THE PROPERTIES OF THE WATTS' POVERTY INDEX UNDER LOGNORMALITY

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Abstract

A priori knowledge about the shape of living standards distribution has not been fully exploited in the literature to investigate properties of poverty indices. The method we propose is to exploit credible distributional assumptions to: generate additional properties for poverty indices; relate the latters to the incidence of poverty and to an inequality index; enrich the sensibility analysis of poverty indices; derive new families of relative poverty lines.

Under lognormality assumption, we show that the Watts' poverty index, one of the major axiomatically sound poverty index, exhibits additional interesting properties. It can be expressed as the product of an inequality index by the cumulating incidence of poverty. We give its explicit formula and present sensitivity and limit analyses. A family of relative poverty lines associated with the Watts' index is derived.

Résumé

Les connaissances a priori des formes des distributions de niveaux de vie n' a pas été pleinement exploité dans la litterature pour investiguer les propriétés des indices de pauvreté. La méthode que nous proposons consiste à exploiter des hypothèses distributionnelles crédibles pour: engendrer des propriétés supplémentaires pour les indices de pauvreté; relier ceux-ci à un indice d'inégalité et à l'incidence de la pauvreté; enrichir l'analyse de sensibilité des indices de pauvreté; dériver de nouvelles familles de lignes de pauvreté relative.

Sous une hypothèse de lognormalité nous montrons que l'indice de pauvreté de Watts, un des indices majeur de pauvreté qui soit axiomatiquement correct, possède des propriétés supplémentaires intéressantes. Il peut être exprimé comme le produit d'un indice d'inégalité par le cumul de l'incidence de la pauvreté. Nous donnons sa formule explicite et nous conduisons des analyses de sensibilité ou de limites. Finalement, nous dérivons une famille de lignes de pauvreté relatives associee à l'indice de Watts.

1 Introduction

To be meaningful, a poverty index must satisfy reasonable properties[1]. Various sets of axioms have been studied in the literature[2] and several formulae have been proposed for axiomatically sound indices, sometimes relating them to social welfare functions[3].

The properties of poverty indices are generally considered independently from a priori assumptions on living standards distributions. Yet, the fact that the observed distributions show generally unimodal strongly asymmetric and leptokurtic shapes, and incorporate only positive values suggests that much relevant and quasi-systematic information resides in the shape of these distributions. Accounting for typical features of these distributions should not limit in practice the validity of poverty indicators while it should provide further restrictions in the form of new proprieties.

Assumptions on the distribution of living standards are commonly used to derive results for aggregate consumption demands (Hildenbrand (1983), Grandmont (1987), Quah (1997)). It is plausible that these assumptions can also help in deriving properties for aggregate property.

Is it possible to derive additional properties of poverty indices using a distribution model? One of the most popular axiomatically sound poverty index is the Watts' index (Watts (1968), Zheng (1993)). The aim of this article is to show that interesting properties of the Watts' poverty index arise under lognormality of living standard distribution.

We de…ne in section 2 the Watts' poverty index. We derive new properties of the Watts' index under lognormality in section 3. Finally, we conclude in section 4.

2 The Watts' poverty index

The Watts' poverty index (Watts (1968)) is de…ned as

[1] Atkinson (1987), Lipton and Ravallion (1993) and Ravallion (1994).

[2] Sen (1976), Takayama (1979), Kakwani (1980), Kindu and Smith (1983), Foster, Greer and Thorbecke (1984), Foster and Shorrock (1988), Zheng (1994), Bourguignon and Fields (1995), Shorrocks (1995), Chakravarty (1997), Thon (1997), Zheng (1997), Foster (1998).

[3] Blackorby and Donaldson (1980), Lambert (1993), Zheng (1993).

$$W = \int_0^z -\ln(y/z)\, d\mu(y) \qquad (1)$$

where μ is the cumulative probability distribution of living standards y, and z is the poverty line.

The Watts' index satisfies the focus, monotonicity, transfer and transfer sensitivity axioms. It is also continuous, subgroup consistent and even decomposable (Foster, Greer, Thorbecke (1984), Zheng(1993)). Moreover, when related to social welfare functions, the Watts index is the unique index giving the absolute amount of social welfare loss due to poverty that as well satisfies monotonicity, continuity, decomposability and scale invariance (Zheng (1993), Zheng (1997)).

In practice, because of its axiomatic properties, it is generally a better representation of poverty than frequently used poverty indicators such as the head-count index (P_0) or the poverty gap index (P_1). For example, Muller (1998) using data from Rwanda finds that most axiomatically sound poverty indices, accounting for the severity of poverty, lead to qualitatively similar results, by contrast with P_0 and P_1.

3 Properties under lognormality

3.1 An explicit formula

The choice of the lognormal distribution as a model for living standards distributions is suggested by the fact that histograms of nominal living standards have generally unimodal, asymmetrical and leptokurtic shapes. Moreover, only positive values of income or living standards are possible.

The lognormal approximation has been frequently used in applied analysis of living standards (e.g. Alaiz and Victoria-Feser (1990), Slesnick (1993)). The assumption of lognormality of income has as well been exploited in theoretical economics (Hildenbrand (1998)). Other distribution models for living standards or incomes (Hirschberg and Slottje (1989)), such that the Pareto distribution or the Gamma distribution (Salem and Mount (1974)) are also used, but will not lead to an explicit expression of the Watts' index.

Explicit formulae of the Gini coefficient and of the inequality measures from the family of the General Entropy, have been calculated when the income distribution is lognormal (e.g. Cowell and Victoria-Feser (1996)). The formula of the head-count index is known as well (e.g. Hanmer, Pyatt and White (1997)) by direct integration. To the best of our knowledge, the formula of the Watts' index is not available and we present it now.

Proposition 1

If the living standards, y, follow a lognormal distribution law such that $\ln(y) \sim N(m, \sigma^2)$, then the Watts' index is equal to:

$$W = (\ln z - m).\Phi\left(\frac{\ln z - m}{\sigma}\right) + \sigma.\phi\left(\frac{\ln z - m}{\sigma}\right) \tag{2}$$

where ϕ and Φ are respectively the p.d.f. and c.d.f. of the standard normal distribution.

The knowledge of $Z = (\ln z - m)/\sigma$ and σ is sufficient for the knowledge of W.

$$W = \sigma.[Z.\Phi(Z) + \varphi(Z)] = \sigma.G(Z) \tag{3}$$

where G is a primitive function of the head-count index under lognormality.

For any variable t, the elasticity of W with respect to t, denoted $e(W|t)$, can be decomposed into

Proposition 2 $\quad e(W|t) = e(\sigma|t) + e(G(Z)|t) \quad$ (4)

Proof: See appendix.

We denote m, the "loglevel" (the mean of the logarithm of living standards), and σ, the "logvariability" (the standard deviation of the logarithm of living standards). σ is a notion close to a well known inequality index, commonly called the "standard deviation of logarithms" (e.g. Sen (1997)), whose formula is $\sqrt{\frac{1}{n}\sum_i (\ln y_i - \ln \mu)^2}$ and makes intervene μ, the arithmetic mean of living standards. This suggests that σ can also be interpreted as a descriptive inequality index.

Consequently, eq. 3 shows that the Watts index can be decomposed in a product of two terms: σ and $G(Z)$, where Z is the standardised logarithm of the poverty line (s.l.p.l.).

Because under lognormality the head-count index (incidence of poverty) is equal to $\Phi(Z)$, function $G(Z)$ is a primitive function with respect to Z of the poverty incidence (with value $\frac{1}{\sqrt{2\pi}}$ at $Z = 0$). We denote $G(Z)$ "cumulating (lognormal) poverty incidence". It is the function to use in second order stochastic dominance analysis of poverty curves (Atkinson (1987)), which corresponds to additive social welfare functionals based on increasing and concave utility functions and therefore is consistent with the basic properties of consumer theory. For this reason $G(Z)$ is itself an interesting poverty index.

Hence, the Watt's poverty index under lognormality of living standards is the product of an inequality index by a poverty index well adapted to second order dominance analysis.

As a direct consequence, eq. 4 shows that the elasticity of W with respect to any variable t is the sum of the elasticity of the inequality index σ and of the elasticity of the cumulating poverty incidence. This decomposition is likely to help the interpretation of poverty changes by separating the direct influence of the proximity to the s.l.p.l. with the interpersonal comparisons involved in poverty severity aspects.

A similar decomposition, although of different origin, occurs with the Sen's poverty index, $S = I.H + I.(I\text{-}H).G_p$ where H is the head-count index, I is the income-gap ratio and G_p the Gini coefficient among the poor (See Sen (1976)). It is also the case for the Foster-Greer-Thorbecke poverty index $P_2 = H.[I^2 + (1\text{-}I)^2 C_p^2]$ where C_p is the coefficient of variation among the poor (e.g. Ravallion (1994)). In all these cases the incidence of poverty, the poverty gap ratio, and an inequality index are combined to produce the poverty index of interest.

As stressed in Sen (1997), different views can be adopted as for the 'relativist' aspect of poverty measurement. The Sen's approach with an explicit additive incorporation of inequality index has been generalised by Anand (1977), Takayama (1979) and Blackorby and Donaldson (1980). However, other ways of introducing the inequality in poverty measures are certainly possible. We have explicited how it can be represented for the Watts' index by using lognormality assumptions, albeit this feature was not originally intended by Watts. This approach is generalisable to other poverty indices and other distribution models, although it should yield explicit formulae only in rare cases.

Sen (1976) generates an inequality index by replacing z in the formula of the sen's index with the mean income of the population. The same procedure in our case yields the following inequality index

$$INEGW = \frac{\sigma^2}{2}\Phi\left(\frac{\sigma}{2}\right) + \sigma\phi\left(\frac{\sigma}{2}\right) \quad (5)$$

which is a simple transformation of the logvariability.

3.2 Sensitivity to distribution parameters

We turn now to the sensitivity analysis of W, which will clarify the role of distribution parameters.

Proposition 3 The gradient of W with respect to distribution parameters has the following components.

$$\frac{\partial W}{\partial m} = -\Phi(Z) < 0 \quad (6)$$

$$\frac{\partial W}{\partial \sigma} = \phi(Z) > 0 \quad (7)$$

Proof: Straightforward differential calculus.

Poverty measured by the Watts' index decreases in the mean level of the logarithm of living standards, m. The corresponding gradient component is equal to minus the incidence of poverty, and is therefore bounded in [-1,0] . By contrast, it does not necessarily decrease in the mean of living standards, $e^{m+\sigma^2/2}$.

W increases with the logvariability, σ, although it does not necessarily increase in the variance of living standards, $e^{2m+\sigma^2}.(e^{\sigma^2} - 1)$. The marginal augmentation of W with σ is bounded upwards by $1/\sqrt{2\pi}$ and downwards by 0, limiting the influence of a variation of the inequality on the poverty measure.

3.3 Sensitivity to the choice of the poverty line

Proposition 4

The marginal variation of W with respect to the s.l.p.l. Z is

$$\frac{\partial W}{\partial Z} = \sigma.\Phi(Z) > 0 \tag{8}$$

Proposition 5 The variation of W with respect to the poverty line itself is

$$\frac{\partial W}{\partial z} = \frac{1}{z}\Phi(Z) > 0 \tag{9}$$

Proof: Straightforward differential calculus.

Eq. 8 implies that the marginal variations of W with respect to Z or G(Z) are positive. The marginal variation of W with the s.l.p.l. is proportional to both the logvariability and the incidence of poverty, and is therefore stronger in populations with many poor people and high inequality.

Of course, as shows eq. 9, an increase in the level of the poverty line increases poverty.

Because the level of the relevant poverty line z is generally unknown, it is important to investigate the sensitivity of the poverty measure to an a priori choice of z. A simple consequence of eq. 9 is that the elasticity of W with respect to the poverty line is equal to a ratio of two poverty indices: the incidence of poverty over the Watts index. If we consider a situation where the number of poor remains fixed and where the severity of poverty increases for one poor person, then the sensitivity of W to z, as described by the elasticity, also augments. One expects a higher sensitivity of W to z when there are many extremely poor individuals in the population, or when the inequality is high.

Until now we have considered only absolute poverty lines. However, one often considers that the appropriate poverty line should be relative to the distribution of living standards. The literature about poverty lines is extensive (Hagenaars and van Praag (1985); Callan and Nolan (1991); Short, Garner, Johnson and Shea (1998); Pradhan and Ravallion (1998), and very varied rules to define these lines, often somewhat arbitrary, have been proposed. Formulae based on the mean or percentiles of the income distribution are sometimes used. Half of the mean, half of the median, the first or the second quintile are popular choices. We propose now a rule such that a change in the poverty line exactly compensates a change in an aggregate living standard notion, so as to conserve the poverty level for different situations of aggregate living standards.

Since the marginal rate of substitution of z to m in the Watts' index is equal to -z, the choice of the poverty line and the level of logarithm of living standards are substitutable when the value of the poverty index is maintained fixed. The strength of the substitutability relation is higher for higher poverty lines. This suggests a rule to define relative poverty line levels adapted to the Watts' index, as functions of the mean of logarithms of living standards.

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Proposition 6

A family of relative poverty lines associated with W is $z = K.e^m$, where K is a constant.

Proof:

$dW = 0$ implies $\frac{dz}{dm} = -\frac{\partial W/\partial m}{\partial W/\partial z} = z$, then by integration $z = K.e^m$, where K is a constant to choose. QED.

For $K = 1/2$ and a low logvariability, the relative poverty line is close to half the mean of living standards. That is no longer the case if the logvariability is high.

The rule can be used to update poverty lines along the development path of a country, parallely to the general growth of income, here represented by m.

A general study of the variation of W with Z provides now an exhaustive information about the sensitivity of W with respect to the s.l.p.l.

Proposition 7 (Etude of the variations of W with Z)

The function $G(Z) = Z.\Phi(Z) + \phi(Z)$ summarises the variations of W with respect to Z.

$G'(Z) = \Phi(Z) > 0$ and $G''(Z) = \phi(Z) > 0$.

G'' is increasing from 0 to $1/\sqrt{2\pi}$ when $Z=0$, then decreasing to 0. It is an even function.

G' is increasing from 0 to 1. G is increasing from 0 to $+\infty$.

In a neighbourhood of $Z = 0$ (i.e. $z = e^{\mu}$), W is equivalent to $\sigma/\sqrt{2\pi} + dz/z$

In the neighbourhood of $z = 0$, W is equivalent to
$\ln(z).\Phi(\ln z/\sigma)$

Moreover,

$$\lim_{z \to 0} W = 0$$

$$\lim_{z \to +\infty} W = +\infty$$

Proof: See appendix.

When the poverty line is low (close to zero), the Watts' index is equivalent to $\ln(z)\Phi(\ln z/\sigma)$, i.e. to the incidence of poverty with a scale factor and tends to 0 with z, as the population of the poor vanishes.

When the poverty line increases to the infinity, the Watts' index goes to the infinity. Indeed, all the population become poor and the impact of the severity of poverty increases dramatically.

Even if an exact value of the poverty line is generally not accepted by all analysts, an interval of values may be the object of an agreement. In that case, local approximations of W can be used to assess the robustness of the poverty measure to a local variation of the poverty line. The study of variations of function G(Z) shows that in particular very different approximations of W are to be used for far apart values of the line.

4 Conclusion

We have shown in this paper, that under lognormality assumption the Watts' poverty index, one of the major axiomatically sound poverty index, possesses additional interesting properties. We give its explicit formula that is the product of an inequality index and the cumulating incidence of poverty.

As a matter of fact, the introduction of a distributional assumption enables us to explicit how inequality considerations are included in poverty indices. This rejoins the Sen's index and other generalisations, which offer simple ways to explicitly incorporate inequality considerations in poverty analysis. To this extent, once distributional assumptions are allowed, the distinction between the Sen's index and its generalisations, and decomposable poverty indices, believed to represent an approach to poverty, much independent from inequality considerations, appear as an arti…cial one.

In this context, we present new properties of the Watts' indices based on variational and limit analysis. In particular, a family of relative poverty lines associated with the Watts' index is derived.

The methods used in this article could be generalised to other poverty indices and other distribution assumptions. This will not generally lead to explicit formulae, although the analysis can be implemented using numerical integration methods, particularly when empirical applications are of interest. The general method would consist in …rstly exploiting credible distributional assumptions on living standards to generate additional properties for poverty indices and relating it to the incidence of poverty and an inequality indicator; secondly to enrich the sensitivity analysis of poverty indices by distributional assumptions; and …nally to derive relative poverty lines families.

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Appendix

Proof of proposition 1:

$\ln(y) \sim N(m, \sigma)$, whose c.d.f. is denoted H. The Watts' index can be decomposed as follows

$$W(z) = \int_0^z \left(-\ln(y) + \ln(z)\right) d\mu(y) \tag{10}$$

which yields using the transfer theorem (Monfort (1980)) with $u = \ln(y)$)

$$W(z) = \ln(z) . H(\ln(z)) - \int_{-\infty}^{\ln z} u \, dH(u) \tag{11}$$

and again with normalisation of u with $t = \frac{u-m}{\sigma}$

$$W(z) = \ln(z) . \Phi\left(\frac{\ln(z) - m}{\sigma}\right) - \int_{-\infty}^{\frac{\ln z - m}{\sigma}} (\sigma t + m) \, d\Phi(t) \tag{12}$$

where Φ is the cumulative distribution function of the standard normal law, N(0,1). Then,

$$W(z) = (\ln(z) - m) . \Phi\left(\frac{\ln(z) - m}{\sigma}\right) - \sigma . J(z) \tag{13}$$

where

$$J(z) = \int_{-\infty}^{\frac{\ln z - m}{\sigma}} t \, d\Phi(t) \tag{14}$$

Integration of eq. 14 yields

$$J(z) = -\frac{1}{\sqrt{2\pi}} e^{-\left(\frac{\ln(z)-m}{\sigma}\right)^2/2} \tag{15}$$

Finally,

$$W = (\ln(z) - m)\cdot\Phi\left(\frac{\ln z - m}{\sigma}\right) + \frac{\sigma}{\sqrt{2\pi}} e^{-\left(\frac{\ln(z)-m}{\sigma}\right)^2/2} \qquad (16)$$

QED.

Proof of Proposition 2:

The only difficulties of the proposition are the local results.

The change in variable, $H(V) = G(1/V)$, leads us to examine the neighbourhood of $V = 0^+$ and of $V = 0^-$, instead of respectively neighbourhoods of $Z = +\infty$ and $Z = -\infty$.

$H(V) = \frac{1}{V}\Phi(\frac{1}{V}) + \varphi(\frac{1}{V})$ is indefinitely differentiable everywhere on $\mathbb{R}\setminus\{0\}$:

$H'(V) = -\frac{1}{V^2}\Phi(\frac{1}{V}) < 0$.

We first examine the situation in the neighbourhood of $V = 0^+$:

If $V > 0$; then $\Phi(\frac{1}{V}) > \frac{1}{2}$ and therefore $|H'(V)| > \frac{1}{2V^2} \to +\infty$ when $V \to 0^+$.

Therefore, $H' \to -\infty$ when $V \to 0^+$. Then, there is no Taylor expansion in a neighbourhood of 0+, even at the first order.

Similarly, with H we have for $V > 0$; $H(V) > \frac{1}{V}\Phi(\frac{1}{V}) > \frac{1}{2V}$. Therefore, $H \to +\infty$ when $V \to 0^+$, and then $W \to +\infty$ when $z \to +\infty$.

For discussing the situation in a neighbourhood of $V = 0^-$, we need to consider that $e^{-\frac{1}{2t^2}} = o(|t|^k); \forall k$.

Then, $\int_{-\infty}^{V} e^{-\frac{1}{2t^2}}dt = o(\left[\frac{1}{k+1}t^{k+1}\right]_{-\infty}^{V}); \forall k$, since the exponential is an analytical function and we consider only positive functions. Then, for k = -3, we obtain $\int_{-\infty}^{V} e^{-\frac{1}{2t^2}}dt = o(\left[\frac{1}{2}|t|^{-2}\right]_{-\infty}^{V}) = o(\frac{1}{3}v^{-2})$. Therefore, $\Phi(\frac{1}{V})\cdot\frac{1}{V^2} \to 0$, which implies

$H(V) \to 0$ when $V \to 0^-$, and $W \to 0$ when $z \to 0$.

Since, H is discontinuous in V=0, no Taylor expansion is possible in a neighbourhood of V = 0.

QED.