Cross-Sectional Effects of Common and Heterogeneous Regressors on Asymptotic Properties of Panel Unit Root Tests

Katsuto Tanaka
Gakushuin University, Tokyo, Japan

Abstract

Our main purpose is to examine cross-sectional effects of regressors on the asymptotic properties of panel unit root tests. For this purpose we compute limiting local powers of various tests for three kinds of error terms: (a) AR (b) MA (c) composite errors. It is found that the existence of common regressors fades away asymptotically. We also prove that the tests that are not asymptotically efficient in the time series case become efficient in the panel case.

Outline of my talk with related references:

1. Panel AR unit root tests

   Moon, Perron and Phillips (2007), Moon and Perron (2008)

2. Panel MA unit root tests

   Tanaka (2017)

3. Panel stationarity tests using error components models

   Hadri (2000), Shin and Snell (2006).

4. Concluding remarks

Panel models to be considered here

$$
\begin{aligned}
\text{Model A:}\quad y_{it} &= \beta_0 + \beta_1 t + \eta_{it} \qquad (i = 1, \ldots, N;\ t = 1, \ldots, T) \\
\text{Model B:}\quad y_{it} &= \beta_{0i} + \beta_1 t + \eta_{it} \\
\text{Model C:}\quad y_{it} &= \beta_0 + \beta_{1i} t + \eta_{it} \\
\text{Model D:}\quad y_{it} &= \beta_{0i} + \beta_{1i} t + \eta_{it}
\end{aligned}
$$

where the error term $\{\eta_{it}\}$ is defined by

$$
\begin{aligned}
\text{AR:}\quad \eta_{it} &= \rho\, \eta_{i,t-1} + \varepsilon_{it}, \qquad H_0 : \rho = 1 \\
\text{MA:}\quad \eta_{it} &= \varepsilon_{it} - \alpha\, \varepsilon_{i,t-1}, \qquad H_0 : \alpha = 1 \\
\text{ECM:}\quad \eta_{it} &= \mu_{it} + \varepsilon_{it},\ \mu_{it} = \mu_{i,t-1} + \xi_{it}, \qquad H_0 : \mathrm{Var}(\xi_{it}) = 0
\end{aligned}
$$

Note: The above models are the same when $N = 1$, tha is, the time series case, but these are different when $N > 1$.

- Panel statistics: Panel statistics belong to a class of double indexed processes $\{S_{NT}\}$. To deal with asymptotics we can use

- Joint limit $S_{NT} \overset{N,T\to\infty}{\Longrightarrow} S$

- Sequential limit $S_{NT} \overset{T\to\infty}{\Longrightarrow} S_N \overset{N\to\infty}{\Longrightarrow} S$

We would like to observe the intermediate situation S_N rather than the extreme end S to examine the cross-sectional effects caused by N.

Note that the joint limit usually requires stronger conditions than the sequential limit [Phillips and Moon (1999)].

1. Panel AR models

$$
\begin{aligned}
\text{Model A:}\quad y_{it} &= \beta_0 + \beta_1\, t + \eta_{it} \\
\text{Model B:}\quad y_{it} &= \beta_{0i} + \beta_1\, t + \eta_{it} \\
\text{Model C:}\quad y_{it} &= \beta_0 + \beta_{1i}\, t + \eta_{it} \\
\text{Model D:}\quad y_{it} &= \beta_{0i} + \beta_{1i}\, t + \eta_{it}
\end{aligned}
$$

where the error term $\{\eta_{it}\}$ is defined by

$$\eta_{it} = \rho\, \eta_{i,t-1} + \varepsilon_{it}, \qquad \{\varepsilon_{it}\} \sim \text{i.i.d.}(0,\, \sigma^2).$$

The panel AR unit root test considered here is

$$H_0 \;:\; \rho = 1 \quad \text{vs} \quad H_1 \;:\; \rho < 1\,.$$

We assume that, under H_1, the value of ρ is given by

$$\rho = 1 - \frac{c}{N^\kappa T} = 1 - \frac{c_N}{T}\,, \qquad c_N = \frac{c}{N^\kappa}\,, \qquad (c > 0,\; 0 < \kappa < 1).$$

1.1 OLSE-based tests

The OLSE-based tests for Model M use the OLSE $\widehat{\rho}^{(M)}$ of ρ given by

$$\widehat{\rho}^{(M)} = \frac{\sum_{i=1}^{N} \sum_{t=2}^{T} \widehat{\eta}_{i,t-1}^{(M)} \widehat{\eta}_{it}^{(M)}}{\sum_{i=1}^{N} \sum_{t=2}^{T} \{\widehat{\eta}_{i,t-1}^{(M)}\}^2}, \qquad \widehat{\eta}_{it}^{(M)}\text{: OLS residual from Model M}$$

Theorem 1. As $T \to \infty$ with N fixed under $\rho = 1 - c_N/T$, the asymptotic distribution of $\widehat{\rho}^{(M)}$ in Model M (M=A, B, C, D) follows

$$T(\widehat{\rho}^{(M)} - 1) = \frac{\sum_{i=1}^{N} U_{iT}^{(M)}}{\sum_{i=1}^{N} V_{iT}^{(M)}} \Rightarrow Q_N^{(M)} = \frac{\sum_{i=1}^{N-1} U_i^{(M)} + U_N^{(D)}}{\sum_{i=1}^{N-1} V_i^{(M)} + V_N^{(D)}},$$

$$
\begin{aligned}
U_i^{(A)} &= \int_0^1 Y_i(r)\, dY_i(r)\,, \quad V_i^{(A)} = \int_0^1 Y_i^2(r)\, dr \\
U_i^{(B)} &= \int_0^1 \left(Y_i(r) - \int_0^1 Y_i(s)\, ds \right) dY_i(r) \\
V_i^{(B)} &= \int_0^1 \left(Y_i(r) - \int_0^1 Y_i(s)\, ds \right)^2 dr \\
U_i^{(C)} &= \int_0^1 \left(Y_i(r) - 3r \int_0^1 sY_i(s)\, ds \right) \left(dY_i(r) - 3 \int_0^1 sY_i(s)\, ds\, dr \right) \\
V_i^{(C)} &= \int_0^1 \left(Y_i(r) - 3r \int_0^1 sY_i(s)\, ds \right)^2 dr \\
U_i^{(D)} &= \int_0^1 \left(Y_i(r) - (4 - 6r) \int_0^1 Y_i(s)\, ds - (12r - 6) \int_0^1 sY_i(s)\, ds \right) dY_i(r) \\
V_i^{(D)} &= \int_0^1 \left(Y_i(r) - (4 - 6r) \int_0^1 Y_i(s)\, ds - (12r - 6) \int_0^1 sY_i(s)\, ds \right)^2 dr,
\end{aligned}
$$

where $\{Y_i(r)\}$ is the O-U process defined by

$$
dY_i(r) = -c_N\, Y_i(r)\, dr + dW_i(r), \quad \{W_i(r)\} : \text{standard Brownian motion}
$$

$$T(\widehat{\rho}^{(M)} - 1) = \frac{\sum_{i=1}^{N} U_{iT}^{(M)}}{\sum_{i=1}^{N} V_{iT}^{(M)}} \Rightarrow Q_N^{(M)} = \frac{\sum_{i=1}^{N-1} U_i^{(M)} + U_N^{(D)}}{\sum_{i=1}^{N-1} V_i^{(M)} + V_N^{(D)}},$$

(a) When $N = 1$, that is, in the time series case, the distribution of $Q_N^{(M)}$ reduces to $Q_1^{(D)} = U_1^{(D)}/V_1^{(D)}$ for M=A, B, C, D. Note also that $U_1^{(A)}/V_1^{(A)}$ corresponds to the popular near-unit root distribution associated with the time series model $y_t = \rho\, y_{t-1} + \varepsilon_t$ with $\rho = 1 - c/T$.

(b) As N becomes large, it holds that

$$Q_N^{(M)} = \frac{\sum_{i=1}^{N-1} U_i^{(M)} + U_N^{(D)}}{\sum_{i=1}^{N-1} V_i^{(M)} + V_N^{(D)}} = \frac{\sum_{i=1}^{N-1} U_i^{(M)} + O_p(1)}{\sum_{i=1}^{N-1} V_i^{(M)} + O_p(1)} \approx \frac{\sum_{i=1}^{N} U_i^{(M)}}{\sum_{i=1}^{N} V_i^{(M)}},$$

where the distribution of this last quantity is obtained from Model M without common regressor, which means that the effect of common regressor fades away as N becomes large.

Appendix (pp.9-17) : proof for simpler cases

$$
\begin{aligned}
\text{Model 1: } y_{it} &= \eta_{it}\,, \qquad\qquad (i = 1, \ldots, N;\ t = 1, \ldots, T) \\
\text{Model 2: } y_{it} &= \alpha + \eta_{it}\,, \\
\text{Model 3: } y_{it} &= \alpha_i + \eta_{it}\,,
\end{aligned}
$$

where

$$
\eta_{it} = \rho\, \eta_{i,t-1} + \varepsilon_{it}\,, \qquad \eta_{i0} = 0, \qquad \{\varepsilon_{it}\} \sim \text{ i.i.d.}(0,\, \sigma^2).
$$

OLS estimates of α and α_i

$$
\min_{\alpha} \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{it} - \alpha)^2 \;\rightarrow\; \hat{\alpha} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} y_{it}
$$

$$
\min_{\alpha_1, \ldots, \alpha_N} \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{it} - \alpha_i)^2 \;\rightarrow\; \hat{\alpha}_i = \frac{1}{T} \sum_{t=1}^{T} y_{it} \qquad (i = 1, \ldots, N)
$$

$$
\hat{\eta}_{it}^{(2)} = y_{it} - \hat{\alpha} = \eta_{it} - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \eta_{it}, \qquad \hat{\eta}_{it}^{(3)} = y_{it} - \hat{\alpha}_i = \eta_{it} - \frac{1}{T} \sum_{t=1}^{T} \eta_{it}\,.
$$

$$\widehat{\eta}_{it}^{(2)} = y_{it} - \widehat{\alpha} = \eta_{it} - \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} \eta_{it}, \qquad \widehat{\eta}_{it}^{(3)} = y_{it} - \widehat{\alpha}_i = \eta_{it} - \frac{1}{T}\sum_{t=1}^{T} \eta_{it}\,,$$

$$\frac{1}{\sqrt{T}\sigma}\widehat{\eta}_{i[Tr]}^{(2)} \;\Rightarrow\; Y_i(r) - \frac{1}{N}\sum_{i=1}^{N}\int_0^1 Y_i(s)\,ds, \qquad \frac{1}{\sqrt{T}\sigma}\widehat{\eta}_{i[Tr]}^{(3)} \;\Rightarrow\; Y_i(r) - \int_0^1 Y_i(s)\,ds\,.$$

Model 1:

$$\widehat{\rho}^{(1)} = \frac{\sum_{i=1}^{N}\sum_{t=2}^{T}\eta_{i,t-1}\eta_{it}}{\sum_{i=1}^{N}\sum_{t=2}^{T}\eta_{i,t-1}^2} \qquad \left(\frac{1}{\sqrt{T}\sigma}\eta_{i[Tr]} \;\Rightarrow\; Y_i(r)\right)\,,$$

$$\begin{aligned} T(\widehat{\rho}^{(1)} - 1) \;&=\; \frac{1}{T\sigma^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\eta_{i,t-1}(\eta_{it} - \eta_{i,t-1}) \Big/ \frac{1}{T^2\sigma^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\eta_{i,t-1}^2 \\ &\Rightarrow\; \frac{\sum_{i=1}^{N}\int_0^1 Y_i(r)\,dY_i(r)}{\sum_{i=1}^{N}\int_0^1 Y_i^2(r)\,dr} = \frac{\sum_{i=1}^{N} U_i^{(1)}}{\sum_{i=1}^{N} V_i^{(1)}}\,. \end{aligned}$$

Model 2:

$$\widehat{\rho}^{(2)} = \frac{\sum_{i=1}^{N} \sum_{t=2}^{T} \widehat{\eta}_{i,t-1}^{(2)} \widehat{\eta}_{it}^{(2)}}{\sum_{i=1}^{N} \sum_{t=2}^{T} \left\{\widehat{\eta}_{i,t-1}^{(2)}\right\}^2} \quad \left(\frac{1}{\sqrt{T}\sigma} \widehat{\eta}_{i[Tr]}^{(2)} \Rightarrow Y_i(r) - \frac{1}{N} \sum_{i=1}^{N} \int_0^1 Y_i(s)\, ds \right),$$

$$\begin{aligned} T(\widehat{\rho}^{(2)} - 1) &= \frac{1}{T\sigma^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \widehat{\eta}_{i,t-1}^{(2)} (\widehat{\eta}_{it}^{(2)} - \widehat{\eta}_{i,t-1}^{(2)}) \Big/ \frac{1}{T^2\sigma^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \left\{\widehat{\eta}_{i,t-1}^{(2)}\right\}^2 \\ &= \sum_{i=1}^{N} U_{iT}^{(2)} \Big/ \sum_{i=1}^{N} V_{iT}^{(2)} . \end{aligned}$$

$$
\begin{aligned}
\sum_{i=1}^{N} U_{iT}^{(2)} &= \sum_{i=1}^{N} \frac{1}{2T\sigma^2} \left[\left\{ \widehat{\eta}_{iT}^{(2)} \right\}^2 - \left\{ \widehat{\eta}_{i1}^{(2)} \right\}^2 - \sum_{t=2}^{T} \left(\widehat{\eta}_{it}^{(2)} - \widehat{\eta}_{i,t-1}^{(2)} \right)^2 \right] \\
&\Rightarrow \sum_{i=1}^{N} U_{i}^{(2)} = \frac{1}{2} \sum_{i=1}^{N} \left[\left(Y_i(1) - \frac{1}{N} \sum_{k=1}^{N} \int_0^1 Y_k(s)\, ds \right)^2 \right. \\
&\qquad \left. - \left(\frac{1}{N} \sum_{k=1}^{N} \int_0^1 Y_k(s)\, ds \right)^2 - 1 \right], \\
\sum_{i=}^{N} V_{iT}^{(2)} &= \frac{1}{T^2\sigma^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \left\{ \widehat{\eta}_{i,t-1}^{(2)} \right\}^2 \\
&\Rightarrow \sum_{i=1}^{N} V_{i}^{(2)} = \sum_{i=1}^{N} \int_0^1 \left(Y_i(r) - \frac{1}{N} \sum_{k=1}^{N} \int_0^1 Y_k(s)\, ds \right)^2 dr\,.
\end{aligned}
$$

$$\sum_{i=1}^{N} U_i^{(2)} = \frac{1}{2}\sum_{i=1}^{N}(Y_i^2(1)-1) - \frac{1}{N}\sum_{i=1}^{N} Y_i(1)\sum_{k=1}^{N}\int_0^1 Y_k(s)\,ds\,,$$

$$\sum_{i=1}^{N} V_i^{(2)} = \sum_{i=1}^{N}\int_0^1 Y_i^2(r)\,dr - \frac{1}{N}\left(\sum_{k=1}^{N}\int_0^1 Y_k(s)\,ds\right)^2 .$$

Define the $N\times N$ orthogonal matrix H with the N-th row being $\boldsymbol{i}_N'/\sqrt{N} = (1, \ldots, 1)/\sqrt{N}$, that is,

$$H = \begin{pmatrix} H_1 \\ \boldsymbol{i}_N'/\sqrt{N} \end{pmatrix}, \quad HH' = I_N, \quad H\boldsymbol{i}_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{N} \end{pmatrix},$$

and put $\boldsymbol{Z}(r) = H\boldsymbol{Y}(r)$, where $\boldsymbol{Z}(r) = (Z_1(r), \ldots, Z_N(r))'$ and $\boldsymbol{Y}(r) = (Y_1(r), \ldots, Y_N(r))'$.

$$
\begin{aligned}
\sum_{i=1}^{N} U_i^{(2)} &\stackrel{\mathcal{D}}{=} \frac{1}{2}\sum_{i=1}^{N-1}(Z_i^2(1)-1)+\frac{1}{2}(Z_N^2(1)-1)-Z_N(1)\int_0^1 Z_N(s)\,ds, \\
&= \sum_{i=1}^{N-1}\int_0^1 Z_i(r)\,dZ_i(r)+\int_0^1\left(Z_N(r)-\int_0^1 Z_N(s)\,ds\right)dZ_N(r), \\
\sum_{i=1}^{N} V_i^{(2)} &\stackrel{\mathcal{D}}{=} \sum_{i=1}^{N-1}\int_0^1 Z_i^2(r)\,dr+\int_0^1 Z_N^2(r)\,dr-\left(\int_0^1 Z_N(s)\,ds\right)^2 \\
&= \sum_{i=1}^{N-1}\int_0^1 Z_i^2(r)\,dr+\int_0^1\left(Z_N(r)-\int_0^1 Z_N(s)\,ds\right)^2 dr,
\end{aligned}
$$

which yields

$$
T(\widehat{\rho}^{(2)}-1) \Rightarrow \frac{\sum_{i=1}^{N-1} U_i^{(1)}+U_N^{(3)}}{\sum_{i=1}^{N-1} V_i^{(1)}+V_N^{(3)}}.
$$

Model 3:

$$\widehat{\rho}^{(3)} = \frac{\sum_{i=1}^{N} \sum_{t=2}^{T} \widehat{\eta}_{i,t-1}^{(3)} \widehat{\eta}_{it}^{(3)}}{\sum_{i=1}^{N} \sum_{t=2}^{T} \left\{\widehat{\eta}_{i,t-1}^{(3)}\right\}^2} \qquad \left(\frac{1}{\sqrt{T}\sigma} \widehat{\eta}_{i[Tr]}^{(3)} \Rightarrow Y_i(r) - \int_0^1 Y_i(s)\, ds \right),$$

$$\begin{aligned} T(\widehat{\rho}^{(3)} - 1) &= \frac{1}{T\sigma^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \widehat{\eta}_{i,t-1}^{(3)} (\widehat{\eta}_{it}^{(3)} - \widehat{\eta}_{i,t-1}^{(3)}) \Big/ \frac{1}{T^2\sigma^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \left\{\widehat{\eta}_{i,t-1}^{(3)}\right\}^2 \\ &= \sum_{i=1}^{N} U_{iT}^{(3)} \Big/ \sum_{i=1}^{N} V_{iT}^{(3)}. \end{aligned}$$

$$
\begin{aligned}
\sum_{i=1}^{N} U_{iT}^{(3)} &= \sum_{i=1}^{N} \frac{1}{2T\sigma^2} \left[\left\{ \widehat{\eta}_{iT}^{(3)} \right\}^2 - \left\{ \widehat{\eta}_{i1}^{(3)} \right\}^2 - \sum_{t=2}^{T} \left(\widehat{\eta}_{it}^{(3)} - \widehat{\eta}_{i,t-1}^{(3)} \right)^2 \right] \\
&\Rightarrow \sum_{i=1}^{N} U_i^{(3)} = \frac{1}{2} \sum_{i=1}^{N} \left[\left(Y_i(1) - \int_0^1 Y_k(s)\, ds \right)^2 \right. \\
&\qquad \left. - \left(\int_0^1 Y_k(s)\, ds \right)^2 - 1 \right], \\
\sum_{i=}^{N} V_{iT}^{(3)} &= \frac{1}{T^2\sigma^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \left\{ \widehat{\eta}_{i,t-1} \right\}^2 \\
&\Rightarrow \sum_{i=1}^{N} V_i^{(3)} = \sum_{i=1}^{N} \int_0^1 \left(Y_i(r) - \int_0^1 Y_k(s)\, ds \right)^2 dr\,.
\end{aligned}
$$

$$\sum_{i=1}^{N} U_i^{(3)} = \frac{1}{2}\sum_{i=1}^{N}\left[\left(Y_i(1) - \int_0^1 Y_k(s)\,ds\right)^2 - \left(\int_0^1 Y_k(s)\,ds\right)^2 - 1\right]$$

$$= \sum_{i=1}^{N}\int_0^1 \left(Y_i(r) - \int_0^1 Y_k(s)\,ds\right) dY_i(r)\,,$$

$$\sum_{i=1}^{N} V_i^{(3)} = \sum_{i=1}^{N}\int_0^1 \left(Y_i(r) - \int_0^1 Y_k(s)\,ds\right)^2 dr\,,$$

which yields

$$T(\hat{\rho}^{(3)} - 1) \Rightarrow \frac{\sum_{i=1}^{N} U_i^{(3)}}{\sum_{i=1}^{N} V_i^{(3)}} = \frac{\sum_{i=1}^{N-1} U_i^{(3)} + U_N^{(3)}}{\sum_{i=1}^{N-1} V_i^{(3)} + V_N^{(3)}}\,.$$

- Computation of the distribution of $Q_N^{(M)} = \dfrac{\sum_{i=1}^{N-1} U_i^{(M)} + U_N^{(D)}}{\sum_{i=1}^{N-1} V_i^{(M)} + V_N^{(D)}}$

$$
\begin{aligned}
P\left(Q_N^{(M)} \le z\right) &= P\left(z\Big(\sum_{i=1}^{N-1} V_i^{(M)} + V_N^{(D)}\Big) - \sum_{i=1}^{N-1} U_i^{(M)} - U_N^{(D)} \ge 0\right) \\
&= P\left(\sum_{i=1}^{N-1} \left(zV_i^{(M)} - U_i^{(M)}\right) + zV_N^{(D)} - U_N^{(D)} \ge 0\right) \\
&= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \frac{1}{\theta} \mathrm{Im}\left[\left\{m^{(M)}(-i\theta,\, i\theta z)\right\}^{N-1} m^{(D)}(-i\theta,\, i\theta z)\right] d\theta \,,
\end{aligned}
$$

where

$$
m^{(M)}(x, y) = \mathrm{E}\left[\exp\left\{x\, U_i^{(M)} + y\, V_i^{(M)}\right\}\right] .
$$

The expressions for $m^{(M)}(x, y)$ are available from Tanaka (2017, Chap. 10) for M=A, B, C, and D. Numerical computation like Simpson's formula can be used by taking care of the computation of square roots of complex-valued quantities.

- Powers of the OLSE-based tests

$$P\left(Q_N^{(M)} < z_\gamma\right) \quad \text{(Powers for finite } N\text{)}$$

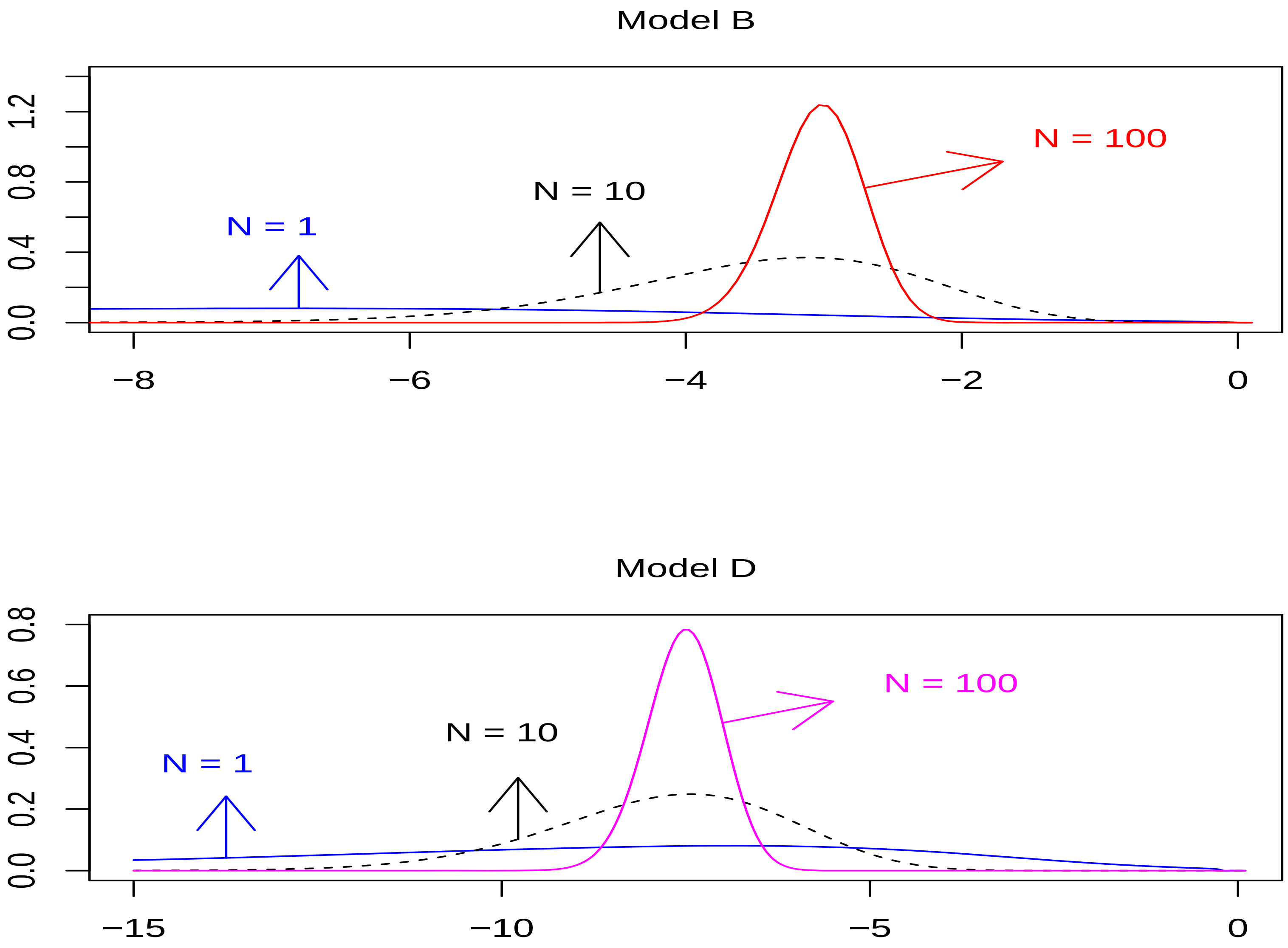
Theorem 2. (Powers for $N = \infty$) The limiting local powers of the tests based on $Q_N^{(M)}$ (M=A, B, C, D) as $N \to \infty$ under $\rho = 1 - c/(N^\kappa T)$ at the $100\gamma\%$ level are given as follows:

$$P\left(\sqrt{\frac{N}{2}} Q_N^{(A)} < z_\gamma\right) \to \Phi(z_\gamma + 0.707\, c), \quad (\kappa = 1/2)$$

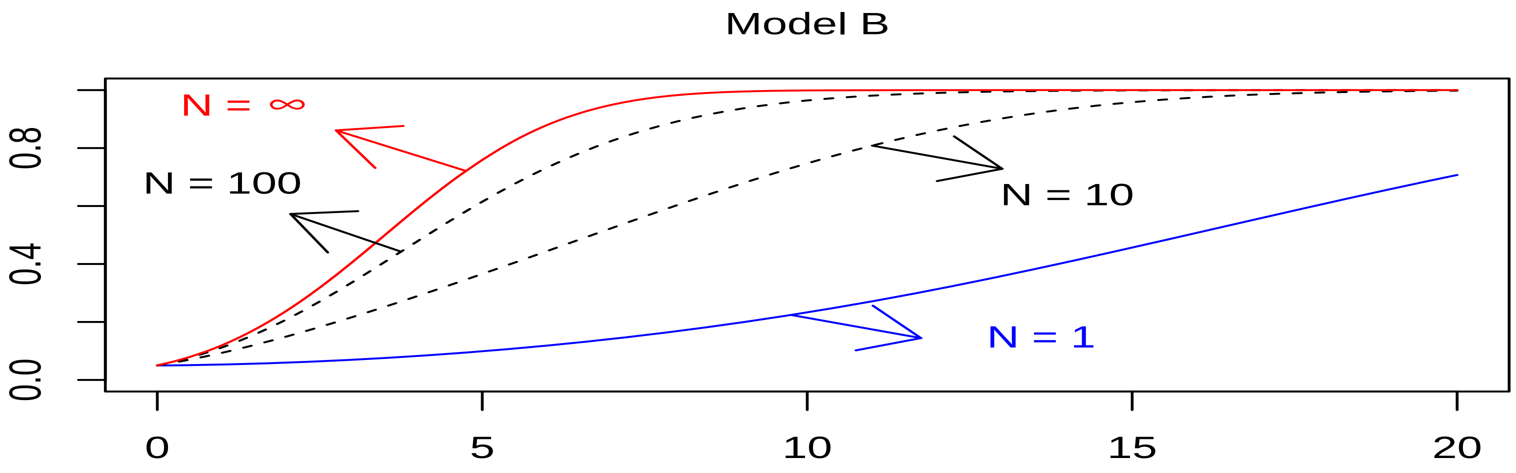
$$P\left(\sqrt{\frac{5N}{51}} (Q_N^{(B)} + 3) < z_\gamma\right) \to \Phi(z_\gamma + 0.470\, c), \quad (\kappa = 1/2)$$

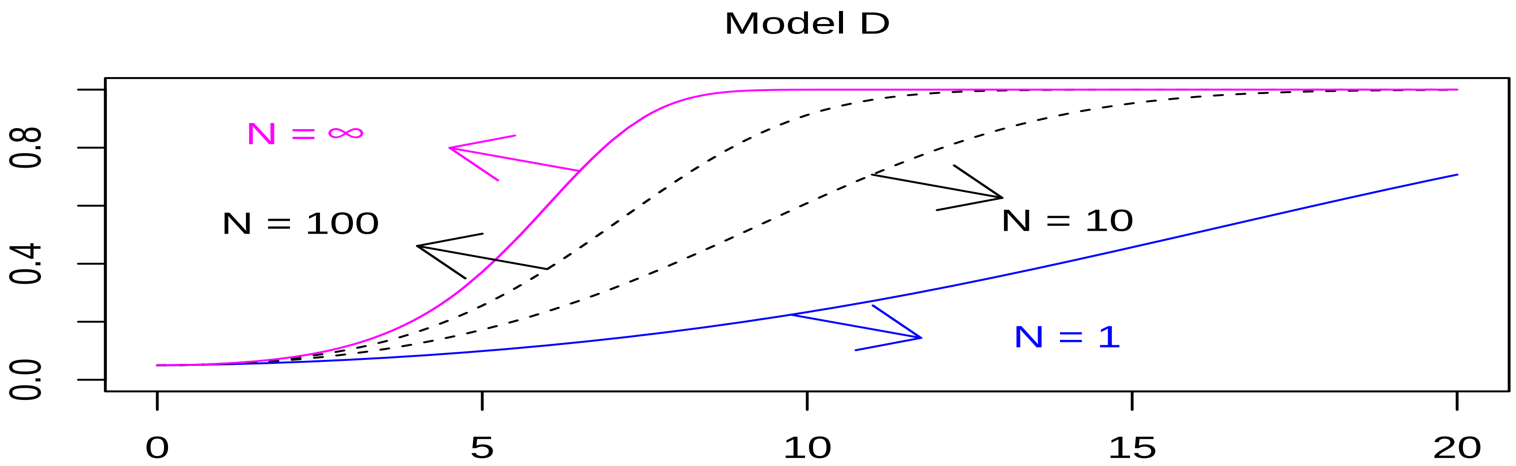
$$P\left(\sqrt{\frac{7N}{110}} (Q_N^{(C)} + 4) < z_\gamma\right) \to \Phi\left(z_\gamma + 0.0721\, c^2\right), \quad (\kappa = 1/4)$$

$$P\left(\sqrt{\frac{112N}{2895}} \left(Q_N^{(D)} + \frac{15}{2}\right) < z_\gamma\right) \to \Phi\left(z_\gamma + 0.0527\, c^2\right), \quad (\kappa = 1/4).$$



Null densities of the OLSE-based statistics





Local powers of the OLSE-based tests

1.2 GLSE-based tests

The GLSE-based tests use the GLSE $\tilde{\rho}^{(M)}$ of ρ given by

$$\tilde{\rho} = \frac{\sum_{i=1}^{N} \sum_{t=2}^{T} \tilde{\eta}_{i,t-1}\tilde{\eta}_{it}}{\sum_{i=1}^{N} \sum_{t=2}^{T} \tilde{\eta}_{i,t-1}^{2}}, \qquad \tilde{\eta}_{it} : \text{GLS residual from M}$$

The GLSE-based tests are LBIU under the normality assumption.

Theorem 3. As $T \to \infty$ with N fixed under $\rho = 1 - c_N/T$, the asymptotic distribution of $\tilde{\rho}^{(M)}$ for Model M (M=A, B, C, D) follows

$$T(\tilde{\rho}^{(M)} - 1) \;\Rightarrow\; R_N^{(M)} = \frac{\sum_{i=1}^{N-1} W_i^{(M)} + W_N^{(D)}}{\sum_{i=1}^{N-1} X_i^{(M)} + X_N^{(D)}},$$

where

$$W_i^{(A)} = W_i^{(B)} = \int_0^1 Y_i(r)\, dY_i(r)\,, \qquad X_i^{(A)} = X_i^{(B)} = \int_0^1 Y_i^2(r)\, dr\,,$$

$$W_i^{(C)} = W_i^{(D)} = -\frac{1}{2}, \qquad X_i^{(C)} = X_i^{(D)} = \int_0^1 (Y_i(r) - rY_i(1))^2\, dr\,.$$

- Powers of the GLSE-based tests

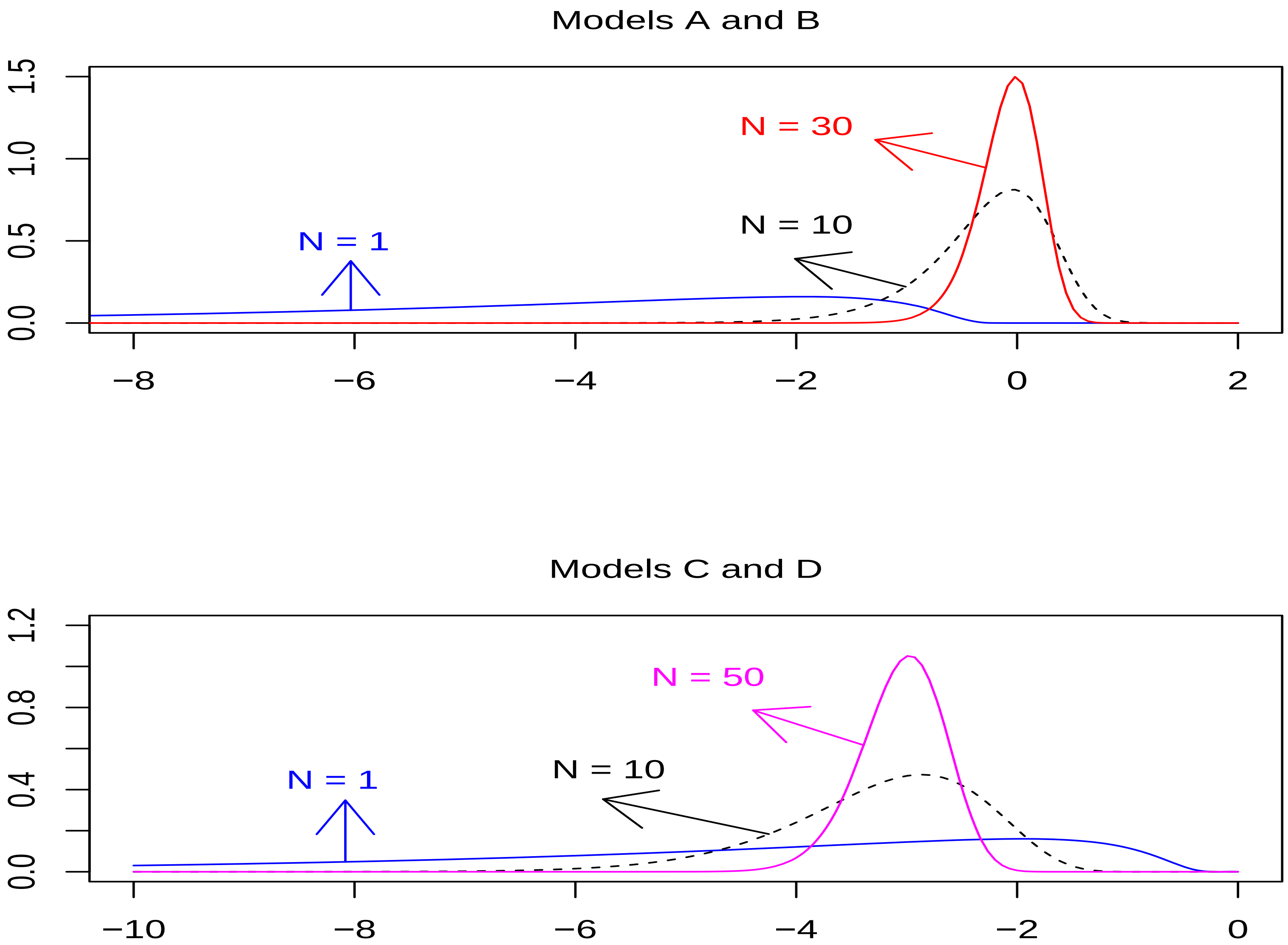
$$P\Big(R_N^{(M)} < z_\gamma\Big) \quad \text{(Powers for finite } N\text{)}$$

Theorem 4. (Powers for $N = \infty$) The limiting powers of the tests based on $R_N^{(M)}$ (M=A, B, C, D) as $N \to \infty$ under $\rho = 1 - c/(N^\kappa T)$ at the $100\gamma\%$ level are given as follows:

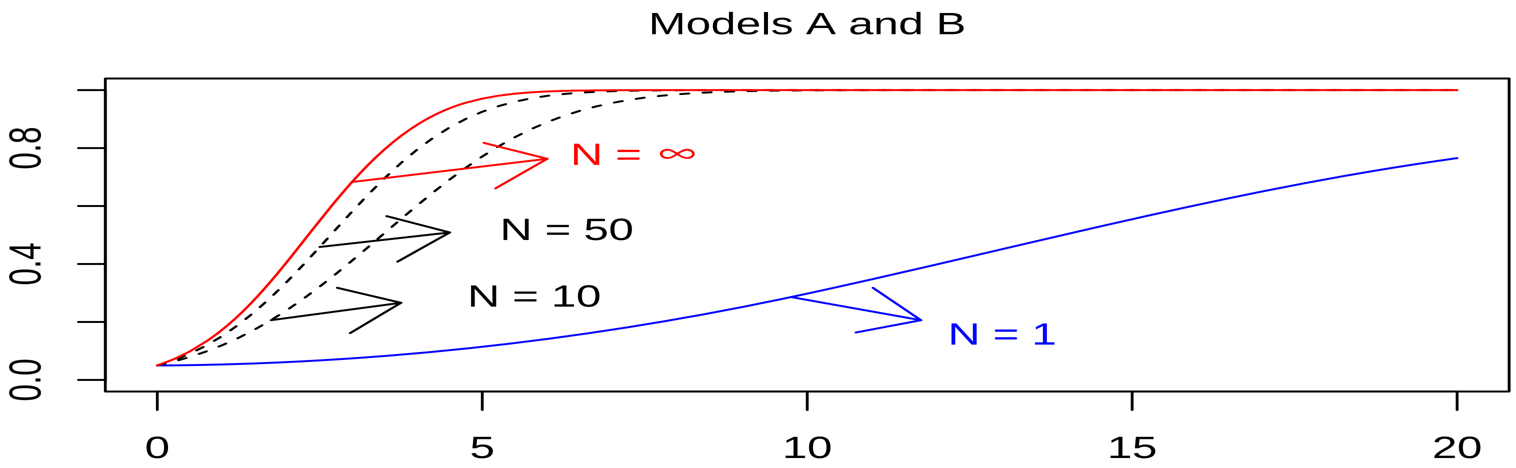
$$P\left(\sqrt{\frac{N}{2}}\, R_N^{(M)} < z_\gamma\right) \to \Phi(z_\gamma + 0.707\, c)\,, \quad (\text{M=A, B})\,,$$

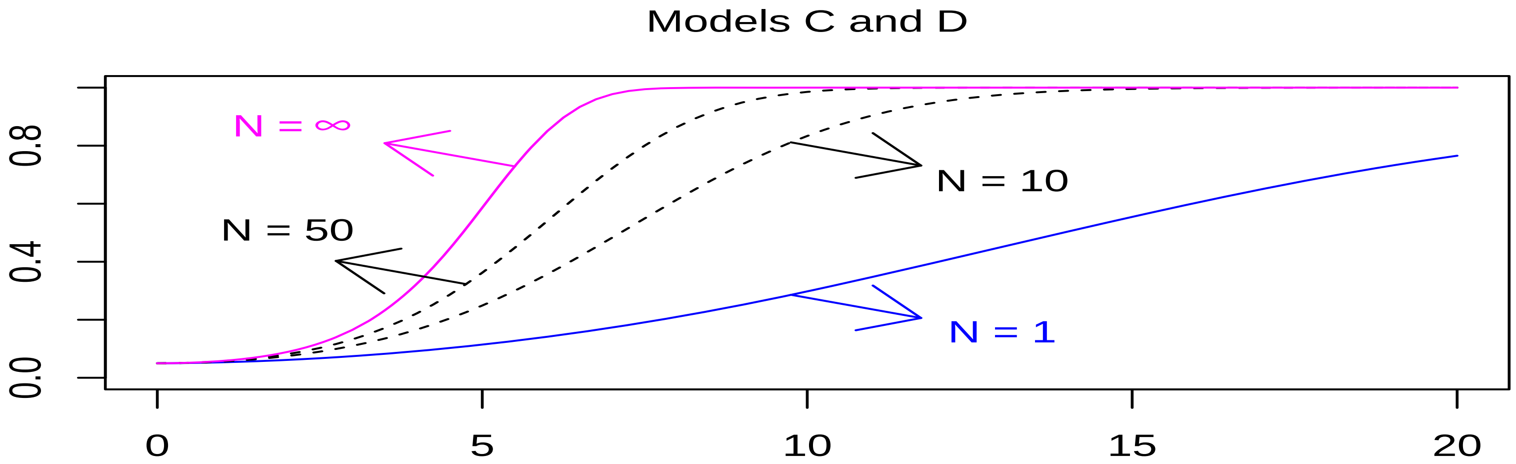
$$P\left(\frac{\sqrt{5N}}{6}(R_N^{(M)} + 3) < z_\gamma\right) \to \Phi(z_\gamma + 0.0745\, c^2)\,, \quad (\text{M=C, D})\,,$$

where $\kappa = 1/2$ for Models A and B, and $\kappa = 1/4$ for Models C and D.



Null densities of the GLSE-based statistics





Local powers of the GLSE-based tests

1.3 Asymptotic efficiency of the GLSE-based tests

Let us consider the testing problem

$$H_0 : \rho = 1 \quad \text{versus} \quad H_1 : \rho = 1 - \frac{\theta_N}{T} = \rho_\theta ,$$

where $\theta_N = \theta/N^\kappa$ with θ being a known positive constant. We assume that the true value of ρ under H_1 is given by $\rho_c = 1 - c_N/T$ with $c_N = c/N^\kappa$. Assuming $\{\varepsilon_{it}\} \sim \text{NID}(0, \sigma^2)$, the Neyman-Pearson lemma tells us that the test rejects H_0 for small values of

$$S_{NT}^{(M)}(\theta) = \frac{\sum_{i=1}^N \sum_{t=1}^T \left[\left(\tilde{\eta}_{it}^{(M)}(1) - \rho_\theta \tilde{\eta}_{i,t-1}^{(M)}(1) \right)^2 - \left(\tilde{\eta}_{it}^{(M)}(0) - \tilde{\eta}_{i,t-1}^{(M)}(0) \right)^2 \right]}{\frac{1}{T} \sum_{i=1}^N \sum_{t=1}^T \left(\tilde{\eta}_{it}^{(M)}(0) - \tilde{\eta}_{i,t-1}^{(M)}(0) \right)^2}$$

is MPI, where $\tilde{\eta}_{it}^{(M)}(0)$ and $\tilde{\eta}_{it}^{(M)}(1)$ are the GLS residuals obtained from Model M under H_0 and H_1, respectively.

Theorem 5. The limiting power envelopes of the MPI tests for Model M (M=A, B, C, D) as $T \to \infty$ and $N \to \infty$ at level γ coincide with the corresponding GLSE-based tests. More sepcifically, we have

$$P\left(\sqrt{\frac{N}{2}}\left(\frac{S_N^{(M)}(\theta)}{\theta_N} - \frac{1}{2}\theta_N\right) < z_\gamma\right) \to \Phi\left(z_\gamma + 0.707c\right), \quad \text{(M=A, B)},$$

$$P\left(3\sqrt{5N}\left(\frac{S_N^{(M)}(\theta) + \theta_N}{\theta_N^2} - \frac{1}{6} + \frac{1}{45}\theta_N^2\right) < z_\gamma\right)$$

$$\to \Phi(z_\gamma + 0.0745c^2), \quad \text{(M=C, D)},$$

where $\kappa = 1/2$ for Models A and B, and $\kappa = 1/4$ for Models C and D.

Note: The GLSE-based tests for $N = 1$ (time series case) are not asymptotically efficient [Tanaka (1996)].

2. Panel MA models

$$\begin{aligned}
\text{Model A:}\quad y_{it} &= \beta_0 + \beta_1 t + \eta_{it}, \qquad \eta_{it} = \varepsilon_{it} - \alpha\, \varepsilon_{i,t-1} \\
\text{Model B:}\quad y_{it} &= \beta_{0i} + \beta_1 t + \eta_{it}, \qquad \{\varepsilon_{it}\} \sim \text{NID}(0, \sigma^2) \\
\text{Model C:}\quad y_{it} &= \beta_0 + \beta_{1i} t + \eta_{it} \\
\text{Model D:}\quad y_{it} &= \beta_{0i} + \beta_{1i} t + \eta_{it}.
\end{aligned}$$

The testing problem is

$$H_0 : \alpha = 1 \quad \text{versus} \quad H_1 : \alpha < 1 \quad \left(\alpha = 1 - c/(N^{1/4}T)\right)$$

We assume here that $\{\varepsilon_{it}\} \sim \text{NID}(0, \sigma^2)$ for $i = 1, \ldots, N$ and $t = 0, 1, \ldots, T$, but the asymptotic results depend on the assumption on ε_{i0}.

2.1 LBIU tests and limiting local powers

The test which rejects H_0 for large values of

$$R_{NT}^{(M)} = \frac{1}{T^2\tilde{\sigma}^2} \sum_{i=1}^{N} \tilde{\boldsymbol{\eta}}_i' \Omega^{-2} \tilde{\boldsymbol{\eta}}_i$$

is LBIU, where $\Omega = \mathrm{Var}(\boldsymbol{\eta}_i)/\sigma^2$ under H_0 and

$\boldsymbol{\eta}_i$: $T \times 1$ error vector of i-th cross section

$\tilde{\boldsymbol{\eta}}_i$: $T \times 1$ residual vector computed from the MLE under H_0

$\tilde{\sigma}^2$: MLE of σ^2 computed under H_0

Theorem 6. It holds that, as $T \to \infty$ under $\alpha = 1 - c_N/T$ for each N,

$$R_{NT}^{(M)} \Rightarrow R_N^{(M)} = \sum_{i=1}^{N-1} U_i^{(M)} + U_N^{(D)}, \qquad \text{(M=A, B, C, D)},$$

where

$$U_i^{(M)} = \int_0^1 \int_0^1 \left[K^{(M)}(r,s) + c_N^2 K_{(2)}^{(M)}(r,s) \right] dW_i(r)\, dW_i(s)\,.$$

Here $\boldsymbol{W}(r) = (W_1(r), \ldots, W_N(r))'$ is the N-dimensional standard Brownian motion and

$$\begin{aligned} K^{(A)}(r,s) &= \min(r,s) - rs\,, \quad K^{(B)}(r,s) = \min(r,s) - rs - 3rs(1-r)(1-s) \\ K^{(C)}(r,s) &= \min(r,s) - rs - \frac{5}{4} rs(1-r^2)(1-s^2)\,, \\ K^{(D)}(r,s) &= \min(r,s) - rs - 2rs(1-r)(1-s)(4-5r-5s+10rs)\,, \end{aligned}$$

whereas $K_{(2)}^{(M)}(r,s)$ is the iterated kernel of $K^{(M)}(r,s)$ defined by

$$K_{(2)}^{(M)}(r,s) = \int_0^1 \int_0^1 K^{(M)}(r,u) K^{(M)}(u,s)\, du\,.$$

- Powers of the LBIU tests

$$P\left(R_N^{(M)} > z_{1-\gamma}\right) \qquad \text{(Powers for finite } N\text{)}$$

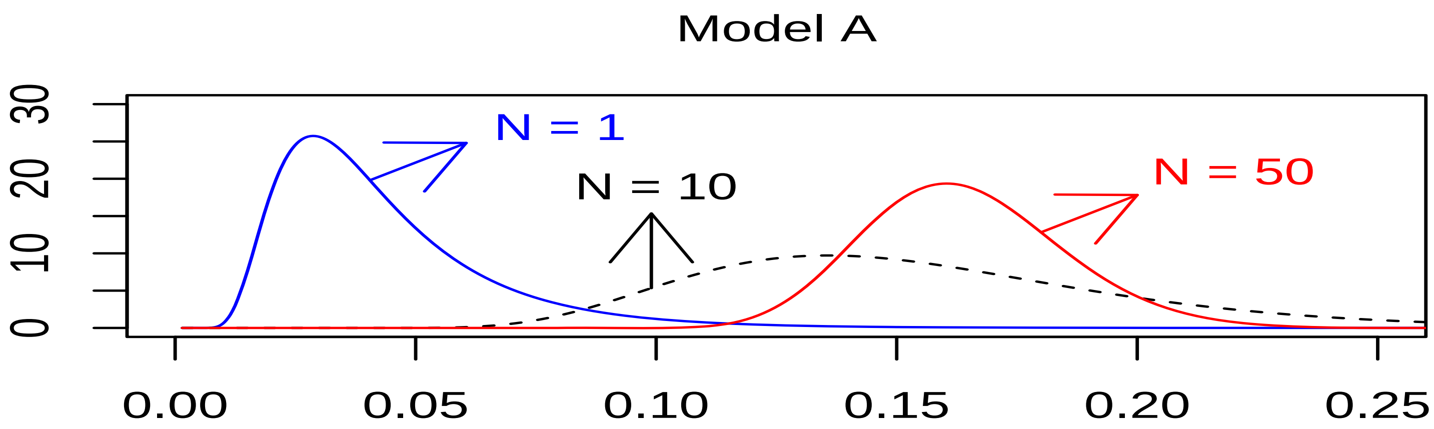
Theorem 7. (Powers for $N = \infty$) The limiting powers of the LBIU tests as $N \to \infty$ under $c_N = c/N^{1/4}$ at the $100\gamma\%$ level are given as follows:

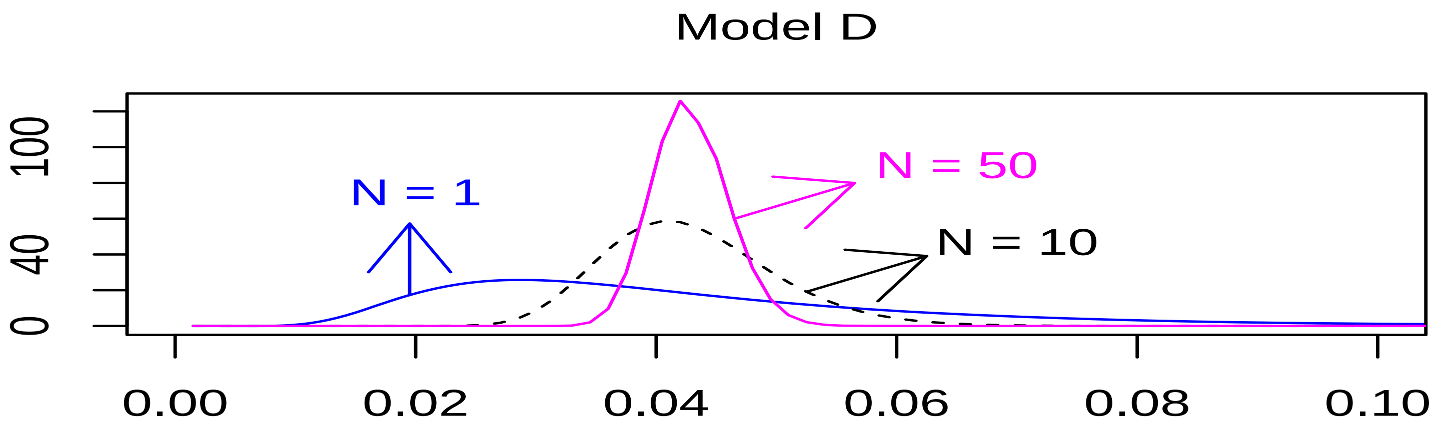
$$P\left(\sqrt{\frac{45}{N}}\left(R_N^{(A)} - \frac{N}{6}\right) > z_{1-\gamma}\right) \to \Phi(z_\gamma + 0.0745\, c^2),$$

$$P\left(\sqrt{\frac{6300}{11N}}\left(R_N^{(B)} - \frac{N}{15}\right) > z_{1-\gamma}\right) \to \Phi(z_\gamma + 0.0209\, c^2),$$

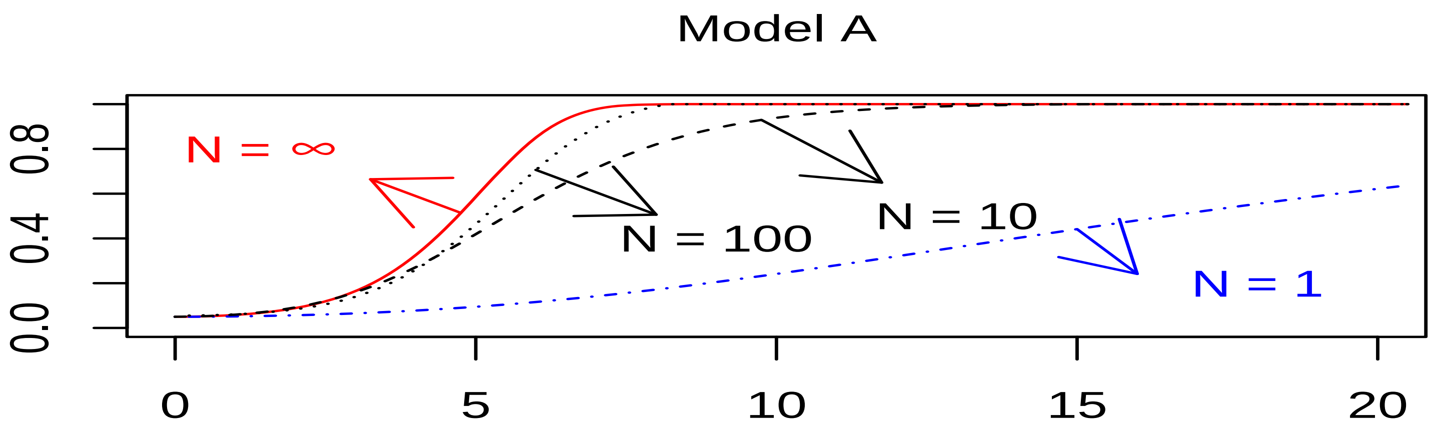
$$P\left(\frac{21}{\sqrt{N}}\left(R_N^{(C)} - \frac{N}{14}\right) > z_{1-\gamma}\right) \to \Phi(z_\gamma + 0.0238\, c^2),$$

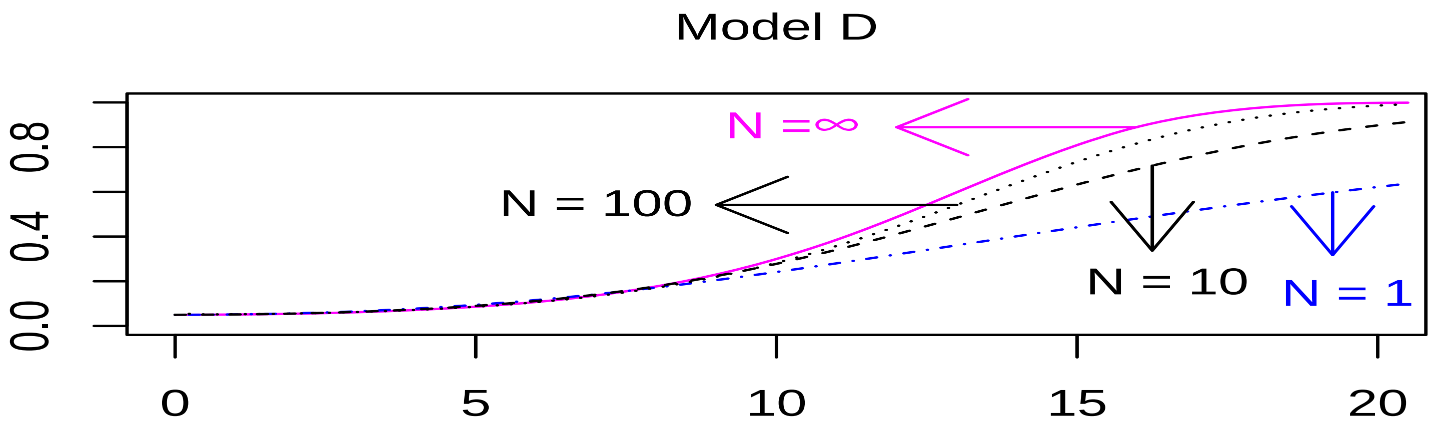
$$P\left(\sqrt{\frac{22050}{11N}}\left(R_N^{(D)} - \frac{3N}{70}\right) > z_{1-\gamma}\right) \to \Phi(z_\gamma + 0.0112\, c^2).$$





Null densities of the LBIU statistics $R_N^{(M)}/N$





Local powers of the LBIU tests based on $R_N^{(M)}$

2.2 Asymptotic efficiency of the LBIU tests

Theorem 8. For Model A, the limiting power envelope of the MPI tests as $T \to \infty$ and $N \to \infty$ at level γ coincides with the corresponding LBIU test.

Note: The proof of the asymptotic efficiency for Models B, C, and D remains to be done. The LBIU tests for $N = 1$ (time series case) are not asymptotically efficient [Tanaka (1996)].

3. Panel stationarity tests for error components models

$$\begin{aligned}
\text{Model E:} \quad y_{it} &= \beta_0 + \beta_1\, t + \eta_{it} \\
\text{Model F:} \quad y_{it} &= \beta_{0i} + \beta_1\, t + \eta_{it} \\
\text{Model G:} \quad y_{it} &= \beta_0 + \beta_{1i}\, t + \eta_{it} \\
\text{Model H:} \quad y_{it} &= \beta_{0i} + \beta_{1i}\, t + \eta_{it},
\end{aligned}$$

where $\eta_{it} = \varepsilon_{it} + \mu_{it},\ \mu_{it} = \mu_{i,t-1} + \xi_{it}$ with $\{\varepsilon_{it}\} \sim \text{NID}(0, \sigma_\varepsilon^2)$ and $\{\xi_{it}\} \sim \text{NID}(0, \sigma_\xi^2)$, $\{\varepsilon_{it}\} \perp \{\xi_{it}\}$, and $\eta_{i0} = 0$.

$$H_0 : \rho = \frac{\sigma_\xi^2}{\sigma_\varepsilon^2} = 0 \quad \text{versus} \quad H_1 : \rho > 0 \quad \left(\rho = c^2/(\sqrt{N}\,T^2)\right)$$

There is a close relationship with MA models, which will be described later.

3.1 LBI tests and limiting local powers

The test which rejects H_0 for large values of

$$S_{NT}^{(M)} = \frac{1}{T^2 \hat{\sigma}_\varepsilon^2} \sum_{i=1}^{N} \sum_{s=1}^{T} \left(\sum_{t=1}^{s} \hat{\eta}_{it}^{(M)} \right)^2$$

is LBI for Model M (M=E, F, G, H), where

$\hat{\eta}_{it}^{(M)}$: OLS residual from Model M

$$\hat{\sigma}_\varepsilon^2 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left\{ \hat{\eta}_{it}^{(M)} \right\}^2 .$$

Theorem 9. It holds that, as $T \to \infty$ under $\rho = c_N^2/T^2$ for each N,

$$S_{NT}^{(M)} \Rightarrow S_N^{(M)} = \sum_{i=1}^{N-1} V_i^{(M)} + V_i^{(H)}, \qquad \text{(M=E, F, G, H)},$$

$$V_i^{(M)} = \int_0^1 \int_0^1 \left[K^{(M)}(r,s) + c_N^2 \, K_{(2)}^{(M)}(r,s) \right] dW_i(r)\, dW_i(s)\,,$$

$$\begin{aligned}
K^{(E)}(r,s) &= 1 - \max(r,s), \\
K^{(F)}(r,s) &= \min(r,s) - rs, \\
K^{(G)}(r,s) &= 1 - \max(r,s) - \frac{3}{4}(1-r^2)(1-s^2), \\
K^{(H)}(r,s) &= \min(r,s) - rs - 3rs(1-r)(1-s).
\end{aligned}$$

It follows that $K^{(F)}(r,s) = K^{(A)}(r,s)$ and $K^{(H)}(r,s) = K^{(B)}(r,s)$.

- Powers of the LBI tests

$$P\left(S_N^{(M)} > z_{1-\gamma}\right) \qquad \text{(Powers for finite } N\text{)}$$

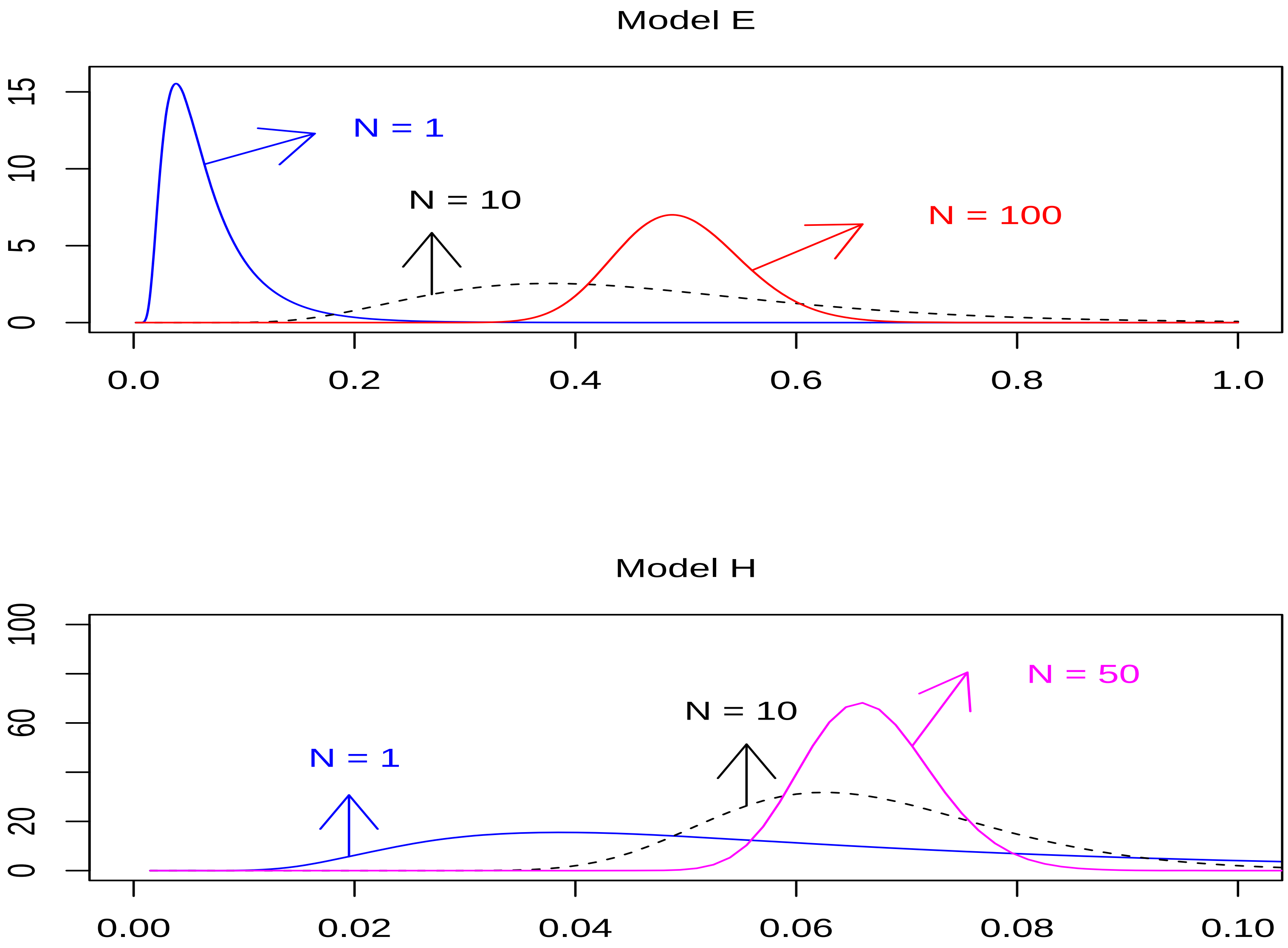
Theorem 10. (Powers for $N = \infty$) The limiting powers of the LBI tests as $N \to \infty$ under $c_N^2 = c^2/N^{1/2}$ at the $100\gamma\%$ level are given as follows:

$$P\left(\sqrt{\frac{3}{N}}\left(S_N^{(E)} - \frac{N}{2}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + 0.2887\,c^2\right),$$

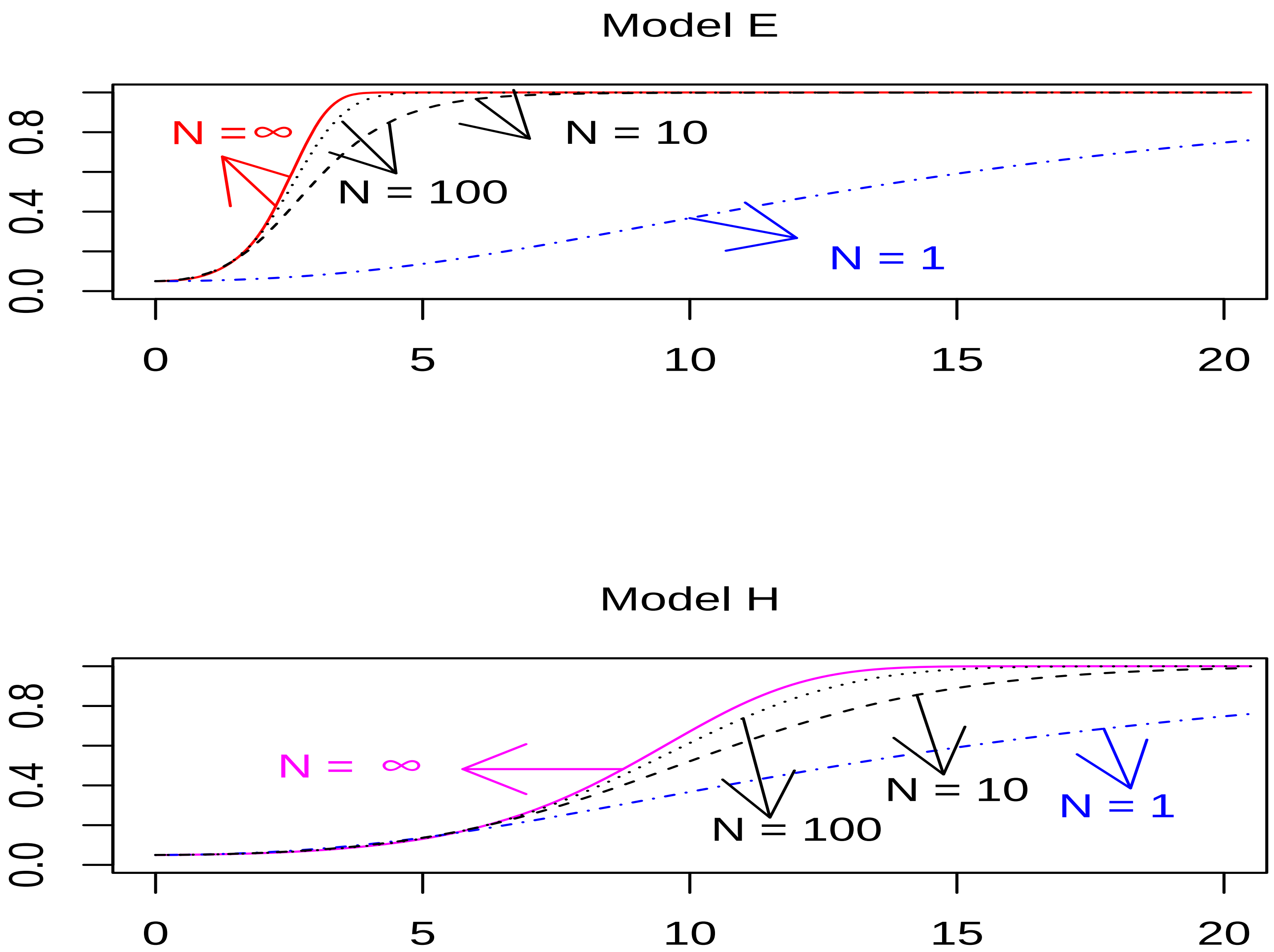
$$P\left(\sqrt{\frac{45}{N}}\left(S_N^{(F)} - \frac{N}{6}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + 0.0745\,c^2\right),$$

$$P\left(\sqrt{\frac{175}{N}}\left(S_N^{(G)} - \frac{N}{10}\right) > z_{1-\gamma}\right) \to \Phi(z_\gamma + 0.0378\,c^2),$$

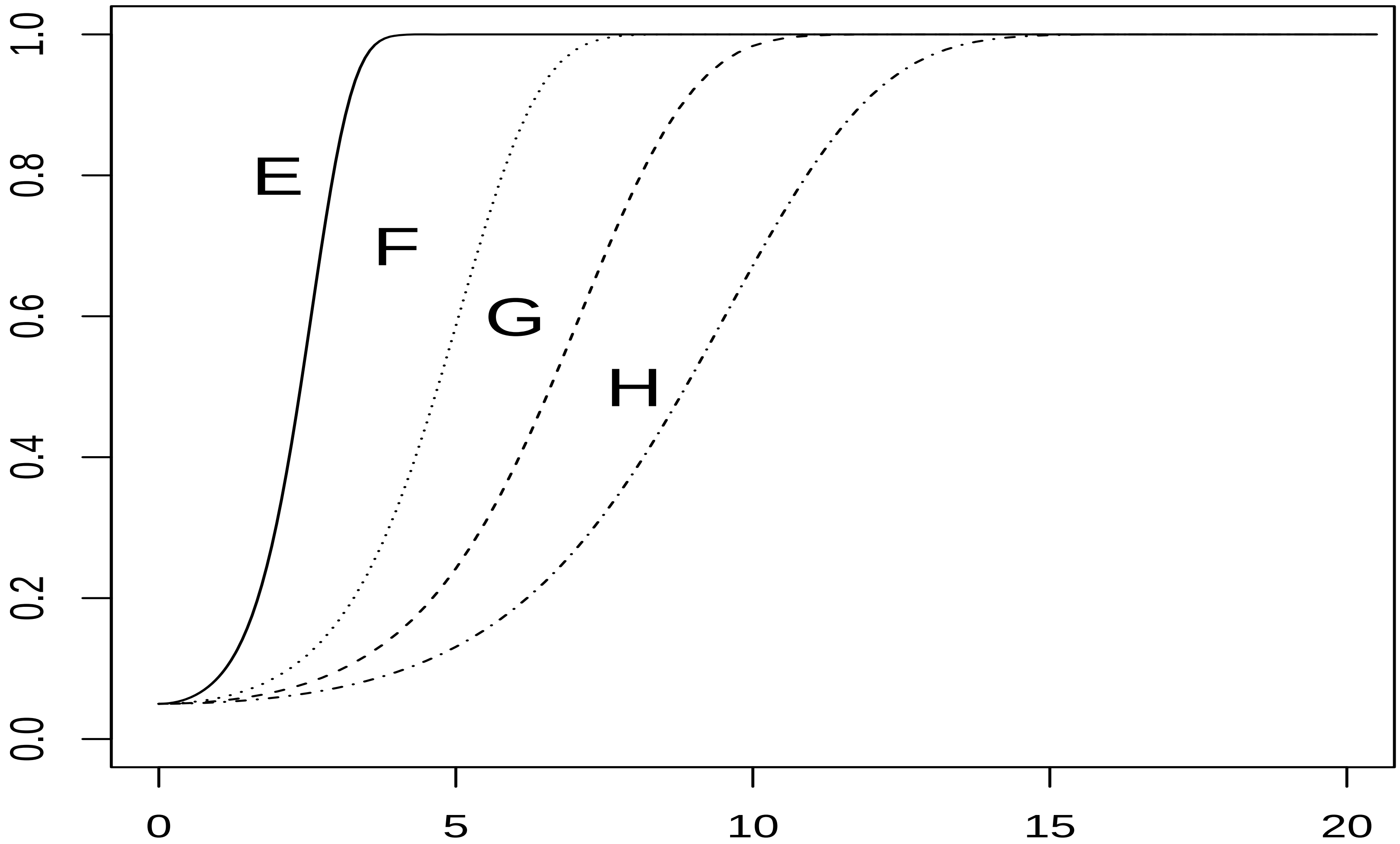
$$P\left(\sqrt{\frac{6300}{11N}}\left(S_N^{(H)} - \frac{N}{15}\right) > z_{1-\gamma}\right) \to \Phi(z_\gamma + 0.0209\,c^2).$$



Null densities of the LBI statistics $S_N^{(M)}/N$



Local powers of the LBI tests based on $S_N^{(M)}$



Limiting power comparison of the LBI tests

3.2 Asymptotic efficiency of the LBI tests

Theorem 11. For Models E and F, the limiting power envelopes of the MPI tests as $T \to \infty$ and $N \to \infty$ at level γ coincide with the corresponding LBI tests.

Note: The proof of the asymptotic efficiency for Models G and H remains to be done. The LBI tests for $N = 1$ (time series case) are not asymptotically efficient [Tanaka (1996)].

4. Concluding remarks

1. The existence of common regressor does not affect the asymptotic property of the tests, although heterogeneous regressor does.

2. The GLSE-based tests for panel AR models are asymptotically efficient, unlike the time series case. The same is true of the LBIU tests for panel MA models and the LBI tests for panel error components models.

3. The performance of the panel unit root tests improves as N becomes large given $T \to \infty$.

4. For MA models and error components models, the assumption on the initial value of the error term in the time series direction affects the asymptotic distribution of test statistics, unlike AR models.