Excess Volatility and Closed-End Fund Discounts

Michael Bleaney
School of Economics
University of Nottingham
Nottingham NG7 2RD,
U.K.
Tel. (+44) 115 951 5464
Fax (+44) 115 951 4159
e-mail: michael.bleaney@nottingham.ac.uk
(corresponding author)
Tel. +44 115 951 5464
Fax. +44 115 951 4159

R. Todd Smith
Department of Economics
University of Alberta
Edmonton, AB T6G 2H4
Canada
e-mail: todd.smith@ualberta.ca

Abstract

It is shown that, with fixed transaction costs in the market for risky assets, investors with wealth below a certain threshold will hold pooled index funds that charge a proportional fee, rather than the market portfolio chosen by wealthier investors. If a portfolio of closed-end index funds yields greater volatility of returns to investors than open-end index funds, and charges the same fees, the closed-end funds need to trade at a discount in equilibrium to attract buyers. The same applies to actively managed funds if higher fees fully reflect extra expected returns from the managers' skill. In this case excess volatility is a sufficient condition for closed-end fund discounts. It is unnecessary for discount risk to be systematic.

Keywords: Closed-end fund, discount, excess volatility.

JEL Nos: G12

1. Introduction

The true explanation of closed-end fund discounts remains a matter of debate (see Dimson and Minio-Kozerski, 1999, for a survey). One possible explanation is that the fees charged exceed the skill of the managers (Berk and Stanton, 2007; Ferguson and Leistikow, 2001, 2004; Ross, 2002). Although there is evidence in favor of this hypothesis, such as the tendency for discounts to be negatively correlated with past portfolio returns, which are likely to be treated as indicators of skill, there is also evidence that is not the whole story. For example closed-end index funds in the U.K. market trade at a discount even though their fees tend to be lower than on open-end index funds (Bleaney and Smith, 2010), and fund fees tend to decline with the age of the fund (Gemmill and Thomas, 2006), in contradiction to the pattern required to generate discounts in the model of Berk and Stanton (2007).

The principal alternative explanation is that discounts represent compensation for the extra risk in holding closed-end fund shares associated with the variability of the discount itself. Lee *et al*. (1991) argue that closed-end fund shares are disproportionately held by noise traders, and that therefore discounts are subject to noise-trader risk. To the extent that this risk is systematic, it will be priced and rewarded with a higher return, which is achieved by trading at a discount in equilibrium, so that dividend payments represent a higher percentage of the investor's outlay. Lee *et al*. (1991) provide some evidence that discount risk is systematic, but this is controversial and has been challenged by other authors (Chen *et al*., 1993; Elton *et al*., 1998).

It is generally accepted that returns on closed-end fund shares are more volatile than returns on the funds' underlying assets, even if much of this excess volatility is uncorrelated with market returns and so represents idiosyncratic rather than systematic risk (Pontiff, 1997; Sias *et al*., 2001). Discounts also tend to move together (Bodurtha *et al*., 1993; Dimson and Minio-Kozerski, 1999; Lee *et al*., 1991), which implies that not only individual funds but also portfolios of closed-end funds experience excess volatility (in what follows we refer to this as "portfolio excess volatility"). In other words excess volatility cannot be diversified away by holding a wide spread of closed-end funds.[1] The issue in previous research has been whether the closed-end fund discount risk is systematic (significantly correlated with market returns), which has been assumed to be a necessary condition for it to be priced. Here we develop an alternative model in which all investors have identical expectations of the probability distribution of returns on risky assets, but face fixed transactions costs in buying and selling them. The transactions costs provide a motivation for the existence of collective investment funds. These funds run passive portfolios for a proportional fee, which is attractive to small investors because the fund enables them to pool transactions costs. We show that, in this model, portfolio excess volatility is a sufficient condition for closed-end funds to trade at a discount, independently of whether the extra risk is idiosyncratic or systematic, provided that its fees are similar to those of an open-end fund. This result arises from the fact that any investor either holds the market portfolio directly or through a fund,

[1] Data supplied by the Association of Investment Companies show that for a portfolio of all UK conventional closed-end funds (excluding venture capital trusts) that are represented by the Association (as most are), the standard deviation of share price returns was 28.2% greater than the standard deviation of net asset value returns over the period 31 December 1999 to 31 December 2009. The portfolio is a weighted average based on market capitalization.

but not both (except in special cases that can only occur when closed-end funds already trade at a discount). If, however, it were the case that discounts did not co-move, the excess volatility on individual closed-end funds could be eliminated by spreading holdings across many funds, so that there would be no portfolio excess volatility and excess volatility in individual funds would not need to be rewarded with higher returns.

2. The Model

We consider an overlapping generations model in which agents live for two periods, working and saving in the first period, and consuming in the second period. Second-period consumption is financed by selling to the next generation financial assets that were bought from the previous generation in the first period. In the first period agents must decide what portfolio of financial assets to hold. They are assumed to have a CARA utility function defined over their consumption (C):

$$U = -e^{-aC} \tag{1}$$

where a (>0) is the coefficient of absolute risk aversion. Each agent chooses her portfolio allocation of her initial wealth (W) so as to maximize E(U).

Assume initially that the available assets consist of one safe asset, with a certain net return of zero, and n risky assets, the net returns on each of which are normally distributed with an expectation of μ (>0) and variance σ^2. For simplicity we assume that all risky asset returns are uncorrelated with each other, so returns on the whole array of risky assets are N($\underline{\mu}$, σ^2I). There is also a transactions cost of *T* in buying and selling each risky asset. In a portfolio which consists of a proportion (1−λ) invested in the safe asset and a proportion λ equally divided amongst *m* risky assets, where $0 \le m \le n$, consumption will be distributed as

$$C = N(W + \lambda W \mu - mT, \frac{\lambda^2 W^2 \sigma^2}{m}). \tag{2}$$

We assume further that agents cannot borrow, so $\lambda \le 1$ (This assumption is an artificial device necessitated by the CARA utility function, because without it small investors would always borrow whatever is necessary to hold the same portfolio of risky assets as large investors).

Clearly there is a trade-off here between the reduction of risk through portfolio diversification and the extra transactions costs involved. The portfolio decision of the agent consists of choosing λ and *m* to maximize E(*U*). Using the formula for the expectation of a lognormal distribution and log-linearizing, we have:

$$-\ln[-E(U)] = a(W + \lambda W \mu - mT) - \frac{a^2 \lambda^2 W^2 \sigma^2}{2m} \tag{3}$$

The first-order conditions of this problem are:

$$\frac{\partial\{-\ln[-E(U)]\}}{d\lambda} = aW\mu - \frac{a^2\lambda W^2\sigma^2}{m} = 0 \tag{4}$$

$$\frac{\partial\{-\ln[-E(U)]\}}{\partial m} = \frac{a\lambda^2 W^2\sigma^2}{2m^2} - aT = 0 \tag{5}$$

Equation (4) yields the solution for the amount invested in risky assets:

$$\lambda W = \frac{m\mu}{a\sigma^2} \tag{6}$$

or exactly $\frac{\mu}{a\sigma^2}$ per asset. Substituting (4) into (5), it becomes clear that, provided that there is an interior solution for λ, only corner solutions are possible for m: either m=0 (if $\mu^2 < 2aT\sigma^2$) or m=n (if $\mu^2 > 2aT\sigma^2$). Since the case of m=0 is uninteresting, we assume that expected returns on risky assets are always sufficient to offset the transactions costs and the risk ($\mu^2 > 2aT\sigma^2$), so that m=n. It follows that an interior solution for λ requires that $W \geq X = \frac{n\mu}{a\sigma^2}$. If $W < X$, then λ=1, and (5) is maximized with W given, which implies that the amount invested per risky asset is:

$$\frac{W}{m} = \frac{1}{\sigma}\sqrt{\frac{2T}{a}} \tag{7}$$

Equation (7) shows that, when wealth is relatively low, more of it is invested per asset if a and σ are smaller (reducing the benefits of portfolio diversification) and if T is larger (increasing the costs of it). Multiplying (7) by n and substituting the definition of X reveals that agents will hold all n risky assets down to a wealth of $\frac{\sigma\sqrt{2aT}}{\mu}X$, which is necessarily less than X if the condition that $m > 0$ is to be met.

2.1 The Fund Industry

Now we introduce a fund industry. This industry offers agents the opportunity to invest in an index fund that holds equal quantities of all n risky assets for a proportional fee of f per dollar invested (with no transaction fee). There is free entry to the industry, so prices are competed down to costs. The assumed proportional fee corresponds to widespread practice in the industry, and also generates the empirically desirable prediction that funds are more attractive to small than to large investors. Assume initially that the only funds offered are open-ended.

A fund has the same portfolio variance as the market portfolio of n risky assets. Because of this there is no incentive to hold a fund *and* the market portfolio, which would simply incur extra fees for no gain in portfolio diversification. So the question is: when do agents hold the market portfolio and when do they hold a fund?

The answer is that the critical factor is the investor's wealth. There are four cases that need to be discussed separately. If $W \geq X$, then an investor who holds the market portfolio (MP) holds X of risky assets. This is, however, not true if she holds a fund instead, because the fees reduce the return for the same risk. The holder of the fund chooses λ to maximize:

$$-\ln[-E(U)]_{fund} = aW + a\lambda W(\mu - f) - \frac{a^2\lambda^2 W^2\sigma^2}{2n} \qquad (8)$$

This is maximized when $\lambda W = \frac{n(\mu - f)}{a\sigma^2} = \frac{\mu - f}{\mu} X$. Taking this lower investment in risky assets into account, we can evaluate the two alternatives:

$$-\ln[-E(U)]_{MP} = a(W - nT) + \frac{n\mu^2}{2\sigma^2} \qquad (9)$$

$$-\ln[-E(U)]_{fund} = aW + \frac{n(\mu - f)^2}{2\sigma^2} \qquad (10)$$

Equation (9) yields a higher expected utility only if

$$aT < \frac{\mu^2 - (\mu - f)^2}{2\sigma^2} \qquad (11)$$

This condition does not contain W because, in the region under consideration ($W \geq X$), 100% of marginal wealth is invested in the safe asset in either case. We assume that (11) holds because otherwise every investor will hold the fund rather than the market portfolio. Under this assumption, all investors with wealth of at least X will hold the market portfolio directly.

Next we consider the second case, where wealth lies in the region

$$\frac{\sigma\sqrt{2aT}}{\mu} X \leq \frac{\mu - f}{\mu} X < W < X \tag{12}$$

In this case the agent either invests 100% in the market portfolio, or holds $\frac{\mu - f}{\mu} X$ in a fund and the remainder in the safe asset.[2] It is shown in the Appendix that the investor holds the market portfolio rather than a fund only if

$$W > X - \frac{n}{a\sigma^2}\sqrt{\mu^2 - (\mu - f)^2 - 2aT\sigma^2} \tag{13}$$

The third case is where $\frac{\sigma\sqrt{2aT}}{\mu} X \leq W \leq \frac{\mu - f}{\mu} X$. Here the investor holds none of the safe asset, and the variance terms are the same, so it is simply a matter of comparing the fund management fees against the extra transactions costs of the market portfolio. Consequently in this case the market portfolio is held if:

[2] The first inequality in (12) is a simplifying assumption which ensures that the agent holds all n risky assets if she does not prefer the fund.

$$fW > nT \tag{14}$$

The final case is where $W < \frac{\sigma\sqrt{2aT}}{\mu} X$, in which case investors holding risky assets directly only hold a subset *m* of them. It is shown in the Appendix that the fund is preferred in this case whenever

$$W < \frac{2n(\sigma\sqrt{2aT} - f)}{a\sigma^2} \tag{15}$$

To summarize the four cases, provided that equation (11) holds, and that fund fees are not so large that the right-hand side of (15) is negative, there is a threshold value of wealth above which investors prefer to hold the market portfolio, or some subset of it, and below which they prefer to hold a fund. This threshold is defined by (13), (14) or (15), depending on the exact values of the parameters.

2.2 Closed-End Index Funds

If closed-end index funds were offered to investors, would they have to trade at a discount to be competitive? As mentioned in the Introduction, the shares of closed-end funds tend to be more volatile than the underlying assets. A possible explanation is the additional element of noise-trader risk, as discussed by Lee *et al*. (1991). Although the asset returns of all closed-end index funds are 100% correlated with one another, the same is not true of the returns to holders of their shares, because the discounts on individual funds may move differently. To the extent that discount movements are uncorrelated with one another, excess volatility can be reduced by holding a spread of funds. Nevertheless we assume, in line with the evidence of discount co-movement referred to earlier, that if excess volatility holds for individual closed-end funds, it also holds for a portfolio of them. The most important point is that we are *not* assuming that any of this discount risk is necessarily systematic (correlated with market returns) rather than idiosyncratic. It is also possible that closed-end funds may charge lower fees than open-end funds, because they do not have to administer net sales and redemptions.[3] Consequently we assume that closed-end funds charge proportional fees of h ($\leq f$). The return on closed-end funds is the market return, plus some element that reflects the initial discount (because this affects the ratio of dividend payments to price), plus a stochastic element related to the change in the discount.

Excess volatility implies that the variance of returns on a portfolio of closed-end fund shares is $\frac{k\sigma^2}{n}$, where $k > 1$, although in general it is possible that closed-end fund shares are less volatile than open-end fund shares, in which case $k < 1$. The expected rate of return on this portfolio may be written as $\delta(\mu - h)$, where δ is an index of the equilibrium ratio of the net asset value (NAV) to price, and is equal to one if they are equal (zero discount), less than one if the price exceeds the NAV, and greater than one if the NAV exceeds the price. As in Lee *et*

[3] In the UK market, closed-end funds do have lower total expense ratios than open-end funds, as discussed by Bleaney and Smith (2010), although there is less evidence of this for index funds.

al. (1991), the greater expected return as δ increases reflects the fact that the same dividends are received for a lower price. So the issue is whether δ has to be greater than one for closed-end funds to compete with open-end funds.[4]

An investor holding a portfolio of closed-end index funds chooses λ to maximize:

$$-\ln[-E(U)]_{CEF} = aW + a\delta(\mu - h)\lambda W - \frac{ka^2\lambda^2W^2\sigma^2}{2n} \tag{16}$$

and the first-order conditions imply that

$$\lambda W = \frac{n\delta(\mu - h)}{ak\sigma^2} = \frac{\delta(\mu - h)}{k\mu} X \tag{17}$$

We shall focus on the two main cases: where W is high enough that agents are not 100% invested in risky assets, whatever type of funds they hold, and where W is low enough that they are 100% invested in risky assets in either case. In the former case, the expected utility from holding an open-end fund is as given in (10):

---

[4] If the fund pays a dividend of d, then $\delta = 1 + \frac{d[(NAV/P) - 1]}{\mu - h}$, where P is the price and NAV the net asset value of the fund in equilibrium.

$$-\ln[-E(U)]_{OEF} = aW + \frac{n(\mu - f)^2}{2\sigma^2} \tag{18}$$

whereas for closed-end funds it is:

$$-\ln[-E(U)]_{CEF} = aW + \frac{n\delta^2(\mu - h)^2}{2k\sigma^2} \tag{19}$$

For closed-end funds to be competitive requires that (19) equals (18), or that:

$$\delta = \frac{(\mu - f)}{(\mu - h)}\sqrt{k} \tag{20}$$

This implies that the portfolio of closed-end funds has the same Sharpe ratio as an open-end fund. Since $k > 1$ by assumption, $\delta > 1$ unless h is sufficiently less than f that $\sqrt{k} \leq \frac{\mu - h}{\mu - f}$.

In the second case (no safe assets held), $\lambda = 1$ and we have:

$$-\ln[-E(U)]_{OEF} = a(1 + \mu - f)W - \frac{a^2\sigma^2W^2}{2n} \tag{21}$$

and

$$-\ln[-(EU)]_{CEF} = a[1+\delta(\mu-h)]W - \frac{a^2 k \sigma^2 W^2}{2n} \tag{22}$$

Equalization of expected utility requires that:

$$(\delta-1)(\mu-h) = \frac{a(k-1)\sigma^2 W}{2n} - (f-h) \tag{23}$$

which again shows that δ will exceed one unless the lower fees on closed-end funds ($f-h$) are sufficient to offset excess volatility ($k-1$). Thus in either case, if closed-end funds charge the same fees as open-end funds, portfolio excess volatility is a sufficient condition for them to trade at a discount. It does not matter whether the excess volatility derives entirely from idiosyncratic discount risk or whether the discount risk is to some degree systematic.

2.3 Large Investors and Closed-End Index Funds

In this model it is possible that closed-end funds can be held by large rather than small investors, unlike open-end funds. Suppose that discount risk is purely idiosyncratic, so that it can be diversified away. Large investors holding the market portfolio can then enhance their expected returns, for no extra risk, by adding some closed-end funds whenever $\delta(\mu-h) > \mu$. Indeed, comparing this condition with (20), large investors will require a lower discount than the small investors for whom (20) is relevant whenever $\sqrt{k} > \dfrac{\mu}{\mu - f}$.[5] If this condition holds, and closed-end funds are held by large investors, the discount will also respond to the systematic component of discount risk, which cannot be diversified away. This does not change the prediction of the model that it is discount risk in general, rather than its systematic component, that explains the existence of discounts.

2.4 Actively Managed Funds

The closed-end fund market, in particular, is dominated by actively managed funds rather than index funds. As noted by Bleaney and Smith (2010), a newly created closed-end index fund is unattractive to investors if it is expected to fall to a discount, and the ones that have existed in the UK market have tended to come into existence through a change of mandate of an actively managed fund that was already trading at a substantial discount. Actively managed funds charge higher fees than index funds for the supposed skill of the managers.

Suppose that actively managed fund j charges additional proportional fees of x_j, and investors expect returns of s_j above that of an index fund. Whether the variance of returns on actively managed funds exceeds that on index funds is something of an open question, but for simplicity we assume that the two variances are the same.

---

[5] It is also true that as the fee on closed-end funds (h) tends to zero, large investors will arbitrage the discount to zero in this case.

In the case of open-end funds, an actively managed fund will attract investors away from index funds if $s_j > x_j$. Unlike index funds, it may also attract large investors who would otherwise hold the market portfolio directly, if the additional return compensates for total fees ($s_j > x_j + f$). It could be argued that, if skill can be identified with any accuracy, and since managers are the monopolists of their own skill, they will raise x_j to the level of s_j, in which case actively managed funds, like index funds, would only be held by small investors.

In the case of closed-end funds, the discount can adjust for any difference between x_j and s_j so as to equalize returns on actively managed and index funds. Denoting η_j as the equilibrium ratio of NAV to price (equivalent to δ as used above for an index fund), the expected return on the actively managed fund is $\eta_j(\mu + s_j - x_j - h)$, compared with $\delta(\mu - h)$ for an index fund. To equalize returns therefore requires

$$\eta_j = \frac{\delta(\mu - h)}{(\mu + s_j - x_j - h)} \qquad (24)$$

Taking the case where wealth is high enough that agents hold a mixture of safe and risky assets, and substituting the solution for δ as given in equation (20), we obtain:

$$\eta_j = \frac{(\mu - f)}{(\mu + s_j - x_j - h)}\sqrt{k} \qquad (25)$$

Thus different actively managed closed-end funds will trade on different discounts depending on the value of ($s_j - x_j$), and only if $s_j = x_j$ will they trade on exactly the same discount as an index fund. Indeed if ($s_j - x_j$) is sufficiently positive, then η_j may be less than one, so that the fund trades at a premium, as happens in reality in a minority of cases.

Theories of the discount based on the pricing of managerial skill, such as those of Berk and Stanton (2007) and Ferguson and Leistikow (2001), essentially argue that the discount exists because on the typical seasoned fund $s_j < x_j$. In our model, any such discount effect is *additional* to the discount that would apply to an index fund because of excess volatility. New funds can be created and sold at net asset value if the perceived skill of the managers is overestimated, as discussed by Ferguson and Leistikow (2004).

3. Conclusions

The widespread belief that discount risk on closed-end funds has to be systematic in order to explain why the discount exists in equilibrium is a misconception that has arisen from the absence of an articulated model of portfolio allocation in the presence of collective investment funds. In the model developed here, the element of discount risk that is common to all closed-end funds is reflected in the pricing of closed-end funds in equilibrium, even if this risk is uncorrelated with market returns and is therefore unsystematic. If excess volatility of the returns to holders of a portfolio of closed-end fund shares (relative to returns to holders of open-end funds) is large enough to offset any cost advantage in operating a closed-end fund, closed-end funds will trade at a discount. The reason why the decomposition of excess volatility into its systematic and idiosyncratic elements is not relevant is that funds are held by small investors who do not hold them in conjunction with the market portfolio; the only circumstances in which large investors might be tempted to hold closed-end funds in conjunction with the market portfolio is when they already trade at a discount.

Any divergence between the skill of the manager of an actively managed fund and the price charged for it will be reflected in the discount. To the extent that charges on seasoned actively managed funds tend to exceed the extra returns implied by managerial skill, as suggested by some theories, actively managed funds will on average trade on higher discounts than index funds.

References

Berk, Jonathan B., and Richard Stanton (2007) Managerial Ability, Compensation, and the Closed-End Fund Discount, *Journal of Finance* 62 (2), 529-56.

Bleaney, Michael F., and R. Todd Smith (2010) Managerial Skill and Closed-End Fund Discounts, *Financial Markets and Portfolio Management* 24, 441-51.

Bodurtha, J., E. Kim and C. Lee (1993) Closed-End Country Funds and US Market Sentiment, *Review of Financial Studies* 8, 879-918.

Chen, N., R. Kan and M.H. Miller (1993) Are Discounts on Closed-End Funds a Sentiment Index? *Journal of Finance* 48, 795-800.

Dimson, Elroy and Carolina Minio-Kozerski (1999) Closed-End Funds: A Survey, *Financial Markets, Institutions and Instruments* 8, 1-41.

Elton, Edwin J., Martin J. Gruber and Jeffrey A. Busse (1998) Do Investors Care About Sentiment? *Journal of Business* 71 (4), 477-500.

Ferguson, Robert, and Dean Leistikow (2001) Valuing Active Managers, Fees, and Fund Discounts, *Financial Analysts Journal* 57, 52-62.

Ferguson, Robert, and Dean Leistikow (2004) Closed-End Fund Discounts and Expected Investment Performance, *The Financial Review* 39, 179-202.

Gemmill, Gordon and Dylan Thomas (2006) The Impact of Corporate Governance on Closed-end Funds, *European Financial Management* 12 (5), 725-46.

Lee, Charles M.C., Andrei Shleifer and Richard H. Thaler (1991) Investor Sentiment and the Closed-End Fund Puzzle, *Journal of Finance* 46, 75-109.

Pontiff, Jeffrey (1997) Excess Volatility and Closed-End Funds, *American Economic Review* 87 (1), 155-69.

Ross, Stephen A. (2002) Neoclassical Finance, Alternative Finance and the Closed-End Fund Puzzle, *European Financial Management* 8 (2), 129-37.

Sias, Richard W., Laura T. Starks and Seha M. Tiniç (2001), Is Noise Trader Risk Priced? *Journal of Financial Research* 24 (3), 311-29.

Appendix

Derivation of (13)

The case under consideration is where

$$\frac{\sigma\sqrt{2aT}}{\mu} X \leq \frac{\mu - f}{\mu} X < W < X \qquad (12)$$

So that the investor either holds a mixture of a fund and the safe asset, or the market portfolio with none of the safe asset. In the former case expected utility is given by (10):

$$-\ln[-E(U)]_{fund} = aW + \frac{n(\mu - f)^2}{2\sigma^2} \qquad (10)$$

In the latter case λ=1 and expected utility is:

$$-\ln[[-E(U)]_{MP} = aW(1+\mu) - anT - \frac{a^2W^2\sigma^2}{2n} \tag{A1}$$

These two will be equalized when

$$\frac{a^2\sigma^2}{2n}W^2 - a\mu W + anT + \frac{n(\mu - f)^2}{2\sigma^2} = 0 \tag{A2}$$

The solution to this equation is:

$$W = \frac{a\mu \pm \sqrt{a^2\mu^2 - a^2(\mu - f)^2 - 2a^3T\sigma^2}}{a^2\sigma^2 / n} \tag{A3}$$

which simplifies to

$$W = X - \frac{n}{a\sigma^2}\sqrt{\mu^2 - (\mu - f)^2 - 2aT\sigma^2} \tag{A4}$$

where we have ignored the solution with $W > X$ because it is outside the range under consideration. The term in the square root is positive because we are assuming that (11) holds. Since in the relevant range

$$\frac{-\partial\{\ln[-E(U)]\}_{MP}}{\partial W} = a[1 + \mu(1 - \frac{W}{X})] > \frac{-\partial\{\ln[-E(U)]\}_{fund}}{\partial W} = a \tag{A5}$$

it follows that investors choose the fund below the threshold value of wealth defined by (A4) and the market portfolio above it.

Derivation of (15)

In this case $W < \frac{\sigma\sqrt{2aT}}{\mu}X$. Since λ=1, from (8) expected utility for holders of a fund is:

$$-\ln[-E(U)]_{fund} = aW(1+\mu-f) - \frac{a^2W^2\sigma^2}{2n} \tag{A6}$$

and for those holding a selection of *m* shares from the market portfolio it is:

$$-\ln[-E(U)]_{MP} = aW(1+\mu) - amT - \frac{a^2\sigma^2W^2}{2m} \tag{A7}$$

Substituting $m = \frac{a\sigma W}{\sqrt{2aT}}$ from (7), (A7) reduces to:

$$\begin{aligned} -\ln[-E(U)]_{MP} &= aW(1+\mu) - \frac{a^2\sigma TW}{\sqrt{2aT}} - \frac{a\sigma W\sqrt{2aT}}{2} \\ &= aW(1+\mu) - a\sigma W\sqrt{2aT} \end{aligned} \tag{A8}$$

Equations (A6) and (A8) are equalized when either *W*=0 or

$$W = \frac{2n}{a^2\sigma^2}(a\sigma\sqrt{2aT} - af) = \frac{2n(\sigma\sqrt{2aT} - f)}{a\sigma^2} \tag{A9}$$

At this value of *W*,

$$\frac{\partial\{-ln[-E(U)]\}_{fund}}{\partial W} = a(1+\mu-f) - \frac{a^2\sigma^2W}{n} = a(1+\mu+f) - 2a\sigma\sqrt{2aT} \tag{A10}$$

and

$$\frac{\partial\{-ln[-E(U)]\}_{MP}}{\partial W} = a(1+\mu) - a\sigma\sqrt{2aT} \tag{A11}$$

which is greater so long as the threshold value of W given by (A9) is positive (i.e. $f < \sigma\sqrt{2aT}$). Consequently the fund is chosen below the value of wealth given by (A9) and a portfolio of m risky assets above it.