Measuring the Standard of Living–An Axiomatic Approach

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Abstract

The standard of living of an agent is viewed as her capability of achieving various functionings (Sen (1985, 1987)). An agent is thus characterized by her capability set that consists of different functioning vectors. The task of measuring the standard of living of an agent formally is therefore to rank different capability sets. This paper explores the problem of ranking capability sets in terms of the standard of living offered to an agent. For this purpose, I consider capability sets that are non-degenerate, compact, comprehensive and convex subsets of the n-dimensional real space, propose several intuitively plausible properties for the ranking and give characterizations of several rules that have some interesting features. Alternative interpretations of some of the results in the paper, for example, in terms of social welfare functions and of optimality of bargaining solutions in bargaining theory, are also discussed.
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1 Introduction

Comparisons of the standard of living have traditionally taken either the opulence or utility approach. In the opulence view, an agent's living standard is judged by her commanding of wealth and income. In the utility view of the standard of living, an agent's standard of living is based on the utility generated by her consumption of commodities. Both views have come under critical scrutiny from Sen (1985, 1987) recently. Sen argues that neither comes close to the issue of measuring the standard of living. Instead, he proposes a functioning-capability approach. In this paper, I use Sen's view to approach the problem of measuring the standard of living.

With the functioning-capability approach, an agent is characterised by the functioning of the agent and the extent of various functionings. A *functioning* is what the agent "succeeds in *doing* with the commodities and characteristics at his or her command" (Sen (1985), pp. 10). It is "different both from (1) having goods (and the corresponding characteristics), to which it is posterior, and (2) having utility (in the form of happiness resulting from that functioning), to which it is, in an important way, prior" (Sen (1985), pp. 11). For example, life expectancy, whether well-clothed, adult literacy rate, child mortality, attending social activities, or well-nourished, are some examples of an agent's functiongings. The extent of various functionings is referred to as the *capability set* and reflects the various combinations of functionings the agent can achieve when the resource allocation is given. The living standard is then viewed as the set of available capabilities of the agent to function.

Given this characterization of an agent and given that functionings are closely connected with actual living, it may be argued that in assessing the living standard, one should concentrate on functionings and pay little, if any, attention to capabilities. This way of viewing the standard of living may initially sound appealing, it does not, however, survive a closer examination. To see why capabilities may play a direct role along with functionings in assessing the living standard, let us consider the following situation. Suppose an agent's functionings are represented by various life styles a, b, c, d, and e. Suppose further that she can choose one from these five different life styles and she chooses c. Consider now that the life styles a, b, d and e become unavailable to the agent, but she can still choose c. It can be argued that the agent's standard of living has been reduced due to the curtailment of the freedom of choice offered to the agent, even though the life style c is what the agent would choose.[1] This suggests that in the assessment of the standard of living of an agent, one should look for both functionings and capabilities. Recall that the capability set of an agent consists of various combinations of functionings the agent can achieve when the resource allocation is given. It is therefore clear that the capability set of an agent summarizes the relevant information about both functionings and capabilities for the agent. The exercise of assessing the agent's standard of living under different situations can thus be viewed as ranking alternative capability sets arising from these different situations. I will adopt this framework to address the issue of measuring the standard of living.

According to Sen (1987), in evaluating the standard of living for an agent, it is useful to make a distinction between two approaches, viz., the *standard-evaluation* and the *self-evaluation*. The standard-evaluation focuses on some general valuation function that reflects widely shared standards in the society. It relies on some uniformity of judgements on the respective importance of different functionings and places an agent's living conditions in a general ranking in terms of some social standards. The self-evaluation, on the other hand, concentrates on the relevant valuation function that of the agent whose living standard is being assessed. It is concerned with the agent's assessment of her own living standard and tells us what the agent judges to be her living standard in comparisons with other situations. As such, the self-evaluation places an important emphasis on the agent's actual living chosen by her, whereas the standard-evaluation places an important emphasis on the agent's living as perceived by the society according to some widely shared standards among the members of the society (henceforth, it will be called the agent's *perceived living*). Depending on the context, each approach has its own merit. In this paper, I will try to address both approaches: Sections 3 and 6 deals with both the self-evaluation and the standard-evaluation, Section 4 deals with the standard-evaluation only and the self-evaluation is left in Section 5.

The plan of the paper is as follows. In Section 2, I lay down the basic notation and definitions. Section 3 gives an axiomatic characterization of the superset-dominance-based quasi ordering for ranking capability sets in terms of the standard of living. The

[1] The issue of measuring freedom of choice has been extensively examined in the recent literature. See, among others, Arrow (1995), Bossert, Pattanaik and Xu (1994), Foster (1992), Gravel (1994, 1998), Jones and Sugden (1982), Pattanaik and Xu (1990, 1998, 1999), Puppe (1996), Sen (1991, 1992), Steiner (1983), Sugden (1998), van Hees (1998) and Xu (1999).

analysis of Section 3 can be regarded as including both the standard-evaluation and the self-evaluation. In Section 4, I extend the quasi ordering analysis to a complete ordering of capability sets with the interpretation of evaluating the standard of living in terms of the standard-evaluation. Section 5 is concerned with the analysis of the standard of living with the interpretation of the agent's self-evaluation. While analyses in Sections 3, 4, and 5 can be regarded as extremes, in Section 6, I examine a class of quasi orderings that lie in between the superset-dominance rule and an ordering. Section 7 contains some concluding remarks.

2 The Basic Notation and Definitions

Let $\Re^n_+$ be the non-negative orthant of the n-dimensional real space. The points in $\Re^n_+$ will be denoted by $x, y, z, a, b, \cdots$ and will be called alternatives. For all $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \mathbb{R}^n_+$, define $x > y$ when $x_i \geq y_i$ for all $i = 1, \cdots, n$ and $x_j > y_j$ for some $j \in \{1, \cdots, n\}$. The alternatives are to be interpreted as functioning vectors *à la* Sen (1985, 1987).

At any given point of time, the set of all alternatives that may be available to the agent is a subset of $\mathbb{R}^n_+$. Such a set will be called the agent's *capability set*. I will use A, B, C etc. to denote the capability sets.

My concern in this paper is to rank different capability sets in terms of the standard of living that they offer to the agent. What are the capability sets that one would like to rank in the current context? In this paper, I confine my attention to capability sets that are

(2.1) non-degenerate: a capability set $A \subseteq \mathbb{R}^n_+$ is non-degenerate iff there exists $x \in A$ such that $x_i > 0$ for all $i = 1, \cdots, n$,

(2.2) compact: a capability set $A \subseteq \mathbb{R}^n_+$ is compact iff A is closed and bounded,

(2.3) comprehensive: a capability set $A \subseteq \mathbb{R}^n_+$ is comprehensive iff, for all $x, y \in \mathbb{R}^n_+$, if $x_i \geq y_i$ for $i = 1, \cdots, n$ and $x \in A$, then $y \in A$,

(2.4) convex: a capability set $A \subseteq \mathbb{R}^n_+$ is convex iff, for all $x, y \in A$ and for all $\alpha \in [0, 1]$, $\alpha x + (1 - \alpha)y \in A$ holds.

Let K be the set of all capability sets that are non-degenerate, compact, comprehensive and convex.[2]

Let Π be the set of all *permutations* of $\{1, \cdots, n\}$. Elements of Π will be denoted by π, π', etc. For all $A \in K$ and for all $\pi \in \Pi$, let $\pi(A) =: \{x \in \mathbb{R}^N_+ | x_i = y_{\pi(i)}, \exists y \in A\}$. For all $A, B \in K$, if $A = \pi(B)$ for some $\pi \in \Pi$, I then say that A and B are *symmetric*. For example, the sets $\{x \in \mathbb{R}^n_+ | x_1/a_1 + x_2/a_2 + x_3 + \cdots + x_n \leq 1\}$ and $\{x \in \mathbb{R}^n_+ | x_1/a_2 + x_2/a_1 + x_3 + \cdots + x_n \leq 1\}$, where $a_1 > 0$ and $a_2 > 0$, are symmetric, and the sets $\{x \in \mathbb{R}^n_+ | x_1 \leq a_1, x_2 \leq a_2, x_3 \leq 1, \cdots, x_n \leq 1\}$ and $\{x \in \mathbb{R}^n_+ | x_1 \leq a_2, x_2 \leq a_1, x_3 \leq 1, \cdots, x_n \leq 1\}$, where $a_1 > 0$ and $a_2 > 0$, are symmetric. For all $A \in K$, all $i \in \{1, \cdots, n\}$ and all $\alpha^i > 0$, let $\alpha^i A =: \{x \in \mathbb{R}^n_+ | x_i = \alpha^i y_i$ and $x_j = y_j$ for all $j \neq i, \forall y \in A\}$. Finally, for all $A, B \in K$, if for all $b \in B$, there exists $a \in A$ such that $a > b$, then I say that B *lies entirely in* A. Note that, if B lies entirely in A, necessarily, B is a subset of A.

Let $\succeq$ be a binary relation over K that satisfies *reflexivity:* [for all $A \in K, A \succeq A$] and *transitivity:* [for all $A, B, C \in K$, if $A \succeq B$ and $B \succeq C$ then $A \succeq C$]. $\succeq$ is thus a *quasi ordering.* When $\succeq$ satisfies *completeness:* [for all $A, B \in K, A \succeq B$ or $B \succeq A$], reflexivity and transitivity, $\succeq$ is called an *ordering.* The intended interpretation of $\succeq$ is the following: for all $A, B \in K, [A \succeq B]$ will be interpreted as "the standard of living offered by A is at least as much as the standard of living offered by B". $\succ$ and $\sim$, respectively, are the asymmetric and symmetric part of $\succeq$. I will use "$\neg$" to denote the logical negation of a statement.

3 Superset-Subset Dominance

In this section, I present a quasi ordering approach that uses the superset-dominance relation to rank capability sets in terms of the standard of living. First, let me define the superset-dominance-based rule formally.

Definition 3.1. $\succeq$ over K is called the *superset-dominance-based rule* iff,

for all $A, B \in K, A \succeq B \Leftrightarrow B \subseteq A$. $\cdots$(3.1)

Clearly, the superset-dominance-based rule is reflexive and transitive, and thus is a

[2]These restrictions on the capability sets in K can be derived from some basic assumptions in a resource allocation model, see Gotoh, Suzumura and Yoshihara (1999).

quasi ordering. I now introduce two axioms that will be used for the characterization of this ranking rule.

Definition 3.2. $\succeq$ over K satisfies

(3.2.1) **Monotonicity** iff, for all $A, B \in K$, if $B \subseteq A$ and $A \neq B$, then $A \succ B$.

(3.2.2) **Non-Dominance** iff, for all $A, B \in K$, if there exist $a \in A - B$ and $b \in B$ such that $a > b$, then $\neg(B \succ A)$.

The property of Monotonicity is simple and easy to explain. It requires that if the capability set B is a proper subset of the capability set A, then A offers a higher level of the standard of living than B. The intuition of Monotonicity can be explained by appealing to the agent's actual living or her perceived living and the agent's capabilities. The functioning-capability approach to the standard of living focuses on two essential parts of an agent's living conditions, namely, her actual living in the self-evaluation exercise or her perceived living in the case of standard-evaluation and her capabilities to achieve either her actual living or her perceived living. The notion of an agent's capabilities is a freedom type notion and reflects the degree of opportunities offered to the agent. In the case that B is a proper subset of A, A dominates B in both these two essential parts of the agent's living conditions. Thus, it makes sense to say that in such a case, A offers a higher standard of living than B. Non-Dominance requires that for two capability sets A and B, if A contains some functioning vectors that are outside of B and that vectorically dominate some functioning vectors in B, then B cannot be ranked (strictly) higher than A in terms of the standard of living. In other words, if some functioning vector a under A cannot be achieved under B, then the living standard offered by B cannot be higher than the living standard offered by A. The intuition of Non-Dominance is this: the functioning vector a, which is achievable under A but not under B, has the potentiality of becoming the agent's actual living or her perceived living; in such an event, it may be argued that B cannot offer a higher standard of living than A.

Theorem 3.3. Suppose $\succeq$ over K is a quasi ordering. Then, $\succeq$ satisfies Monotonicity and Non-Dominance if and only if it is the superset-dominance-based rule.

Proof. It can be checked that the binary relation $\succeq$ over K induced by the superset-dominance-based rule is a quasi ordering that satisfies Monotonicity and Non-Dominance.

Therefore, I have only to show that if $\succeq$ over K is a quasi ordering that satisfies Monotonicity and Non-Dominance, then it is the superset-dominance-based rule, i.e., (3.1) holds.

Let $\succeq$ over K be a quasi ordering and satisfy Monotonicity and Non-Dominance. To show that (3.1) holds, I have to show that for all $A, B \in K$, (i) if $A = B$, then $A \sim B$, (ii) if $B \subseteq A$ and $A \neq B$, then $A \succ B$, and (iii) if A is not a subset of B and B is not a subset of A, then $\neg(A \succeq B)$ and $\neg(B \succeq A)$. In what follows, I proceed to show the above three cases.

Case (i) follows from the reflexivity of $\succeq$ easily, while case (ii) follows from Monotonicity. To show case (iii), let $A, B \in K$ be such that neither A is a subset of B, nor B is a subset of A. Since K is the set of all capability sets that are non-degenerate, compact, comprehensive and convex, the following must be true:

there exist $a, a' \in A$ and $b, b' \in B$ such that $a > b$ and $b' > a'$, $\cdots$(3.2)

there exists $C \in K$ such that $A \subseteq C$, $C \neq A$, C is not a subset of B and B is not a subset of C, $\cdots$(3.3)

and

there exist $c \in C$ and $x \in B$ such that $x > c$. $\cdots$(3.4)

From (3.2), by Non-Dominance, it follows that $\neg(A \succ B)$ and $\neg(B \succ A)$. Therefore, $[\neg(A \succeq B)$ and $\neg(B \succeq A)]$ or $[A \succeq B$ and $B \succeq A]$ follow from the definition of the asymmetric part, $\succ$, of $\succeq$. I now need to show that it cannot be true that $[A \succeq B$ and $B \succeq A]$. Suppose to the contrary that $[A \succeq B$ and $B \succeq A]$, that is, $A \sim B$. From (3.3), by Monotonicity, it follows that $C \succ A$. Then, by the transitivity of $\succeq$, it follows that $C \succ B$. However, from (3.4), it must be true that $\neg(C \succ B)$, a logical contradiction. Therefore, it cannot be true that $[A \succeq B$ and $B \succeq A]$. Thus, $[\neg(A \succeq B)$ and $\neg(B \succeq A)]$ must hold.

Note that cases (i), (ii) and (iii) exhaust all possibilities. Therefore, Theorem 3.3 is proved. ■

The superset-dominance-based rule defined in (3.1) was first suggested by Sen (1985,1987) in the same context of ranking capability sets in terms of the standard of living. However, he did not provide an axiomatic characterization. The characterization result provided here suggests that the superset-dominance-based rule has two essential parts: Monotonicity, which reflects that more opportunities promote the agent's standard of living, and Non-Dominance, which captures the idea that a capability set B cannot offer a higher level of the standard of living than a capability set A if some functioning vector in A cannot be achieved under B.

I now discuss the issue of redundancy of the axioms used in Theorem 3.3. The following examples, Examples 3.4 and 3.5, establish the independence of the axioms used in Theorem 3.3.

Example 3.4. Define the binary relation $\succeq_1$ over K as follows: for all $A, B \in K, A \sim_1 B$. Clearly, $\succeq_1$ is an ordering (thus a quasi ordering) that satisfies Non-Dominance, but violates Monotonicity.

Example 3.5. Define the binary relation $\succeq_2$ over K as follows: for all $A, B \in K, A \succeq_2 B \Leftrightarrow \text{vol}(A) \geq \text{vol}(B)$, where $\text{vol}(\cdot)$ denotes the volume of a capability set in K. Clearly, $\succeq_2$ is a (quasi) ordering that satisfies Monotonicity, but violates Non-Dominance.

4 The Elementary Standard-Evaluation

The superset-dominance-based rule defined in (3.1) may be very appealing. However, it leaves "too many" capability sets unranked in terms of the standard of living. Yet, one can argue that some of these unranked capability sets can be ranked in terms of the standard of living. For example, consider the capability sets A and B illustrated in Figure 1.

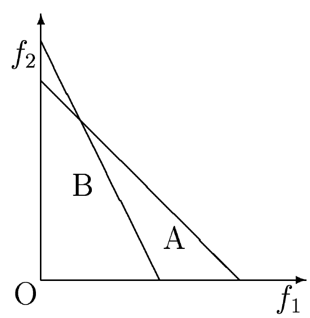

Figure 1

Recall that Sen emphasizes the importance of freedom of choice in determining the standard of living and that freedom of choice means having a capability set that offers plenty of opportunity for choosing. It therefore has some reason to believe that the capability set A offers a higher level of standard of living than the capability set B.

Faced with the problem of incompletenss of the binary relation defined in (3.1), in this section as well as in Section 5, I require that the binary relation $\succeq$ over K be an ordering. In Section 6, I examine a class of quasi orderings that are in between the super-set dominance rule and an ordering. To start with, in this section, I consider the exercise of the standard-evaluation. I try to formulate some normative properties imposed on the binary relation $\succeq$. These normative properties have some intuitive appeals in the current context and some of which appeal directly to the notion of commonly shared values in the society in evaluating the standard of living. I show that they, together, characterize the rule defined below:

<u>Definition 4.1.</u> $\succeq$ over K is called the <u>elementary standard-evaluation</u> rule if and only if

for all $A, B \in K$, $A \succeq B \Leftrightarrow \max_{x \in A}(x_1 \cdot ... \cdot x_n) \geq \max_{y \in B}(y_1 \cdots y_n)$. $\cdots$ (4.1)

The elementary standard-evaluation rule ranks capability sets in terms of the standard of living according to the maximum value of $(x_1 \cdots x_n)$ in a capability set. For the characterization purpose, I introduce four axioms that put restrictions on $\succeq$.

<u>Definition 4.2.</u> $\succeq$ over K satisfies

(4.2.1) **Symmetry** iff, for all $A, B, C \in K$, if A and B are symmetric, then $A \succeq C \Leftrightarrow B \succeq C$.

(4.2.2) **Weak Monotonicity** iff, for all $A, B \in K$, if $B \subseteq A$ then $A \succeq B$, and if B lies entirely in A, then $A \succ B$.

(4.2.3) **Invariance of Scaling Effects** iff, for all $A, B \in K$ and for all $\alpha^i > 0$, $A \succeq B \Leftrightarrow \alpha^i A \succeq \alpha^i B$.

(4.2.4) **Weak Indifference** iff, for all $A \in K$, there exists $a^* \in A$ such that $B = \{x \in \mathbb{R}^n_+ | x_i \leq a^*_i, i = 1, \cdots, n\} \in K$ and $B \sim A$.

The property of Symmetry requires that, if two capability sets A and B are symmetric, then, the ranking of A and a capability set C in terms of the standard of living corresponds to the ranking of B and C. It essentially requires that there be no *discrimination* among different functionings and these functionings be treated equally in evaluating the standard of living offered to the agent by a capability set. This non-discriminatory treatment of different functionings can be regarded as a reflection of some commonly shared social standards in valuing different functionings and thus reflects the idea of the standard-evaluation. If different functionings are treated equally, then the life style achieved by a functioning vector a can be regarded as equivalent to any functioning vector b which is symmetric to a. Thus, in terms of the two essential ingredients of the standard of living, namely, the agent's perceived living and her capabilities to achieve different functionings, symmetric capability sets should be ranked exactly the same in terms of the standard of living.

Weak Monotonicity is a weaker property than Monotonicity introduced in Section 3. It requires that if B is subset of A, then the standard of living offered by A is at least as much as B; further, if B lies entirely in A, then A offers more standard of living than B. It reflects that the agent is not averse to more opportunities.

The property of Invariance of Scaling Effects requires that, by re-scaling the unit of measurement for any functioning while maintaining others unchanged, the relative standard of living offered by two capability sets A and B should be the same: if A offers a higher level of the standard of living than B before rescaling, then A should be offer a higher level of the standard of living than B after re-scaling and *vice versa*. The idea is that the choice of "measuring rod" for a functioning is not relevant for ranking capability sets in terms of the standard of living.

Finally, Weak Indifference requires that for all capability set $A \in K$, there is a functioning vector a^* in A such that the standards of living offered by A and the capability set $A^H = \{x \in \mathbb{R}^n_+ | x_i \leq a^*_i, i = 1, \cdots, n\}$ are the same. The idea of Weak Indifference is the following. The living standard offered by a capability set A can always be achieved by suitably choosing a functioning vector a^* in A and forming a capability set that is the comprehensive hull of a^* in $\mathbb{R}^n_+$. Therefore, it reflects the idea that in certain cases, the loss of opportunities as reflected in the change of the capability set A to the capability set A^H does not reduce the standard of living offered by A provided that the functioning vector a^* in A is properly chosen and is also available in A^H. The intuition of Weak Indifference fits well with Sen's argument that special attention should be given to the agent's actual living or her perceived living.

I now show that the four axioms introduced above, together, characterize the elementary standard-evaluation.

Theorem 4.3. Suppose $\succeq$ over K is an ordering. Then, $\succeq$ on K satisfies Symmetry, Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference if and only if it is the elementary standard-evaluation rule.

To prove Theorem 4.3, the following two lemmas, Lemmas 4.4 and 4.5, will be useful. First, define $K_1 = \{A \in K | A = \{x \in \mathbb{R}^n_+ | \sum_{i=1}^n (x_i/a_i) \leq 1\}$, for all $a \in \mathbb{R}^n_+$ such that $a_i > 0, i = 1, \cdots\}$ and $K_2 = \{A \in K | A = \{x \in \mathbb{R}^n_+ | x_i \leq a_i\}$, for all $a \in \mathbb{R}^n_+$ with $a_i > 0, i = 1, \cdots, n\}$. In the following two lemmas, Lemmas 4.4 and 4.5, the properties, Symmetry, Weak Monotonicity and Invariance of Scaling Effects, of $\succeq$ over K_1 and K_2, respectively, should be re-defined accordingly.

Lemma 4.4. Suppose $\succeq$ is an ordering. $\succeq$ on K_2 satisfies Symmetry, Weak Monotonicity and Invariance of Scaling Effects if and only if for all $A, B \in K_2$, $A \succeq B \Leftrightarrow \max_{x \in A}(x_1 \cdot ... \cdot x_n) \geq \max_{y \in B}(y_1 \cdot ... \cdot y_n)$.

Proof. It can be checked that if for all $A, B \in K_2$, $A \succeq B \Leftrightarrow \max_{x \in A}(x_1 \cdot ... \cdot x_n) \geq \max_{y \in B}(y_1 \cdot ... \cdot y_n)$, then $\succeq$ satisfies Symmetry, Weak Monotonicity and Invariance of Scaling Effects. I have only to show that if $\succeq$ on K_2 satisfies Symmetry, Weak Monotonicity and Invariance of Scaling Effects, then for all $A, B \in K_2$, $A \succeq B \Leftrightarrow \max_{x \in A}(x_1 \cdot ... \cdot x_n) \geq \max_{y \in B}(y_1 \cdot ... \cdot y_n)$.

Let $\succeq$ on K_2 satisfy Symmetry, Weak Monotonicity and Invariance of Scaling Effects. Let $A = \{x \in \mathbb{R}^n_+ | x_i \leq a_i, a_i > 0, i = 1, \cdots, n\}, B = \{y \in \mathbb{R}^n_+ | y_i \leq b_i, b_i > 0, i = 1, \cdots, n\} \in K_2$. Note that $\max_{x \in A}(x_1 \cdot ... \cdot x_n) = a_1 \cdot ... \cdot a_n$ and $\max_{y \in A}(y_1 \cdot ... \cdot y_n) = b_1 \cdot ... \cdot b_n$.

First, I show that

if $a_1 \cdot ... \cdot a_n = b_1 \cdot ... \cdot b_n$, then $A \sim B$. $\cdots(4.2)$

Let $A^* = \{x \in \mathbb{R}^n_+ | x_i \leq 1, i = 1, \cdots, n\}$. Consider the sets $C_1, C_2, \cdots, C_n \in K_2$ defined as follows:

$C_1 = \{x \in \mathbb{R}^n_+ | x_1 \leq c_1, x_2 \leq 1, x_3 \leq 1, \cdots, x_n \leq 1\}$;

$C_2 = \{x \in \mathbb{R}^n_+ | x_1 \leq a_1, x_2 \leq c_2, x_3 \leq 1, \cdots, x_n \leq 1\}$;

$\vdots$

$C_k = \{x \in \mathbb{R}^n_+ | x_1 \le a_1, x_2 \le a_2, \cdots, x_{k-1} \le a_{k-1}, x_k \le c_k, x_{k+1} \le, \cdots, x_n \le 1\}$;

$\cdots$

$C_n = A$,

where $c_1 = a_1 a_2 \cdots a_n, c_2 = a_2 a_3 \cdots a_n, \cdots, c_k = a_k a_{k+1} \cdots a_n,\ c_{n-1} = a_{n-1} a_n$, and $c_n = a_n$. By construction, we have $c_1 = a_1 c_2 = \cdots = a_1 \cdots a_{k-1} a_k = \cdots = a_1 a_2 \cdots a_n$. In what follows, I show that $C_1 \sim C_2 \sim \cdots \sim C_n = A$.

Consider C_1 and C_2 first. Let $\alpha^1 = c_1$ (recall that α^1 re-scales the unit of measurement for functioning 1). By Invariance of Scaling Effects, it follows that

$$C_1 \succeq C_2 \text{ iff } \alpha^1 C_1 \succeq \alpha^1 C_2. \qquad \cdots(4.3)$$

Note that $\alpha^1 C_1 = A^*$. Therefore, the following holds:

$$C_1 \succeq C_2 \text{ iff } A^* \succeq \alpha^1 C_2 = \{x \in \mathbb{R}^n_+ | x_1 \le a_1/c_1, x_2 \le c_2, x_3 \le 1, \cdots, x_n \le 1\}. \quad \cdots(4.4)$$

Now, consider $\alpha'^1 = a_1$ and $\alpha'^2 = c_2$. By Invariance of Scaling Effects, it follows that

$$C_1 \succeq C_2 \text{ iff } \alpha'^1 C_1 \succeq \alpha'^1 C_2. \qquad \cdots(4.5)$$

By another application of Invariance of Scaling Effects, it follows that

$$\alpha'^2(\alpha'^1 C_1) \succeq \alpha'^2(\alpha'^1 C_2) \text{ iff } \alpha^1 C_1 \succeq \alpha^1 C_2. \qquad \cdots(4.6)$$

From (4.5) and (4.6), the following is true:

$$C_1 \succeq C_2 \text{ iff } \alpha'^2(\alpha'^1 C_1) \succeq \alpha'^2(\alpha'^1 C_2). \qquad \cdots(4.7)$$

Note that $\alpha'^2(\alpha'^1 C_2) = A^*$. Thus, the following can be obtained:

$$C_1 \succeq C_2 \text{ iff } \alpha'^2(\alpha'^1 C_1) = \{x \in \mathbb{R}^n_+ | x_1 \le c_1/a_1, x_2 \le 1/c_2, x_3 \le 1, \cdots, x_n \le 1\} \succeq A^*. \quad \cdots(4.8)$$

Note that $a_1/c_1 = 1/c_2$. Therefore, (4.4) can be rewritten as:

$$C_1 \succeq C_2 \text{ iff } A^* \succeq \alpha^1 C_2 = \{x \in \mathbb{R}^n_+ | x_1 \le (1/c_2), x_2 \le (c_1/a_1), x_3 \le 1, \cdots, x_n \le 1\}. \quad \cdots(4.9)$$

Note that $\alpha^1 C_2$ and $\alpha'^2(\alpha'^1 C_1)$ are symmetric. Then, Symmetry implies

$\alpha'^2(\alpha'^1 C_1) = \{x \in \mathbb{R}^n_+ | x_1 \leq c_1/a_1, x_2 \leq 1/c_2, x_3 \leq 1, \cdots, x_n \leq 1\} \sim \alpha^1 C_2 = \{x \in \mathbb{R}^n_+ | x_1 \leq 1/c_2, x_2 \leq c_1/a_1, x_3 \leq 1, \cdots, x_n \leq 1\}$. $\cdots(4.10)$

By the transitivity of $\succeq$, from (4.8), (4.9) and (4.10), it then follows that $C_1 \sim C_2$. Consider now C_2 and C_3. Recall that

$C_2 = \{x \in \mathbb{R}^n_+ | x_1 \leq a_1, x_2 \leq c_2, x_3 \leq 1, \cdots, x_n \leq 1\}$,

and

$C_3 = \{x \in \mathbb{R}^n_+ | x_1 \leq a_1, x_2 \leq a_2, x_3 \leq c_3, x_4 \leq 1, \cdots, x_n \leq 1\}$,

where $c_2 = a_2 \cdots a_n$ and $c_3 = a_3 \cdots a_n$. By Invariance of Scaling Effects, it follows that (consider $\alpha'^1 = a_1$)

$C_2 \succeq C_3$ iff $\{x \in \mathbb{R}^n_+ | x_1 \leq 1, x_2 \leq c_2, x_3 \leq 1, \cdots, x_n \leq 1\} \succeq \{x \in \mathbb{R}^n_+ | x_1 \leq 1, x_2 \leq a_2, x_3 \leq c_3, x_4 \leq 1, \cdots, x_n \leq 1\}$. $\cdots(4.11)$

Noting that $\{x \in \mathbb{R}^n_+ | x_1 \leq 1, x_2 \leq c_2, x_3 \leq 1, \cdots, x_n \leq 1\}$ and $\{x \in \mathbb{R}^n_+ | x_1 \leq c_2, x_2 \leq 1, x_3 \leq 1, \cdots, x_n \leq 1\}$ are symmetric, and $\{x \in \mathbb{R}^n_+ | x_1 \leq 1, x_2 \leq a_2, x_3 \leq c_3, x_4 \leq 1, \cdots, x_n \leq 1\}$ and $\{x \in \mathbb{R}^n_+ | x_1 \leq a_2, x_2 \leq c_3, x_3 \leq 1, x_4 \leq 1, \cdots, x_n \leq 1\}$ are symmetric, by Symmetry, we have

$\{x \in \mathbb{R}^n_+ | x_1 \leq 1, x_2 \leq c_2, x_3 \leq 1, \cdots, x_n \leq 1\} \sim \{x \in \mathbb{R}^n_+ | x_1 \leq c_2, x_2 \leq 1, x_3 \leq 1, \cdots, x_n \leq 1\}$,

$\cdots(4.12)$

and

$\{x \in \mathbb{R}^n_+ | x_1 \leq 1, x_2 \leq a_2, x_3 \leq c_3, x_4 \leq 1, \cdots, x_n \leq 1\} \sim \{x \in \mathbb{R}^n_+ | x_1 \leq a_2, x_2 \leq c_3, x_3 \leq 1, x_4 \leq 1, \cdots, x_n \leq 1\}$. $\cdots(4.13)$

Therefore, by the transitivity of $\succeq$, from (4.11), (4.12) and (4.13), the following holds

$C_2 \succeq C_3$ iff $\{x \in \mathbb{R}^n_+ | x_1 \leq c_2, x_2 \leq 1, x_3 \leq 1, \cdots, x_n \leq 1\} \succeq \{x \in \mathbb{R}^n_+ | x_1 \leq a_2, x_2 \leq c_3, x_3 \leq 1, x_4 \leq 1, \cdots, x_n \leq 1\}$. $\cdots(4.14)$.

Note that $c_2 = a_2 c_3$. Then, by following the same procedures as in the proof for $C_1 \sim C_2$, from (4.14), $C_2 \sim C_3$ can be obtained.

Similarly, $C_3 \sim C_4, \cdots, C_{n-1} \sim C_n$ can be proved. Then, the transitivity of $\succeq$ implies $C_1 \sim C_n = A$. Noting that $c_1 = a_1 \cdots a_n = b_1 \cdots b_n$, $C_1 \sim B$ can be established similarly. By the transitivity of $\succeq$, it then follows that $A \sim B$. Thus, (4.2) holds.

Next, I show that

$$\text{if } a_1 \cdots a_n > b_1 \cdots b_n, \text{ then } A \succ B. \qquad \cdots(4.15)$$

Consider $D, E \in K_2$:

$D = \{x \in \mathbb{R}^n_+ | x_i \leq (a_1 a_2 \cdots a_n)^{(1/n)}, i = 1, \cdots, n\}$,

and

$E = \{x \in \mathbb{R}^n_+ | x_1 \leq (b_1 b_2 \cdots b_n)^{(1/n)}, i = 1, \cdots, n\}$.

From (4.2), $A \sim D$ and $B \sim E$. By Weak Monotonicity, noting that E lies entirely inside D, $D \succ E$ holds. Hence, the transitivity of $\succeq$ implies $A \succ B$. That is, (4.15) holds.

Given (4.2), (4.15) completes the proof. ■

<u>Lemma 4.5.</u> Suppose $\succeq$ is an ordering. $\succeq$ on K_1 satisfies Symmetry, Weak Monotonicity and Invariance of Scaling Effects if and only if for all $A, B \in K_1$, $A \succeq B \Leftrightarrow \max_{x \in A}(x_1 \cdots x_n) \geq \max_{y \in B}(y_1 \cdots y_n)$.

<u>Proof.</u> The proof of Lemma 4.5 is similar to that of Lemma 4.4 and is therefore omitted. ■

Proof of Theorem 4.3. It can be checked that if (4.1) holds, then $\succeq$ on K satisfies Symmetry, Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference. Therefore, I have only to show that if $\succeq$ on K satisfies Symmetry, Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference, then it is the elementary standard-evaluation rule, that is, (4.1) holds.

Let $\succeq$ on K satisfy Symmetry, Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference. I first show that

$$\text{for all } A, B \in K_1 \cup K_2, \max_{x \in A}(x_1 \cdots x_n) = \max_{y \in B}(y_1 \cdots y_n) \Rightarrow A \sim B. \qquad \cdots(4.16)$$

By Lemmas 4.4 and 4.5, I need only to show that for all $A \in K_1$ and all $B = K_2$, if $\max_{x \in A}(x_1 \cdots x_n) = \max_{y \in B}(y_1 \cdots y_n)$, then $A \sim B$. Let $A = \{x \in \mathbb{R}^n_+ | \sum_{i=1}^n x_i/a_i \leq 1\}$ where $a_i > 0, i = 1, \cdots, n$ and $B = \{x \in \mathbb{R}^n_+ | x_i \leq b_i, b_i > 0, i = 1, \cdots, n\}$. Note that if $\max_{x \in A}(x_1 \cdots x_n) = \max_{y \in B}(y_1 \cdots y_n)$, then $b_i = a_i/n, i = 1, \cdots, n$. Let $A^* = \{x \in \mathbb{R}^n_+ | \sum_{i=1}^n x_i \leq 1\}$ and let $A^{**} = \{x \in \mathbb{R}^n_+ | x_i \leq 1/n, i = 1, \cdots, n\}$. Then, by the repeated use of Invariance of Scaling Effects, it must be true that

$A \succeq B \Leftrightarrow A^* \succeq A^{**}$. Therefore, I have only to show that $A^* \sim A^{**}$. Let $a^* \in A^*$ be such that $a^* \neq (1/n, 1/n, \cdots, 1/n)$. Then, it can be checked easily that $A^{**} \succ \{x \in \mathbb{R}^n_+ | x_i \leq a^*_i, i = 1, \cdots, n\}$. If $A^* \succ A^{**}$, then, by the transitivity of $\succeq$, it follows that $A^* \succ \{x \in \mathbb{R}^n_+ | x_i \leq a^*_i, i = 1, \cdots, n\}$. Note that a^* is arbitrary. Together with $A^* \succ A^{**}$, it then follows that $A^* \succ \{x \in \mathbb{R}^n_+ | x_i \leq a_i, i = 1, \cdots, n\}$ where $a \in A^*$, a contradiction with Weak Indifference. Therefore, $A^* \sim A^{**}$. Thus, (4.16) holds.

From (4.16), by Lemmas 4.4 and 4.5, the following can be obtained easily:

for all $A, B \in K_1 \cup K_2, \max_{x \in A}(x_1 \cdots x_n) > \max_{y \in B}(y_1 \cdots y_n) \Rightarrow A \succ B$. $\cdots$(4.17)

From (4.16) and (4.17), noting that $\succeq$ is an ordering, it must be true that

for all $A, B \in K_1 \cup K_2, \max_{x \in A}(x_1 \cdots x_n) \geq \max_{y \in B}(y_1 \cdots y_n) \Leftrightarrow A \succeq B$. $\cdots$(4.18)

I now show that

for all $A, B \in K, \max_{x \in A}(x_1 \cdots x_n) = \max_{y \in B}(y_1 \cdots y_n) \Rightarrow A \sim B$. $\cdots$(4.19)

Let $A \in K$ and $\max_{x \in A}(x_1 \cdots x_n) = a_1 \cdots a_n$ where $a = (a_1, \cdots, a_n) \in A$. Note that since A is compact (and convex) and the function $x_1 \cdots x_n$ is continuous and quasi-concave, the existence of a is guaranteed. Further, since A is non-degenerate, $a_i > 0$ for all $i = 1, \cdots, n$. Consider $G = \{x \in \mathbb{R}^n_+ | x_i \leq a_i, i = 1, \cdots, n\}$ and $H = \{x \in \mathbb{R}^n_+ | \sum_{i=1}^n x_i/(na_i)\}$. Then, it can be checked that $\max_{x \in G}(x_1 \cdots x_n) = \max_{x \in H}(x_1 \cdots x_n) = a_1 \cdots a_n$. Also, by construction, since $A \in K$ (A is non-degenerate, compact, comprehensive and convex), it must be true that $G \subseteq A \subseteq H$. From $\max_{x \in G}(x_1 \cdots x_n) = \max_{x \in H}(x_1 \cdots x_n)$, by (4.18), $G \sim H$ holds. From $G \subseteq A \subseteq H$, by Weak Monotonicity, it follows that $H \succeq A$ and $A \succeq G$. Therefore, by the transitivity of $\succeq$, $A \sim G \sim H$.

Let $B \in K$ and $\max_{x \in A}(x_1 \cdots x_n) = b_1 \cdots b_n$ where $b = (b_1, \cdots, b_n) \in B$. Again, note that since B is compact (and convex) and the function $x_1 \cdots x_n$ is continuous and quasi-concave, the existence of b is guaranteed. Further, since B is non-degenerate, $b_i > 0$ for all $i = 1, \cdots, n$. Consider $G' = \{x \in \mathbb{R}^n_+ | x_i \leq b_i, i = 1, \cdots, n\}$ and $H' = \{x \in \mathbb{R}^n_+ | \sum_{i=1}^n x_i/(nb_i)\}$. Following similar procedures as the last paragraph, $B \sim G' \sim H'$ can be obtained. Note that $a_1 \cdots a_n = b_1 \cdots b_n$. By (4.18), $G \sim G' \sim H \sim H'$. Therefore, $A \sim B$ follows from the transitivity of $\succeq$; that is, (4.19) holds.

From (4.19) and (4.18), by Weak Monotonicity, it follow that for all $A, B \in K, \max_{x \in A}(x_1 \cdots x_n) > \max_{y \in B}(y_1 \cdots y_n) \Rightarrow A \succ B$. $\cdots(4.20)$

Noting that $\succeq$ is an ordering, (4.20), together with (4.19), completes the proof of Theorem 4.3. ■

Remark 4.6. The following examples, Examples 4.7, 4.8, 4.9 and 4.10, establish the independence of the axioms used in Theorem 4.3.

Example 4.7. Define the binary relation $\succeq_1$ over K as follows: for all A and $B \in K$, $A \succeq_1 B$ iff $\max_{x \in A} x_1^2 x_2 \cdots x_n \geq \max_{y \in B} y_1^2 y_2 \cdots y_n$. Note that $\succeq_1$ is an ordering that satisfies Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference, but violates Symmetry.

Example 4.8. Define the binary relation $\succeq_2$ over K as follows: for all $A, B \in K, A \sim_2 B$. Clearly, $\succeq_2$ is an ordering that satisfies Symmetry, Invariance of Scaling Effects and Weak Indifference, but violates Weak Monotonicity.

Example 4.9. Define the binary relation $\succeq_3$ over K as follows: for all for all A and $B \in K$, $A \succeq_3 B$ iff $\max_{x \in A}(x_1 + x_2 + \cdots + x_n) \geq \max_{y \in B}(y_1 + y_2 + \cdots + y_n)$. It can be checked that $\succeq_3$ is an ordering that satisfies Symmetry, Weak Monotonicity and Weak Indifference, but violates Invariance of Scaling Effects.

Example 4.10. For all $A \in K$, let $a_i(A) = \max\{x_i | (x_1, \cdots, x_i, \cdots, x_n) \in A\}$. Define the binary relation $\succeq_4$ over K as follows: for all $A, B \in K, A \succeq_4 B \Leftrightarrow a_1(A) \cdots a_n(A) \geq a_1(B) \cdots a_n(B)$. Then, $\succeq_4$ is an ordering that satisfies Symmetry, Weak Monotonicity and Invariance of Scaling Effects, but violates Weak Indifference.

Remark 4.11. If K is interpreted as a class of all utility possibities sets that are non-degenerate, compact, convex and comprehensive[3] in an n-person society, then the result of Theorem 4.3 can be regarded as a characterization of the Nash social welfare function.

[3] If each individual has a concave utilility function, then a utility possibilities set has these four properties.

5 The Elementary Self-Evaluation

The assessment of the capability sets in terms of the standard of living based on the ranking rule characterized in the previous section, Section 4, is interpreted as the following. There is a commonly accepted value judgment in the society regarding different functionings, namely, different functionings should be treated equally. This value judgment is reflected in the axiom of Symmetry. Based on this value judgement, when interpreting Weak Indifference, the functioning vector a^* figured in the axiom is subsequently interpreted as the agent's perceived living. Thus, any measure of capability sets in terms of the standard of living based on Symmetry and Weak Indifference with the interpretation of the functioning vector a^* being the agent's perceived living may be deemed as an exercise of the standard-evaluation.

In this section, I drop the axiom of Symmetry and re-interpret the functioning vector a^* used in Weak Indifference as the agent's actual living. Imposing no value judgment and attaching a special importance to the agent's actual living in assessing the living standard, I examine the issue of the self-evaluation. Specifically, I prove that the axioms Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference characterize the following ranking rule.

Definition 5.1. $\succeq$ over K is called the *elementary self-evaluation* rule iff, there exist $s_i > 0$ $(i = 1, \cdots, n)$ with $\sum_{i=1}^{n} s_i = 1$ such that

for all $A, B \in K$, $A \succeq B \Leftrightarrow \max_{x \in A}(x_1^{s_1} \cdots x_n^{s_n}) \geq \max_{y \in B}(y_1^{s_1} \cdots y_n^{s_n})$. $\cdots$ (5.1)

Theorem 5.2. Suppose $\succeq$ over K is an ordering. Then, $\succeq$ on K satisfies Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference if and only if it is the elementary self-evaluation rule.

To prove Theorem 5.2, I first prove the following lemmas.

Lemma 5.3. Suppose $\succeq$ over K is an ordering and satisfies Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference. Then, for all $A \in K$, there exists a unique $a^* \in A$ such that $a_i^* > 0$ for all $i = 1, \cdots, n$ and $A \sim \{x \in \mathbb{R}_+^n | x_i \leq a_i^*, i = 1, \cdots, n\}$.

Proof. Let $\succeq$ over K be an ordering and satisfy Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference. Given Invariance of Scaling Effects, it is sufficient to show that for $S = \{x \in \mathbb{R}_+^n | \sum_{i=1}^{n} x_i \leq 1\}$, there exists a unique $s^* \in S$

such that $s_i^* > 0$ for $i = 1, \cdots, n$ and $S \sim \{x \in \mathbb{R}_+^n | x_i \leq s_i^*, i = 1, \cdots, n\} = S^*$. By Weak Indifference, suppose that there exist $s^*, t^* \in S$ such that $s^* \neq t^*$, $s_i > 0$, $t_i^* > 0$ for all $i = 1, \cdots, n$ and $S \sim S^* \sim \{x \in \mathbb{R}_+^n | x_i \leq t_i^*, i = 1, \cdots, n\} = T^*$. Let $z^* = \gamma s^* + (1-\gamma) t^*$ where $\gamma \in (0,1)$. Clearly, by Weak Monotonicity, $s_1 + \cdots + s_n = 1 = t_1^* + \cdots + t_n^*$. Since $s^* \neq t^*$ and $\sum_{i=1}^n s_i^* = \sum_{i=1}^n t_i^* = 1$, there exist j, k such that $s_j^* < t_j^*$ and $s_k^* > t_k^*$. Without loss of generality, let $s_1^* < t_1^*$. Consider the sets $Z = \{x \in \mathbb{R}_+^n | \sum_{i=1}^n (s_i^* x_i / z_i^*) \leq 1\}$ and $Z^* = \{x \in \mathbb{R}_+^n | x_i \leq z_i^*, i = 1, \cdots, n\}$. By Invariance of Scaling Effects, from $S \sim S^*$, it follows that $Z \sim Z^*$. Note that $Z^* \subseteq S$ and T^* lies entirely inside Z. By Weak Monotonicity, $S \succeq Z^*$ and $Z \succ T^*$. From $T^* \sim S$ and $S \succeq Z^*$, by the transitivity of $\succeq$, it follows that $T^* \succeq Z^*$. On the other hand, from $Z \sim Z^*$ and $Z \succ T^*$, by the transitivity of $\succeq$, $Z^* \succ T^*$ holds, a contradiction. Therefore, by Weak Indifference, there exists a unique $s^* \in S$ such that $s_i^* > 0$ for $i = 1, \cdots$, n, $\sum_{i=1}^n s_i = 1$ and $S \sim S^*$. ■

Lemma 5.4. Suppose $\succeq$ over K is an ordering and satisfies Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference. Then, for all $A = \{x \in \mathbb{R}_+^n | \sum_{i=1}^n (x_i/p_i) \leq 1\} \in K$, $A \sim \{x \in \mathbb{R}_+^n | x_i \leq s_i p_i, i = 1, \cdots, n\}$, where $s \in \mathbb{R}_+^n$ is such that $\sum_{i=1}^n s_i = 1$ and $S = \{x \in \mathbb{R}_+^n | \sum_{i=1}^n x_i \leq 1\} \sim \{x \in \mathbb{R}_+^n | x_i \leq s_i, i = 1, \cdots, n\} = S^*$.

Proof. Let $\succeq$ over K be an ordering and satisfy Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference. Let $S = \{x \in \mathbb{R}_+^n | \sum_{i=1}^n x_i \leq 1\} \in K$. From Lemma 5.3, there exists a unique $s \in S$ such that $\sum_{i=1}^n s_i = 1$ and $S \sim \{x \in \mathbb{R}_+^n | x_i \leq s_i, i = 1, \cdots, n\} = S^*$. Let $A = \{x \in \mathbb{R}_+^n | \sum_{i=1}^n (x_i/p_i) \leq 1\} \in K$. From $S \sim S^*$, by Invariance of Scaling Effects, it follows that

$$\{x \in \mathbb{R}_+^n | x_1/p_1 + \textstyle\sum_{i=2}^n x_i \leq 1\} \sim \{x \in \mathbb{R}_+^n | x_1 \leq s_1 p_1, x_i \leq 1, i = 2, \cdots, n\}. \quad \cdots (5.2)$$

From (5.2), by the repeated use of Invariance of Scaling Effects, one obtains that $A = \{x \in \mathbb{R}_+^n | \sum_{i=1}^n (x_i/p_i) \leq 1\} \sim \{x \in \mathbb{R}_+^n | x_i \leq s_i p_i, i = 1, \cdots, n\}$. ■

Lemma 5.5. Suppose $\succeq$ over K is an ordering and satisfies Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference. Then, for all $A = \{x \in \mathbb{R}_+^n | x_i \leq a_i, i = 1, \cdots, n\}, B = \{x \in \mathbb{R}_+^n | x_i \leq b_i, i = 1, \cdots, n\} \in K$, if $A \sim B$, then for all $\lambda \in (0,1), C = \{x \in \mathbb{R}_+^n | x_i \leq \lambda a_i + (1-\lambda) b_i, i = 1, \cdots, n\} \succeq A$.

Proof. Let $\succeq$ over K be an ordering and satisfy Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference. Let $A = \{x \in \mathbb{R}_+^n | x_i \leq a_i, i = 1, \cdots, n\}, B =$

$\{x \in \mathbb{R}^n_+ | x_i \leq b_i, i = 1, \cdots, n\} \in K$, and $A \sim B$. Let $\lambda \in (0,1)$ and $C = \{x \in \mathbb{R}^n_+ | x_i \leq \lambda a_i + (1-\lambda) b_i, i = 1, \cdots, n\}$. If $A = B$, then $C = A$. Hence, $C \succeq A$. Now, let $A \neq B$. Given Weak Monotonicity and Lemma 5.4, it is straightforward to check that there exist j and k such that $a_j < b_j$ and $a_k > b_k$. Without loss of generality, let $a_1 < b_1$. Then, following a similar argument as in the proof of Lemma 5.3, it can be shown that $C \succeq B \sim A$. ■

With the help of Lemmas 5.3, 5.4 and 5.5, I present the proof of Theorem 5.2 below.

Proof of Theorem 5.2. It can be checked that if $\succeq$ over K is the elementary self-evaluation rule, then it satisfies Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference. Therefore, I have only to show that if $\succeq$ on K satisfies Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference, then it is the elementary self-evaluation rule, that is, (5.1) holds.

Let $\succeq$ over K satisfy Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference. Consider the capability set $S = \{x \in \mathbb{R}^n_+ | \sum_{i=1}^n x_i \leq 1\}$. By Lemma 5.3, there exists a unique $s \in S$ such that $s_i > 0$ for all $i = 1, \cdots, n$, $\sum_{i=1}^n s_i = 1$ and $S \sim \{x \in \mathbb{R}^n_+ | x_i \leq s_i, i = 1, \cdots, n\} = S^*$. For all $A \in K$, if $s^* = \arg\max_{x \in A}(x_1^{s_1} \cdots x_n^{s_n})$ (given that A is compact and convex and the function $x_1^{s_1} \cdots x_n^{s_n}$ is quasi-concave, $\arg\max_{x \in A}(x_1^{s_1} \cdots x_n^{s_n})$ is therefore unique), then, it must be true that $S^* \subseteq A \subseteq S$. By Weak Monotonicity and $S \sim S^*$, $A \sim S$ follows from the transitivity of $\succeq$ immediately. By Invariance of Scaling Effects, the following can be obtained easily:

$$\text{for all } B, C \in K, \text{ if } \arg\max_{x \in B}(x_1^{s_1} \cdots x_n^{s_n}) = \arg\max_{x \in C}(x_1^{s_1} \cdots x_n^{s_n}), \text{ then } B \sim C. \quad \cdots (5.3)$$

Clearly, for all $A \in K$, the function $f(A) = \max_{x \in A}(x_1^{s_1} \cdots x_n^{s_n})$ is a representation of $\succeq$ over K satisfing Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference. In what follows, I shall prove that if a function g represents $\succeq$ over K satisfying the three axioms specified in Theorem 5.2, then for all $A, B \in K$, $g(A) \geq g(B) \Leftrightarrow \max_{x \in A}(x_1^{s_1} \cdots x_n^{s_n}) \geq \max_{x \in B} x_1^{s_1} \cdots x_n^{s_n})$.

Let $g : K \rightarrow \mathbb{R}_+$ be such that for all $A, B \in K$, $g(A) \geq g(B) \Leftrightarrow A \succeq B$ and $g(A) > 0$ for all $A \in K$. Define $h : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$ as follows: for all $x \in \mathbb{R}^n_+$, if $x_i > 0$ for all $i = 1, \cdots, n$, then $h(x) = g(\{z \in \mathbb{R}^n_+ | z_i \leq x_i, i = 1, \cdots, n\})$ and if $x_i = 0$

for some $i = 1, \cdots, n$, then $h(x) = 0$. Clearly, the function h is well-defined. I now show that h is quasi-concave and homothetic. To show that h is quasi-concave, I need only to show that the set $L(v) = \{x \in \mathbb{R}^n_+ | h(x) \geq v\}$ is convex. Let $x^*, y^* \in \mathbb{R}^n_+$ be such that $h(x^*) = h(y^*)$. From Lemma 5.4, $\{x \in \mathbb{R}^n_+ | \sum_{i=1}^n (x_i/(x_i^*/s_i)) \leq 1\} \sim \{x \in \mathbb{R}^n_+ | x_i \leq x_i^*, i = 1, \cdots, n\}$ and $\{x \in \mathbb{R}^n_+ | \sum_{i=1}^n (x_i/(y_i^*/s_i)) \leq 1\} \sim \{x \in \mathbb{R}^n_+ | x_i \leq y_i^*, i = 1, \cdots, n\}$. It is also clear that $x_i^* > y_i^*$ and $x_j^* < y_j^*$ for some i, j. Let γ be such that $1 > \gamma > 0$ and consider $z^* = \gamma x^* + (1-\gamma) y^*$. By Lemma 5.5 and the definition of $h(\cdot)$, $h(z^*) > h(x^*) = h(y)$. Hence, h is quasi-concave. For all $A = \{x \in \mathbb{R}^n_+ | \sum_{i=1}^n (x_i/p_i) \leq 1\}$, by Lemma 5.4, $A \sim \{x \in \mathbb{R}^n_+ | x_i \leq s_i p_i, i = 1, \cdots, n\}$. By Lemmas 5.4 and 5.5, h achieves the maximum at $a^* = (s_1 p_1, \cdots, s_n p_n)$ over A. It is then clear that for all $\lambda > 0$, from Lemma 5.4, $\arg\max_{x \in \lambda A} = \lambda a^*$, where $\lambda A = \{x \in \mathbb{R}^n_+ | \sum_{i=1}^n (x_i/\lambda p_i) \leq 1\}$. Therefore, h is homothetic (see Färe and Shephard (1977, 1977a)). That is, $h(x) = H(r(x))$ where $H : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and $r : \mathbb{R}^n_+ \to \mathbb{R}_+$ is homogenous of degree 1. From Weak Monotonicity, H is strictly increasing. Thus, for all $x, y \in \mathbb{R}^n_+$, $r(x) \geq r(y)$ iff $h(x) \geq h(y)$. By the definition of h, it follows that

for all $A = \{x \in \mathbb{R}^n_+ | x_i \leq a_i, i = 1, \cdots, n\}$ and $B = \{x \in \mathbb{R}^n_+ | x_i \leq b_i, i = 1, \cdots, n\} \in K$, $A \succeq B \Leftrightarrow r(a) \geq r(b)$. $\cdots$(5.4)

Clearly, $r(x) > 0$ for all $x \in \mathbb{R}^n_+$ such that $x_i > 0, i = 1, \cdots, n$. Let $r(e) = 1$, where $e = (1, \cdots, 1)$. Consider $p = (p_1, p_2, \cdots, p_n)$ and $x^i = (x_1^i, \cdots, x_n^i)$ where $i = 1, \cdots, n$, $x_i^i = p_i$ and $x_j^i = 1$ for all $j \neq i$. Let $r_i = r(x^i)$. From $r(x^1) = r(e) r_1$, noting that $r_1 > 0$ and r is homogeneous of degree 1, it follows that $r(x^1/r_1) = r(e)$. From (5.4), by Invariance of Scaling Effects, one obtains $r(p_1/r_1, p_2/r_1, 1/r_1, \cdots, 1/r_1) = r(1, p_2, 1, \cdots, 1) = r(x^2)$. Note that r is homogeneous of degree 1. Therefore, $r(p_1, p_2, 1, \cdots, 1) = r(x^1) r(x^2)$. Following similar arguments, given the homogeneity of degree 1 of $r(\cdot)$, (5.4) and Invariance of Scaling Effects, it can be shown that $r(p) = r(x^1) \cdots r(x^n)$. Thus,

for all $p \in \mathbb{R}^n_+$ with $p_i > 0, i = 1, \cdots, n$, $r(p) = r(x^1) \cdots r(x^n)$. $\cdots$(5.5)

I now show that, for all $i = 1, \cdots, n$, there exists $v_i > 0$, for all $p_i > 0$, $r(x^i) = p_i^{v_i}$. From $r(x^i) = r(e) r_i$, since $r(\cdot)$ is homogeneous of degree 1, $r(x^i/r_i) = r(e)$ holds. Then, for all $q_i > 0$, by (5.4) and Invariance of Scaling Effects, it follows that

$r(y^i/r_i) = r(z^i)$, where $y^i = (y^i_1, \cdots, y^i_n)$ is such that $y^i_i = p_i q_i$ and $y^i_j = 1$ for all $j \neq i$, and $z^i = (z^i_1, \cdots, z^i_n)$ is such that $z^i_i = q_i$ and $z^i_j = 1$ for all $j \neq i$. By the homogeneity of degree 1 of $r(\cdot)$, it then follows that $r(y^i) = r(x^i)r(z^i)$. For all $p_i > 0$, define $u(p_i) = r(x^i)$. Then, I have obtained

for all $p_i > 0, q_i > 0$, $u(p_i q_i) = u(p_i)u(q_i)$. $\cdots$(5.6)

Equation (5.6) is the Cauchy equation and its general solution is (see Aczél (1966)): for all $p_i > 0$, $u(p_i) = p_i^{c_i}$ for some $c_i > 0$ (noting that $r(\cdot)$ is strictly increasing, and therefore $u(\cdot)$ is strictly increasing as well). It then follows that for all $p_i > 0, r(x^i) = w_i p_i^{c_i}$ where $w_i > 0$ is a constant. From (5.5) and by the homogeneity of degree 1 of $r(\cdot)$, it follows that for all $p \in \mathbb{R}^n_+$ with $p_i > 0$, $r(p) = w p_1^{c_1} \cdots p_n^{c_n}$ and $c_1 + \cdots + c_n = 1$, where w is a constant. From the definition of $r(\cdot)$ and $h(\cdot)$, it then follows that for all $A = \{x \in \mathbb{R}^n_+ | x_i \leq a_i, i = 1, \cdots, n\}, B = \{x \in \mathbb{R}^n_+ | x_i \leq b_i, i = 1, \cdots, n\} \in K, A \succeq B \Leftrightarrow a_1^{c_1} \cdots a_n^{c_n} \geq b_1^{c_1} \cdots b_n^{c_n}$. From Lemma 5.4, by the definition of $g(\cdot)$ and (5.3), it then follows that for all $X, Y \in K, X \succeq Y \Leftrightarrow \max_{x \in X}(x_1^{c_1} \cdots x_n^{c_n}) \geq \max_{y \in Y}(y_1^{c_1} \cdots y_n^{c_n})$. By Lemma 5.4, it must be true that $c_i = s_i$ for $i = 1, \cdots, n$. That is, for all $X, Y \in K$, $g(X) \succeq g(Y) \Leftrightarrow \max_{x \in X}(x_1^{s_1} \cdots x_n^{s_n}) \geq \max_{y \in Y}(y_1^{s_1} \cdots y_n^{s_n})$.
■

Remark 5.6. From Examples 4.8, 4.9 and 4.10, clearly, the axioms used in Theorem 5.2 are independent.

6 A Quasi Ordering Approach

Both the elementary standard-evaluation rule and the elementary self-evaluation rule may be criticized for their "elementary nature". As our axiomatic results (Theorems 4.3 and 5.2) demonstrate, the major failure of these two rules lies with the axiom of Weak Indifference. Weak Indifference essentially attaches exclusive importance to the agent's perceived living or actual living in evaluating the standard of living offered by a capability set and ignores the freedom type information contained in the set. This does not go along well with Sen's notion of the standard of living in which he emphasizes the opportunity/freedom aspect of a capability set. In this section, I try to relax Weak Indifference and present a quasi ordering approach to the evaluation of the standard of living offered by capability sets. It turns out that the quasi ordering characterized in this section lies in between the superset-dominance rule and the elementary self-evaluation rule.

Let $\succeq$ be a quasi ordering over K. As shown by Donaldson and Weymark (1998) (see also Bossert (1999), Duggan (1999) and Suzumura and Xu (1999)), a quasi ordering is always the intersection of some orderings. That is, for all quasi ordering $\succeq$ over K, there exists a class Ω of orderings over K such that $\succeq = \bigcap_{\succeq_t \in \Omega} \succeq_t$, where each $\succeq_t \in \Omega$ is an ordering. I now impose a few properties on the quasi ordering $\succeq$ over K; or equivalently, on orderings in Ω.

Definition 6.1. For all $\succeq_t \in \Omega$, $\succeq_t$ over K satisfies

(6.1.1) **Weak Monotonicity** iff, for all $A, B \in K$, if $B \subseteq A$ then $A \succeq_t B$, and if B lies entirely in A, then $A \succ_t B$.

(6.1.2) **Invariance of Scaling Effects** iff, for all $A, B \in K$ and for all $\alpha^i > 0$, $A \succeq_t B \Leftrightarrow \alpha^i A \succeq \alpha^i B$.

(6.1.3) **Weak Invariance** iff, for all $A \in K$, there exists $a^* \in A$ such that $B = \{x \in \mathbb{R}^n_+ | x_i \leq a^*_i, i = 1, \cdots, n\} \in K$ and $B \sim_t A$.

The above axioms are the exact counterparts of the axioms proposed in Section 4 and their intuitions have already explained. With the help of these properties and recalling Theorem 5.2, the following result is immediate.

Theorem 6.2. Suppose $\succeq$ over K is a quasi ordering and let $\succeq = \bigcap_{\succeq_t \in \Omega} \succeq_t$ where each and every $\succeq_t \in \Omega$ is an ordering. Then, [for all $\succeq_t \in \Omega : \succeq_t$ over K satisfies Weak Monotonicity, Invariance of Scaling Effects and Weak Indifference] if and only if there exists a set Γ of positive scalars $\{s^t_1, \cdots, s^t_n\}$ such that for all $A, B \in K$, $A \succeq B \Leftrightarrow \max_{x \in A}(x_1^{s^t_1} \cdots x_n^{s^t_n}) \geq \max_{y \in B}(y_1^{s^t_1} \cdots y_n^{s^t_n})$ for all $\{s^t_1, \cdots, s^t_n\} \in \Gamma$.

Proof. The proof follows from Theorem 5.2 and the fact that $\succeq$ over K is a quasi ordering—$\succeq = \bigcap_{\succeq_t \in \Omega} \succeq_t$ for some collection Ω of orderings over K. ■

Remark 6.3. Clearly, the axioms used in Theorem 6.2 are independent.

Remark 6.4. If the quasi ordering $\succeq$ over K in Theorem 6.2 is complete, then the rule characterized there becomes the elementary self-evaluation rule. If, on the other hand, the set Γ consists of all possible positive scalars, then the rule characterized in Theorem 6.2 becomes the superset-dominance rule. In view of these observations, the rule axiomatized in Theorem 6.2 lies in between the superset-dominance rule and the elementary self-evaluation rule.

7 Concluding Remarks

In this paper, I have used Sen's functioning-capability approach to analyse the issue of the standard of living offered to an agent. For this purpose, an agent is characterized by her capability sets each of which summarizes her different functionings and her capabilities to achieve these different functionings. Thus, the formal exercise of evaluating an agent's standard of living is to rank her capability sets. I have considered both the standard-evaluation and the self-evaluation. With the standard-evaluation, the crucial things are to suppose that there exist some widely acceptable value judgment regarding different functionings and that a special importance of the agent's perceived living is attached to the evaluation exercise. On the other hand, the essential part of the self-evaluation is to place an importance of the agent's actual living in assessing her capability sets in terms of the standard of living. The quasi orderings induced by the superset-dominance-based-rule and characterized in Theorem 6.2, respectively, can be regarded as an exercise of both the standard-evaluation and the self-evaluation. The rule based on (4.1) that gives an index for a capability set in terms of the standard of living by its maximum value of $(x_1 \cdot ... \cdot x_n)$ in this capability set has been regarded as an exercise of the standard-evaluation since certain value judgment regarding different functionings is involved in the characterization. The rule based on (5.1) that ranks capability sets according to the maximum value of $(x_1^{s_1} \cdot ... \cdot x_n^{s_n})$ in a capability set, where each s_i is positive and the sum of these s_i's is equal to 1, is interpreted as the self-evaluation since the weights s_i's are determined by the agent's actual choice of the life style from the capability set $\{x \in \mathbb{R}_+^n | \sum_{i=1}^n s_i \leq 1\}$.

Clearly, the superset-dominance-based rule is the smallest class of all ranking rules each of which is an ordering and respects a type of monotonicity property. In a sense, it is the "coarsest" quasi ordering in the context of ranking capability sets in terms of the standard of living. On the other hand, both the elementary standard-evaluation rule and the elementary self-evaluation rule go to the other extreme by requiring a complete ranking of the capability sets and are therefore particular orderings for ranking capability sets in terms of the living standard. The rule characterized in

Theorem 6.2 is an interesting intermediate ranking rule that lies in between and is more general than the two extremes. As remarked in Section 6, both the extremes, the superset-dominance rule and the elementary self-evaluation (or standard-evaluation) rule are special cases of the rule axiomatized in Theorem 6.2.

Finally, it is interesting to note that if K is interpreted as the class of all utility possibilities sets that are non-degenerate, comprehensive, compact and convex for an n-person society, then the results of Theorems 4.3 and 5.2 can be regarded as characterizations of symmetric and asymmetric Nash social welfare function, respectively. If, on the other hand, K is interpreted as the class of n-person bargaining games, then the results of Theorems 4.3 and 5.2 suggest that the Nash bargainging solution (both symmetric and asymmetric) is optimal.

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