_2/\partial q_1}{\partial F_2/\partial q_2}$ equals the slope of the profit level curve $-\frac{\partial \pi_1/\partial q_1}{\partial \pi_1/\partial q_2}$. For Player 1, having a conjecture r_{12} means that the reaction function of Player 1 cuts profit level curves at points where its slope equals r_{12} (by Claim 1). Therefore, conjecture r_{12} equals the slope of the profit level curves, which at the tangency point equals the slope of the reaction function of Player 2, which means that r_{12} is consistent.

3.2 Differentiated Goods Bertrand Duopoly

Consider a symmetric differentiated goods Bertrand duopoly where the demand of Firm i is given by $D_i(p_i, p_j) = A - p_i + kp_j$, with $A > 0$ and $k \in (0, 1)$. Firms choose prices from the strategy set $X = [0, \infty)$. Suppose that costs are zero. The profit of Firm i is $\pi_i(p_i, p_j) = p_i(A - p_i + kp_j)$. Let $Y = (-1, 1)$.

Since $\frac{\partial \pi_i}{\partial p_j} = kp_i \neq 0$ in the interior of X, Condition 1 is satisfied. Given conjecture r_{ij}, Player i's problem is $\max_{p_i} p_i(A - p_i + kp_j(p_i))$, and $\frac{dp_j}{dp_i} = r_{ij}$. The first order condition is $F_i = A - 2p_i + kp_j + kp_i r_{ij} = 0$, which implies $p_i^* = \frac{A+kp_j}{2-kr_{ij}} > 0$. Since the second order condition $2(kr_{ij} - 1) < 0$ is satisfied for all p_i, Condition 2 is satisfied. Therefore the reaction function of Player i is given by $F_i = 0$.

Condition 3 is satisfied since $\frac{\partial F_i}{\partial p_i} = -2 + kr_{ij} \neq 0$ on Y. Since $\frac{\partial F_i}{\partial p_i}\frac{\partial F_j}{\partial p_j} - \frac{\partial F_i}{\partial p_j}\frac{\partial F_j}{\partial p_i} = (2 - kr_{ij})(2 - kr_{ji}) - k^2 \neq 0$ when $r_{ij}, r_{ji} \in (-1, 1)$, Condition 4 is satisfied too.

Consistent symmetric conjecture is found from $r = \frac{dp_i^*}{dp_j} = -\frac{\partial F_i/\partial p_j}{\partial F_i/\partial p_i}$. Then $r = \frac{k}{2-kr}$, or $kr^2 - 2r + k = 0$. The solutions of this equation are $r^C = \frac{1 \pm \sqrt{1-k^2}}{k}$, of which only the root $r^C = \frac{1-\sqrt{1-k^2}}{k} > 0$ is between -1 and 1. There is unique consistent conjecture $r^C \in (0, 1)$. It is unique candidate for an evolutionarily stable conjecture by Proposition 2.

The equilibrium for given r_{ij}, r_{ji} is $p_i^* = \frac{A(2-kr_{ji}+k)}{(2-kr_{ij})(2-kr_{ji})-k^2}$. Then $\frac{\partial \pi_i(p_i^*, p_j^*)}{\partial r_{ij}} = \frac{\partial \pi_i}{\partial p_i^*}\frac{\partial p_i^*}{\partial r_{ij}} + \frac{\partial \pi_i}{\partial p_j^*}\frac{\partial p_j^*}{\partial r_{ij}} = (A - 2p_i^* + kp_j^*)\frac{kp_i^*(2-kr_{ji})}{(2-kr_{ij})(2-kr_{ji})-k^2} + kp_i^*\frac{-Ak+p_j^*(2-kr_{ji})k}{(2-kr_{ij})(2-kr_{ji})-k^2}$. From the reaction functions, $p_j^* = \frac{A+kp_i^*}{2-kr_{ji}}$ and $A - 2p_i^* + kp_j^* = -kp_i^* r_{ij}$, so $\frac{\partial \pi_i(p_i^*, p_j^*)}{\partial r_{ij}} = (p_i^*)^2 \frac{k^2(k-r_{ij}(2-kr_{ji}))}{(2-kr_{ij})(2-kr_{ji})-k^2}$. The first order condition $k - r_{ij}(2 - kr_{ji}) = 0$ has unique solution $r_{ij} = \frac{k}{2-kr_{ji}}$ for any r_{ji}.

For the second order condition, $\frac{\partial^2 \pi_i(p_i^*, p_j^*)}{\partial r_{ij}^2} = \frac{k^2\left(p_i^*\right)^2(2-kr_{ji})}{((2-kr_{ij})(2-kr_{ji})-k^2)^2}(3k(k - r_{ij}(2 - kr_{ji})) - (2 - kr_{ij})(2 - kr_{ji}) + k^2)$. At $r_{ij} = r_{ji} = r^C$ $3k(k - r_{ij}(2 - kr_{ji})) - (2 - kr_{ij})(2 - kr_{ji}) + k^2 = k^2(1 - \frac{1}{r^C}) < 0$, therefore by Proposition 4

Proposition 6 *There exist unique consistent conjecture in the linear differentiated goods Bertrand duopoly of this section and it is the unique evolutionarily stable conjecture.*

3.3 Semi-Public Good Games

Consider the following symmetric two player public good provision game. Players have endowments of private good w. They can contribute x_i to the public good, and leave $y_i = w - x_i$ of private good for consumption. Let the strategy set be $X = [-w, w]$, which is needed to guarantee interior best response and can be interpreted as players having the opportunity to contribute as well as to take out of a common pool of public good. The contribution of Player j enters Player i's utility with weight $0 < \beta < 1$, thus for Player i the total supply of public good is $X_i = x_i + \beta x_j$. Players' utility functions are $u_i(y_i, X_i)$. This is the model of semi-public goods considered in Costrell (1991). Let $Y = (-1, 1)$.

Suppose that the utility functions are $u_i(y_i, X_i) = y_i^\alpha X_i^{1-\alpha}$, where $0 < \alpha < 1$. The payoff function of Player i is then $u_i(x_i, x_j) = (w - x_i)^\alpha (x_i + \beta x_j)^{1-\alpha}$. Given conjecture r_{ij}, the first order condition of the maximization problem is $F_i = -\alpha \left(\frac{X_i}{y_i}\right)^{1-\alpha} + (1-\alpha)\left(\frac{y_i}{X_j}\right)^{\alpha}(1 + \beta r_{ij}) = 0$. Let $v = \frac{X_i}{y_i}$. Then $-\alpha v^{1-\alpha} + (1-\alpha)v^{-\alpha}(1 + \beta r_{ij}) = 0 \Rightarrow v^{-\alpha}(-\alpha v + (1-\alpha)(1 + \beta r_{ij})) = 0 \Rightarrow v = \frac{1-\alpha}{\alpha}(1 + \beta r_{ij})$. This implies that $\frac{x_i + \beta x_j}{w - x_i} = \frac{1-\alpha}{\alpha}(1 + \beta r_{ij})$, or $x_i + \beta x_j = \frac{1-\alpha}{\alpha}(1 + \beta r_{ij})(w - x_i) \Rightarrow x_i^* = \frac{(1-\alpha)(1+\beta r_{ij})}{1+(1-\alpha)\beta r_{ij}} w - \frac{\alpha\beta}{1+(1-\alpha)\beta r_{ij}} x_j$. This x_i^* is interior for all $x_j \in [-w, w]$. Because second order conditions $-\alpha(1-\alpha)\left(\frac{X_i{}^{1-\alpha}}{y_i^{2-\alpha}} + 2\frac{1}{X_i^\alpha y_i^{1-\alpha}}(1 + \beta r_{ij}) + \frac{y_i^\alpha}{X_i^{1+\alpha}}(1 + \beta r_{ij})^2\right) < 0$ are satisfied, reaction functions are given by $F_i = 0$, and Condition 2 is fulfilled.

From the reaction function $\frac{dx_i^*}{dx_j} = -\frac{\beta\alpha}{1+(1-\alpha)\beta r_{ij}}$. With a consistent symmetric conjecture $r = -\frac{\beta\alpha}{1+(1-\alpha)\beta r} \Rightarrow (1-\alpha)\beta r^2 + r + \alpha\beta = 0$. Let $G(r) = (1-\alpha)\beta r^2 + r + \alpha\beta$. Since $G(-1) = \beta - 1 < 0$ and $G(0) = \alpha\beta > 0$, there is an r between -1 and 0 so that $G(r) = 0$. Such r is consistent, and let it be denoted by r^C. From $r^C = -\frac{\partial F_i/\partial x_j}{\partial F_i/\partial x_i}$ it follows that $r^C = -\beta\frac{y_i}{X_i + y_i}$.

Equilibrium for given r_{ij}, r_{ji} is $x_i^* = \frac{(1-\alpha)[(1+\beta r_{ij})(1+(1-\alpha)\beta r_{ji}) - (1+\beta r_{ji})\alpha\beta]}{(1+(1-\alpha)\beta r_{ij})(1+(1-\alpha)\beta r_{ji}) - \alpha^2\beta^2} w$ and analogous expression for x_j^*. Though it is possible that $x_i^* < 0$, it holds that $y_i^* = w - x_i^* > 0$ and $X_i^* = x_i^* + \beta x_j^* > 0$.

It holds that $\frac{\partial u_i}{\partial x_j} = (1-\alpha)\beta\left(\frac{y_i}{X_i}\right)^\alpha \neq 0$ in the interior, as required by Condition 1. Also $\frac{\partial F_i}{\partial x_i} = \alpha(1-\alpha)(y_i + X_i)[-\left(\frac{X_i}{y_i}\right)^{-\alpha}\left(\frac{1}{y_i^2}\right) - \left(\frac{y_i}{X_i}\right)^{\alpha-1}\left(\frac{1}{X_i^2}\right)(1 +$

$\beta r_{ij})] \neq 0$ in equilibrium, thus Condition 3 is also fulfilled. Finally, $\frac{\partial F_i}{\partial x_i}\frac{\partial F_j}{\partial x_j} - \frac{\partial F_i}{\partial x_j}\frac{\partial F_j}{\partial x_i} = \alpha^2(1-\alpha)^2 \frac{(X_i+(1+\beta r_{ij})y_i)(X_j+(1+\beta r_{ji})y_j)}{X_i^{\alpha+1}y_i^{2-\alpha}X_j^{\alpha+1}y_j^{2-\alpha}}[X_iX_j + X_iy_j + y_iX_j + (1-\beta^2)y_iy_j] \neq 0$, satisfying Condition 4. By Proposition 2 the consistent conjecture is the unique interior candidate for an evolutionary stable conjecture.

Since in equilibrium $X_i^* = \frac{1-\alpha}{\alpha}(1+\beta r_{ij})y_i^*$, the utility function can be rewritten as $u_i(x_i^*, x_j^*) = \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}(1+\beta r_{ij})^{1-\alpha}y_i^*$. Then $\frac{\partial u_i(x_i^*,x_j^*)}{\partial r_{ij}} = \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}(1+\beta r_{ij})^{-\alpha}\left[(1-\alpha)\beta y_i^* + (1+\beta r_{ij})\frac{\partial y_i^*}{\partial r_{ij}}\right]$. It holds that $\frac{\partial y_i^*}{\partial r_{ij}} = -\frac{\partial x_i^*}{\partial r_{ij}} = -\frac{\beta(1-\alpha)(1+(1-\alpha)\beta r_{ji})}{(1+(1-\alpha)\beta r_{ij})(1+(1-\alpha)\beta r_{ji})-\alpha^2\beta^2}y_i^*$. Then $\frac{\partial u_i(x_i^*,x_j^*)}{\partial r_{ij}} = -\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}(1-\alpha)\beta^2\alpha\frac{r_{ij}(1+(1-\alpha)\beta r_{ji})+\alpha\beta}{(1+(1-\alpha)\beta r_{ij})(1+(1-\alpha)\beta r_{ji})-\alpha^2\beta^2}(1+\beta r_{ij})^{-\alpha}y_i^*$. The first order maximization condition $\frac{\partial u_i(x_i^*,x_j^*)}{\partial r_{ij}} = 0$ has unique solution $r_{ij} = -\frac{\alpha\beta}{1+(1-\alpha)\beta r_{ji}} = r^C \in (-1, 0)$ when $r_{ji} = r^C$.

Let $K = -\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha}(1-\alpha)\beta^2\alpha < 0$. For the second order condition, $\frac{\partial^2 u_i(x_i^*,x_j^*)}{\partial r_{ij}^2} = K[\frac{\partial}{\partial r_{ij}}((1+\beta r_{ij})^{-\alpha}y_i^*) \cdot \frac{r_{ij}(1+(1-\alpha)\beta r_{ji})+\alpha\beta}{(1+(1-\alpha)\beta r_{ij})(1+(1-\alpha)\beta r_{ji})-\alpha^2\beta^2} + (1+\beta r_{ij})^{-\alpha}y_i^* \cdot \frac{\partial}{\partial r_{ij}}\frac{r_{ij}(1+(1-\alpha)\beta r_{ji})+\alpha\beta}{(1+(1-\alpha)\beta r_{ij})(1+(1-\alpha)\beta r_{ji})-\alpha^2\beta^2}]$. At $r_{ij} = r_{ji} = r^C$ the first term in square brackets is 0. Since $\frac{\partial}{\partial r_{ij}}\frac{r_{ij}(1+(1-\alpha)\beta r_{ji})+\alpha\beta}{(1+(1-\alpha)\beta r_{ij})(1+(1-\alpha)\beta r_{ji})-\alpha^2\beta^2} = \frac{1+(1-\alpha)\beta r^C}{(1+(1-\alpha)\beta r^C)^2-\alpha^2\beta^2}$ at $r_{ij} = r_{ji} = r^C$, the sign of the second term is determined by the signs of $1+(1-\alpha)\beta r^C$ and $(1+(1-\alpha)\beta r^C)^2 - \alpha^2\beta^2$. Since $1+(1-\alpha)\beta r^C = -\frac{\beta\alpha}{r^C}$, $1+(1-\alpha)\beta r^C > 0$ and $(1+(1-\alpha)\beta r^C)^2 - \alpha^2\beta^2 = \alpha^2\beta^2(\frac{1}{(r^C)^2} - 1) > 0$. Therefore $\frac{\partial^2 u_i(x_i^*,x_j^*)}{\partial r_{ij}^2} < 0$, and the consistent conjecture r^C is also evolutionarily stable.

Proposition 7 *In the semi-public good game of this section the unique consistent conjecture r^C is unique evolutionarily stable conjecture.*

4 Conclusion

The observations of Muller and Normann (2003) and Dixon and Somma (2003) about evolutionary stability of consistent conjectures for a particular duopoly case generalize to other games because they are based on coincidence of first order conditions. Apart from the examples considered in the paper, other games to which the results can be applied include common pool resource exploitation games and rent-seeking games. Furthermore, it should be possible to generalize the results to n-player aggregative games, i.e. games in which payoffs depend on own strategy and on an aggregate of strategies of other players, treating the conjecture as conjecture about the aggregate reaction of other players.

The intuition for evolutionary stability of consistent conjecture is that a player with such conjecture correctly estimates the response of the other player and thus maximizes the "right" function, outperforming in evolutionary terms players with other conjectures. Though this result may appear obvious, it certainly was not before the analysis. It is interesting that 'rational' approach (consistency) and 'evolutionary' approach lead to the same outcome in many games. Evolutionary approach can provide a justification for consistent conjectures as emerging from a dynamic process. Consistent conjecture, on the other hand, is often easier to find, simplifying evolutionary analysis. Depending on the questions asked about a game, the two approaches complement each other.

References

[1] Boyer, M., Moreaux, M. (1983) "Consistent versus Non-Consistent Conjectures in Duopoly Theory: Some Examples", *Journal of Industrial Economics*, 32, 97-110.

[2] Bresnahan, T. (1981) "Duopoly Models with Consistent Conjectures", *American Economic Review*, 934-945.

[3] Costrell, R. (1991) "Immiserizing Growth with Semi-Public Goods under Consistent Conjectures", *Journal of Public Economics*, 45, 383-389.

[4] Dixon, H., Somma, E. (2003) "The Evolution of Consistent Conjectures", *Journal of Economic Behavior & Organization*, 51, 523-536.

[5] Guth, W., Yaari, M. (1992) "An Evolutionary Approach to Explain Reciprocal Behavior in a Simple Strategic Game", in: Witt, U. (ed) *Explaining Process and Change - Approaches to Evolutionary Economics*, University of Michigan Press, pp.23-34.

[6] Laitner, J. (1980) ""Rational" Duopoly Equilibria", *Quarterly Journal of Economics*, 95, 641-662.

[7] Muller, W., Normann, H.-T. (2003) "Conjectural Variations and Evolutionary Stability: A New Rationale for Consistency", CentER DP 2003-44, Tilburg University.

[8] Selten, R. (1980) "A Note on Evolutionarily Stable Strategies in Asymmetric Animal Conflicts", *Journal of Theoretical Biology*, 84, 93-101.