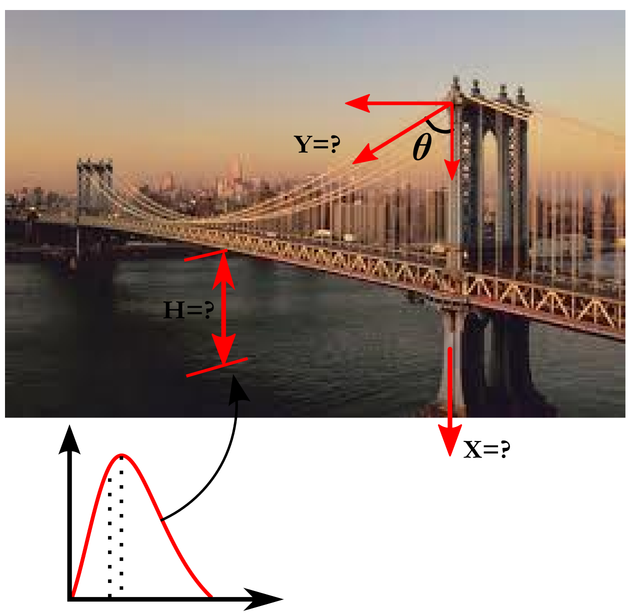
DEPARTMENT OF CIVIL ENGINEERING

The Maths Booklet



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Chapter 1
Introduction

1.1 Purpose

Civil Engineering is a discipline greatly relying on Mathematics. Although you will be having taught Maths modules throughout your course, we must assume that everyone arrives with a certain level of understanding. This booklet has been designed with the purpose of allowing Year 1 students to self-assess their Math skills.

1.2 How to read this booklet

This booklet contains basic A-Level Math material that we expect you to be proficient in at Day 1. We therefore advise you to go through the material presented herein and either

1. Verify that you are able to go through the self-assessment questions. Try the questions and compare your answers with the solutions provided at the end. Please be advised that this is for your own benefit and self-assessment. There is no reason to cheat as you will be only cheating yourself.

2. If you find that you are not able to answer some of the questions, use the background material provided in each chapter to refresh your memory. Use this booklet as a guide to the sort of material you need to cover and to be used in conjunction with a text book. If you encounter difficulties, please do not panic, but try to revise points that you feel you are weak on. Seek advise from your Tutor and your Maths lecturer.

1.3 Acknowledgements

I would like to thank Dr. Barbara Turnbull, Dep. of Civil Engineering and Dr. Tom Wicks, School of Mathematical Sciences for their support and help in writing this booklet.

Chapter 2
Solving Equations

Solving an equation involving the variable x means finding all possible values of x that satisfy the equation. Equations can take different forms and the techniques used in solving them vary with the nature of the equation (polynomial, exponential, logarithmic,...). In this chapter we will only deal with polynomial equations and equations involving absolute values. Once exponential, logarithmic and trigonometric functions are reviewed, we will be able to deal with equations involving these functions.

2.1 Polynomial Equations

Generally speaking, these are probably the easiest to deal with. Linear, quadratic and cubic equations are just particular cases of polynomial equations. In general, a polynomial equation is one that can written under the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{2.1}$$

where each $a_i, i = 1, 2, \ldots, n$ is a real number and is termed a *coefficient* of the polynomial equation whereas n is a positive integer. For $n = 1, 2$ and 3 in (2.1), we obtain linear, quadratic and cubic equations respectively. The main idea behind solving a polynomial equation (of degree 2 or more) of the form P(x) = 0 is to factor the polynomial P(x) completely and then solve for x in each factor.

2.2 Linear Equations

These are equations of the form $p(x) = 0$ where $p(x)$ is a polynomial of degree 1. In other words, they are equations of the form

$$\alpha x + \beta = 0 \tag{2.2}$$

where $\alpha, \beta \in \mathbb{R}$ and a $\alpha \neq 0$.

The solution of such an equation is, of course,

$$x = -\frac{\beta}{\alpha} \tag{2.3}$$

(thus it is essential for α to not equal zero.)

Example 2.1. Derive the solution of the equation

$$\sqrt{2}x - \sqrt{8} - \sqrt{8} = 0 \tag{2.4}$$

Solution 2.1. Regardless of the fact that the coefficients of the equation are square roots, this is still a linear equation with respect to x. Thus, the solution of the equation is readily derived as

$$\sqrt{2}x - \sqrt{8} - \sqrt{8} = 0 \Rightarrow \tag{2.5}$$

$$\sqrt{2}x - \sqrt{8} = \sqrt{8} \Rightarrow \tag{2.6}$$

$$\sqrt{2}x = 2\sqrt{8} \Rightarrow \tag{2.7}$$

$$x = 2\frac{\sqrt{8}}{\sqrt{2}} = 2\frac{\sqrt{4}\sqrt{2}}{\sqrt{2}} = 4 \tag{2.8}$$

Remark Although not stated, when you try to solve an equation, you are actually assuming that there is a solution. If after some steps in attempting to find a solution, you end up with a nonsense like 1 = 2, this indicates that your initial assumption that a solution exists was wrong and in fact, there is no solution. For example, if you try to solve the linear equation $1 + x = 3x2x + 2$ the usual way, then you will end up with the equation 1 = 2. Of course, the statement ”1 = 2” is clearly false, which means that there is no value of x that could ever make this true. Therefore, equation $1 + x = 3x2x + 2$ has no solution.

2.3 Quadratic Equations

A quadratic equation of the variable x is an equation in the form

$$\alpha x^2 + \beta x + \gamma = 0 \tag{2.9}$$

where α, β and $\gamma \in \mathbb{R}$ and $\alpha \neq 0$. To solve this equation is to identify the roots of the corresponding polynomial. Identifying the roots of the polynomial is equivalent to identifying the positions along the x axis where the function $f(x) = \alpha x^2 + \beta x + \gamma$ crosses.

A second order polynomial curve, can either

[i] Cross the x axis at two points, in which case the corresponding quadratic equation has exactly two solutions

[ii] Be tangent to the x axis at a single point, in which case the corresponding quadratic equation has a single solution

[iii] Does not cross the x axis at all, in which case the corresponding quadratic equation has no solution at all

Although solutions to the quadratic equation have been known in various forms since 2000BC, this has been put in the formalism we know today by René Descartes in 1637. Descartes' solution is based on the definition of the discriminant Δ

$$\Delta = \beta^2 - 4\alpha\gamma \tag{2.10}$$

Three cases are then identified

- if $\Delta > 0$ then the quadratic equation has two real and unequal roots

$$\rho_1 = \frac{-\beta + \sqrt{\Delta}}{2\alpha} \qquad \rho_1 = \frac{-\beta - \sqrt{\Delta}}{2\alpha} \tag{2.11}$$

- if $\Delta = 0$ then the quadratic equation has a double real root (two equal real roots)

$$\rho_1 = \rho_2 = -\frac{\beta}{2\alpha} \tag{2.12}$$

- if $\Delta < 0$ then the quadratic equation has no real roots

Example 2.2. Solve the equation

$$2x^2 - 3x + 1 = 0 \tag{2.13}$$

Solution 2.2. The discriminant of the equation is

$$\Delta = (-3)^2 - 4 \cdot 2 \cdot 1 = 1 > 0 \tag{2.14}$$

Thus, the quadratic equation has two real roots which are

$$\rho_1 = \frac{-(-3) + \sqrt{(1)}}{2 \cdot 2} = 1 \tag{2.15}$$

and

$$\rho_2 = \frac{-(-3) - \sqrt{(1)}}{2 \cdot 2} = \frac{1}{2} \tag{2.16}$$

2.4 Self-assessment

Question 1: *Derive the solution for the following equations*

[i] $\frac{3}{2}x + \frac{7}{6} = \frac{2}{5}x - \frac{5}{4}$

[ii] $\frac{2x-1}{3} - \frac{1-x}{4} = \frac{2x+1}{8}$

[iii] $x \ln 5 + \ln 2 = 0$

[iv] $15x = 3$

[v] $4(x+7) = 6(x-3)$

[vi] $\frac{x}{4} + 6 = \frac{x}{3} - 4$

[vii] $\frac{3x-1}{4} = \frac{3}{4}$

Question 2: *Derive the real solution (if any) for the following equations*

[i] $2x^2 - 3x + 5 = 0$

[ii] $x^2 - 2x + 1 = 0$

[iii] $5x^2 - 15x + 3 = 0$

[iv] $13x^2 - 21 = 0$

[v] $4x^2 + 6x + 2 = 0$

[vi] $x^2 - 5x - 50 = 0$

[vii] $x^2 - x = 0$

Chapter 3
Solving Systems of Linear Equations

3.1 The method of elimination

In the elimination method, one tries to reformulate the equations so that when these are added (or subtracted) term by term, one of the two unknown variables is eliminated. This will result in a single equation with a single unknown which can be solved as a linear equation (see also 1)

Example 3.1. Using the elimination method, solve the following system of linear equations

$$x + y = 6 \tag{3.1}$$
$$3x + 8y = 3 \tag{3.2}$$

Solution 3.1. Consider for example in this case, that we multiply the first equation by -3, then the system assumes the following form

$$(-3)x + (-3)y = -18$$
$$3x + 8y = -5$$

Adding the two equations together and collecting terms will result in

$$3x + (-3)x + 8y + (-3)y = -18 + 3 \Rightarrow 5y = -15$$

This is a linear equation with respect to y that results in

$$y = -3$$

Substituting $y = -3$ in either one of the two initial equations will provide us with the value of x, for example

$$x + y = 6 \Rightarrow x = 6 - y = 6 - (-3) \Rightarrow x = 9$$

3.2 Solution by substitution

In this case, one tries to solve one of the two equations in terms of one unknown and then substitute to the second. Again, this will result in a single equation with just one unknown that can be easily solved. It is worth noting however, that the more the equations involved in the system the more cumbersome this method becomes.

Example 3.2. Using the method of substitution, solve the following system of linear equations

$$\begin{aligned} x+y&=6 \\ 3x+8y&=3 \end{aligned} \tag{3.3}$$

Solution 3.2. The first equation can be solved with respect to y resulting in

$$y=6-x \tag{3.4}$$

Substituting this equation to the second of (3.3) results in

$$3x+8\cdot(6-x)=3 \Rightarrow 3x+48-8x=3 \Rightarrow -5x=-45 \Rightarrow x=9 \tag{3.5}$$

Finally, substituting this solution back into equation (3.4) will result in

$$y=6-9=-3 \tag{3.6}$$

3.3 Self-assessment

Question 3: *Solve the following pairs of equations for x and y.*

(a)
$$\begin{aligned} 20x + y &= 81 \\ 2x - y &= 7 \end{aligned}$$

(b)
$$\begin{aligned} 2x - 7y &= 8 \\ 4x - 6y &= 0 \end{aligned}$$

(c)
$$\begin{aligned} 2y - 6x &= 6 \\ 20y + 12x &= 0 \end{aligned}$$

(d)
$$\begin{aligned} 3y + 2x &= 9 \\ 5x - y &= -3 \end{aligned}$$

Chapter 4
Basic Trigonometry

4.1 Measuring Angles

Consider the two lines, **O**1 and **O**2 that meet at point **0** (Fig. 4.1). An angle is formally defined as the set of all points in space that lay between these two lines. The latter are called the *rays* of the angle whereas point **O** is termed the angle vertex. In most cases, we are concerned with defining and measuring angles with respect to specific points in space, for example points **A** and **B** in lines **O**1 and **O**2 respectively. In this case, we denote the corresponding angle as ∠**AOB** and describe it as the angle between the linear segments **OA** and **OB**.

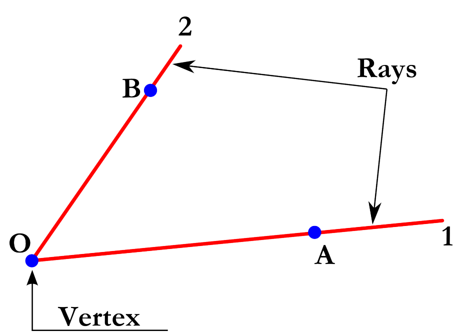


Fig. 4.1: Angle definition

Highly relevant to the notion of an angle is that of *rotation*. Consider an angle ∠**AOB** to be living in the cartesian plane xy and that **OA** coincides with the x axis, as shown in Fig. 4.2. We say that the angle ∠**AOB** is identical to the angle by which the x axis needs to be rotated about **O** to meet axis x'. If this rotation is counter-clockwise we will be considering the angle to be positive. If the rotation is clock-wise we will be considering the corresponding angle to be negative.

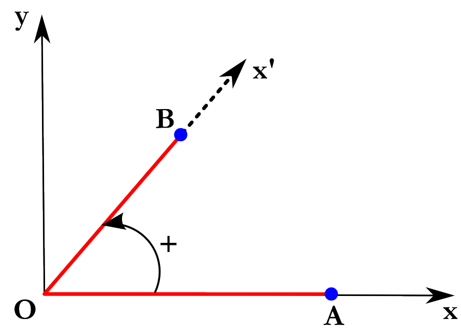


Fig. 4.2: Angle through rotation

4.1.1 Radians

Calculus-based university courses rarely use degrees as measure of an angle, rather a second unit called Radian (rad). Using radians would allows a more smooth extension of the domain of trigonometric functions to real numbers. Also, later on you will see that known formulas involving the rate of change of trigonometric functions are only valid when the angles are expressed in radians rather than degrees. For now, let us, once and for all, try to understand this notion of radian so we can move on. Imagine a straight radius of a circle is lifted off and placed on the curve of the circumference of the circle. The angle at the centre of circle bounded by the two end points of the curve, called a central angle, has a measure of one radian, written 1 rad for short.

Using the above definition of radian allows us to establish a relationship between the two units of measurement of angles (radian versus degree). To start, consider a circle of radius r . Recall that a complete tour of the circle corresponds to a central angle of 360^o. Since the circumference of the circle is equal to 2r , it follows that the number of times one can ”wrap” the radius around the circumference is 2 (just over 6 times) and the central angle corresponding to a full revolution is then equal to 2π radians. In other words, 2π radians is equivalent to 360^o or, π radians is equivalent to 180^o. This is the basic relation between the two units of measurement of an angle:

- If θ is an angle measured in degrees, then $\frac{\pi}{180}\theta$ is the value of the angle in radians.
- If θ is an angle measured in rads, then $\frac{180}{\pi}\theta$ is the value of the angle in radians

4.1.2 Some angles to remember - Part I

There are cases in your future Engineering career that you will be meeting the radian expression of a very specific set of angles, time after time. It’s thus good practice to actually remember the following conversions

Table 4.1: Very Useful Angles

Angle in Degrees	Angle in Radians
30	$\pi/6$
45	$\pi/4$
60	$\pi/3$
90	$\pi/2$
180	π
270	$3\pi/2$

4.2 Trigonometric Functions

4.2.1 The right-angled triangle definitions

If θ is the acute angle shown in the right-angled triangle in Fig. 4.3 the fundamental trigonometric functions, i.e., $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ are defined as

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{a}{c} \tag{4.1}$$

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{b}{c} \tag{4.2}$$

and

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{a}{b} \tag{4.3}$$

respectively.

4.2.2 The unit circle definition

The right angle triangle definition of the previous section does not offer any insight when the angles involved are larger than π. The more general and intuitive unit-circle however offers a much better alternative. In fact, if you remember the definition of the unit-circle you can easily derive the sine and cosine of angles frequently met in engineering. The unit-circle is merely a circle defined in the XY plane whose centre is at the origin. The radius of the circle is $R = 1$.

If we consider a $P_1(x, y)$ on the unit circle laying on the first quadrant Fig. 4.4, then the radius of the circle at the point forms an angle θ_1 with the horizontal x axis. Then, based on the definition of the sine function one may write that

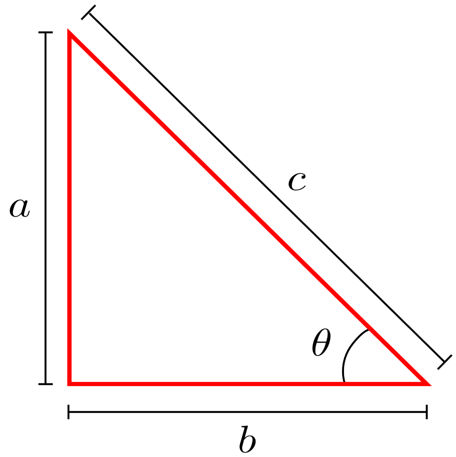


Fig. 4.3: Trig function definitions - The normal Triangle case

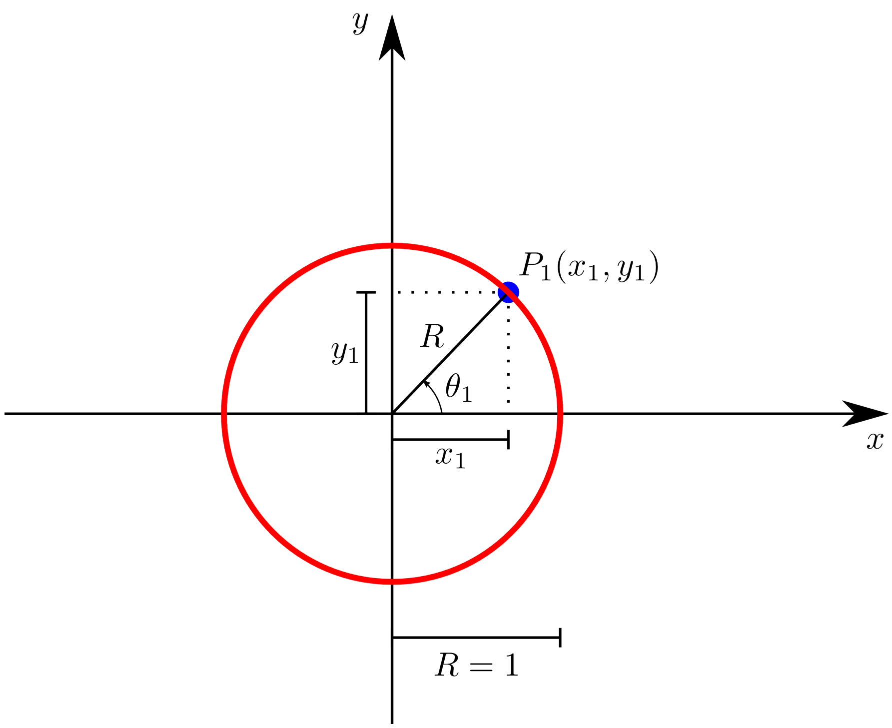


Fig. 4.4: Trig function definitions - The Unit Circle case

$$\cos(\theta_1) = \frac{x_1}{R} = \frac{x}{1} = x_1 \tag{4.4}$$

Similarly, the cosine function will be

$$\sin(\theta_1) = \frac{y_1}{R} = \frac{y}{1} = y_1 \tag{4.5}$$

Notice that in the limit case where $y_1 = 0$, i.e. P_1 lies on the x-axis one has that $\sin(0) = 1$ and $\cos(0) = 0$ as expected. Similarly, when P_1 lies on the y-axis one has $\sin(\pi/2) = 0$ and $\cos(\pi/2) = 1$.

This hints that we can *redefine* the sine and cosine functions as being equal to the x and y coordinates of a point laying on the unit-cirle. This is rather intriguing as in that case not only do we define the cosine and sine for any angle larger than $\pi/2$ but also we get a very intuitive way of remembering what the signs of these functions are for different angles.

For example, just by drawing the trigonometric circle, we can see that any angle $\pi/2 < \theta \leq \pi$ has a negative sine and a positive cosine value Fig. 4.5.

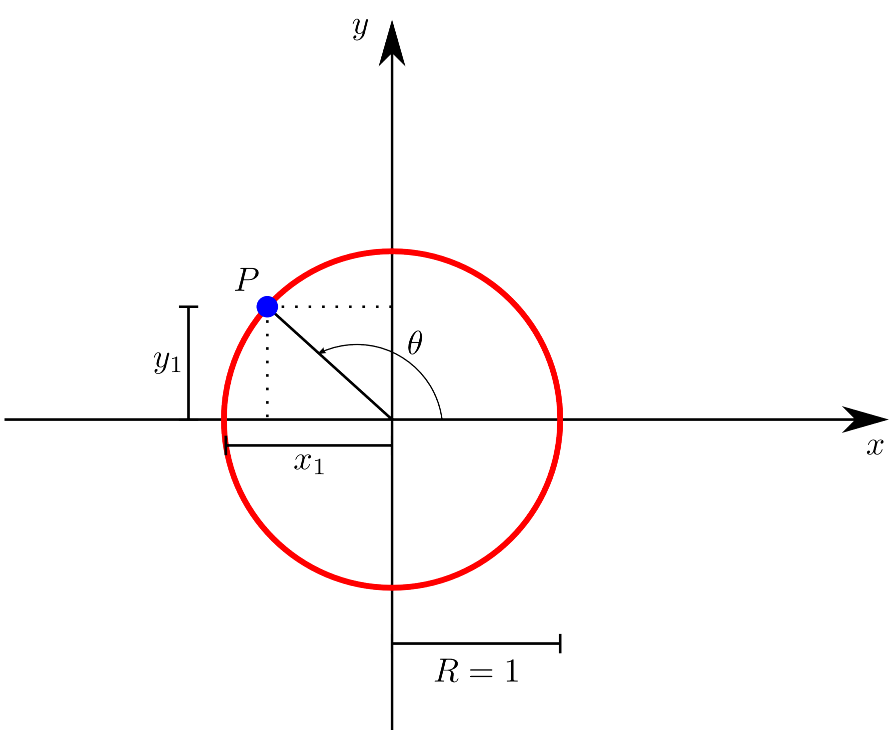


Fig. 4.5: Trig function definitions - The Unit Circle case

Furthermore, using the unit-circle we can directly derive some very useful trigonometric relations. For example, we know that for any two given supplementary angles (e.g., $\theta_1 = 30$ and $\theta_2 = 150$) it holds that

$$\cos(\theta_1) = \cos(\theta_2) \tag{4.6}$$

and

$$\sin(\theta_1) = -\sin(\theta_2) \tag{4.7}$$

Indeed, by referring back to the unit-circle Fig. 4.6 we can directly see that for these two angles it holds that

$$x_2 = -x_1 \qquad\qquad y_2 = y_1 \tag{4.8}$$

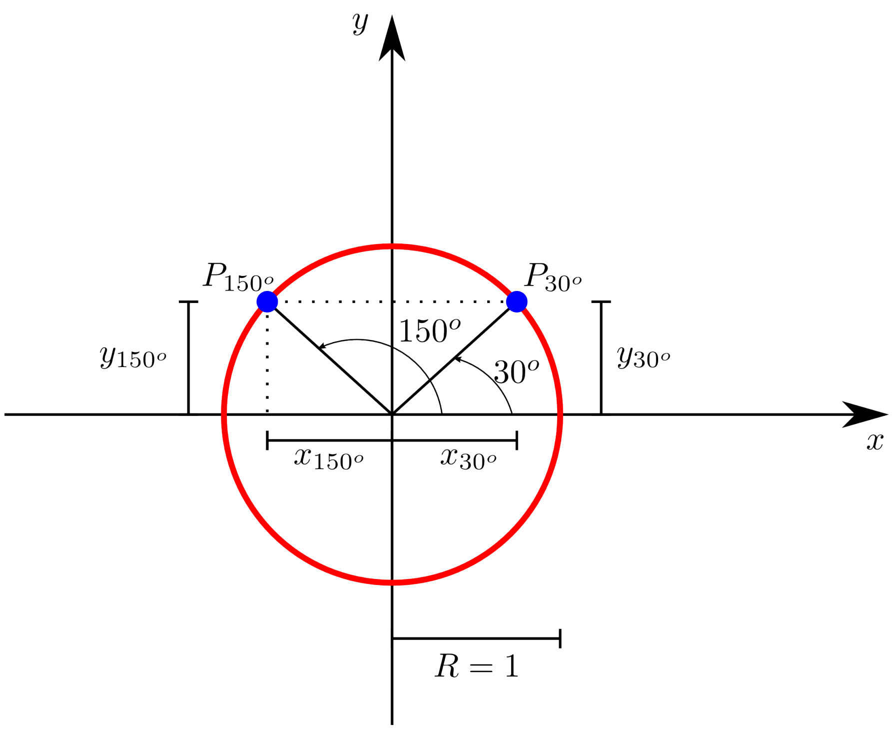


Fig. 4.6: Supplementary angles

Example 4.1. Using the unit-circle, write down the values of the following expressions in terms of either $\sin(\theta)$ or $\cos(\theta)$ alone

$$\sin(-\theta)$$
$$\cos(-\theta)$$

Solution 4.1. Consider the radius the point P^+ corresponding to the angle θ and the point P^- corresponding to the angle $-\theta$ in Fig. 4.7. It holds that

$$x_{P^-} = x_{P^+} \qquad\qquad y_{P^-} = -y_{P^+}$$

Thus, the requested relation is

$$\sin(-\theta) = -\sin(\theta) \quad \cos(-\theta) = \cos(\theta)$$

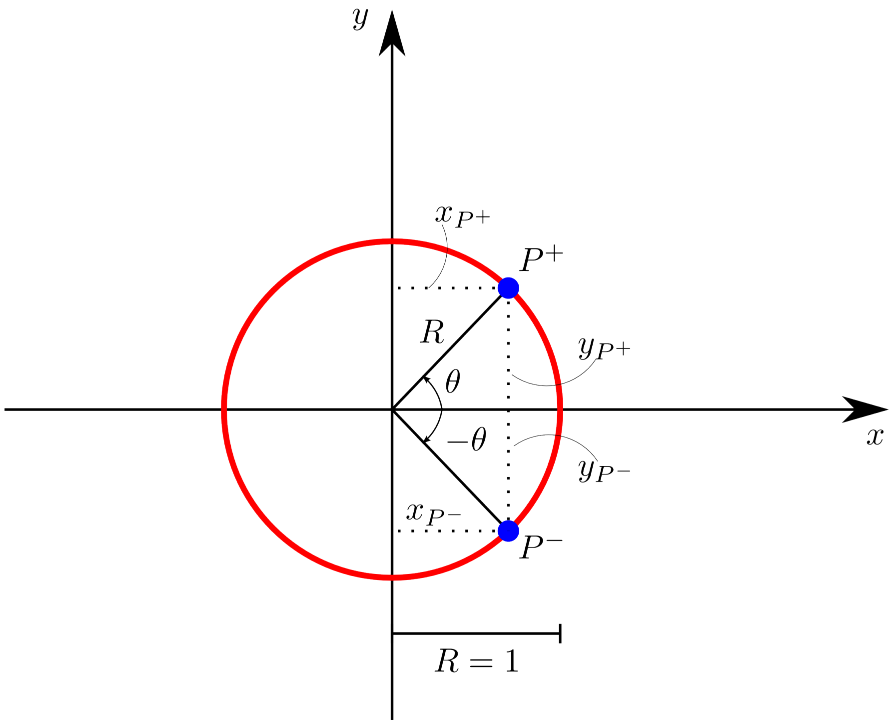


Fig. 4.7: Equal and opposite angles

4.2.3 Trigonometric Equations

Trigonomentric equations are equations of the form

$$F(x) = 0 \tag{4.9}$$

where $F(x)$ is an expression comprising trigonometric components. In cases, such equations can be difficult to solve, or a solution, at least in the form of an analytical expression, might not even exist. However there are several cases where such equations can be easily solved and the unit-circle can again be an intuitive way of treating the problem. Such equations are frequently met in engineering, especially in problems pertaining to stability analysis and dynamics.

Example 4.2. What is the value of the angle x when we know that $0 < x < \pi$ for which the following equation holds

$$\sin(2x) = 0$$

Solution 4.2. Looking at the trigonometric circle (Fig. 4.8), we notice that there are two points on that circle where the sine of an angle equals 0. This is either when the angle involved is equal to 0 or the angle involved equals π. Notice that we can rotate around the circle as much as we want. So, if we start from zero and rotate by 2π we end up again at zero. If we start at zero and rotate by 3π we end up at π (see also, Fig. 4.9).

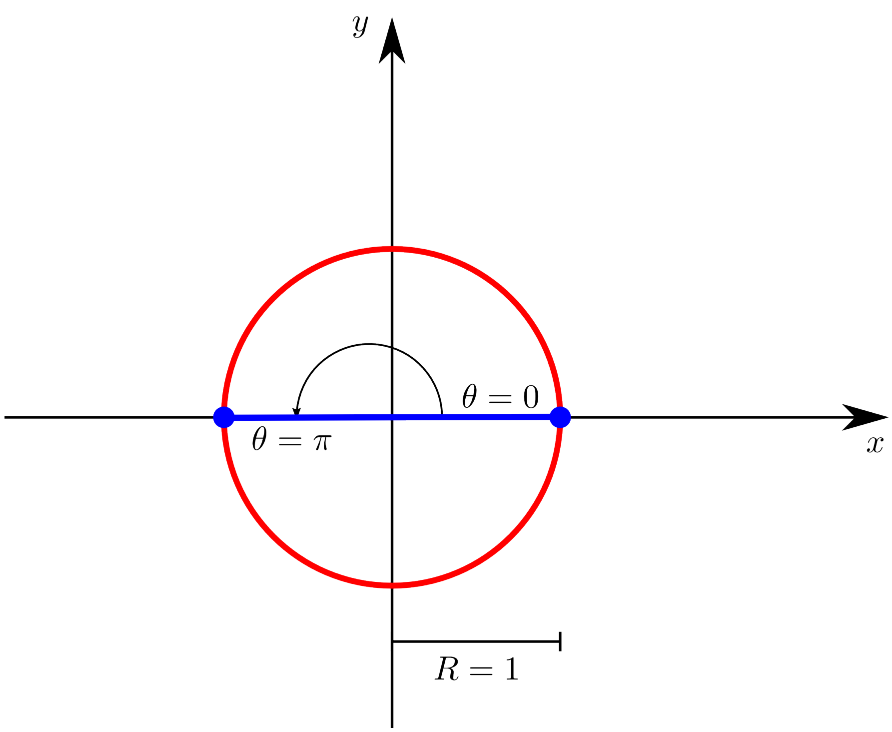


Fig. 4.8: Equal and opposite angles

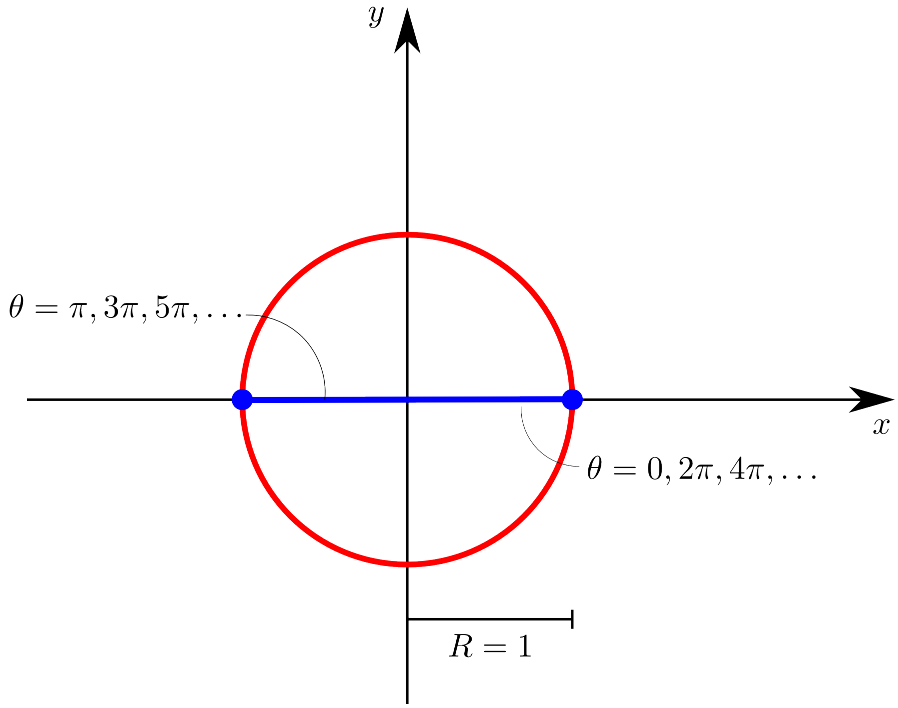


Fig. 4.9: Equal and opposite angles

This means that, if we want to define all the angles θ with zero sine, we need to introduce a recursive expression that can assume the following form

$$\theta_i = k\pi, k = 0, 1, 2 \ldots \tag{4.10}$$

This means that the solution of the equation provided can be written as

$$2x_i = k\pi, k = 0, 1, 2 \cdots \Rightarrow x = \frac{k}{2}\pi \tag{4.11}$$

that is, there does not exist only one solution, but several values of the unknown which x satisfy the equation. However, we are told that $0 < x < \pi$ which means that the only solution that is relevant to the question is

$$x = \pi/2 \tag{4.12}$$

for $k = 1$.

4.3 Self-assessment

Question 4: *Write down the angle in degrees whose measurements in radians are*

[i] π

[ii] $\frac{\pi}{3}$

[iii] $\frac{2\pi}{3}$

Question 5: *Write down the angle in multiples of* π *whose measurements in degrees are*

[i] 90^o

[ii] 135^o

[iii] 30

Question 6: *Using the unit-circle, fill in the sign values for the sine and cosine functions for the different angles in table 4.2*

Question 7: *Using the unit-circle, write down the values of the following expressions as a function of* θ

[i] $\sin(\pi/2 - theta) =$

Table 4.2: Find the signs of the cosine and sine functions

Angle	cosine	sine
$\pi/3$	$+$	$+$
$\pi/4$		
$2\pi/3$		
$4\pi/3$		
$8\pi/3$		

[ii] $\cos(\pi/2 - theta) =$

[iii] $\sin(\pi/2 + theta) =$

[iv] $\cos(\pi/2 + theta) =$

[v] $\sin(\pi + theta) =$

[vi] $\cos(\pi + theta) =$

[vii] $\sin(2\pi + theta) =$

[viii] $\cos(2\pi + theta) =$

Question 8: *Solve the following equations*

[i] $\sin(x) = -1$

[ii] $\cos(3x) = 1$

Chapter 5
Functions

5.1 Definition

A function is nothing more than a piece of mathematical machinery that takes some input in the form of a variable acts on it and produces a unique output number. For example, if t is the amount of time a car travels at constant velocity v, then the distance travelled is provided by a function of the form

$$f(t) = v \cdot t$$

In this case, one denotes t as the function variable, $f(t)$ as the function output, and v as a function parameter. Function parameters are constants.

5.2 Plotting Functions

Functions are visually represented using plots. In the common two-dimensional case, we use the horizontal x-axis to account for the values of the function variable (for example the length or the time). We use the vertical y-axis to account for the values of the function $f(x)$. Considering a set of values $x_i, i = 1, 2, \ldots$ we can calculate the corresponding set of $f_i = f(x_i), i = 1, 2, \ldots$ function values. These pairs, i.e., $(x_i, f(x_i))$ correspond to specific points on the function plot. Connecting these points, one creates the function graph.

Example 5.1. Plot the function $f(x) = x^2$ for $-8 \leq x \leq 8$.

Solution 5.1. The plot is shown in Fig. 5.1. In the left, specific points are plotted corresponding to pairs shown in Table 5.1. The function graph is shown on the right.

Table 5.1: Plotted points

x	$f(x)$
-8	64
-7	49
-6	36
-5	25
-4	16
-3	9
-2	4
-1	1
0	0
1	1
⋮	⋮
8	64

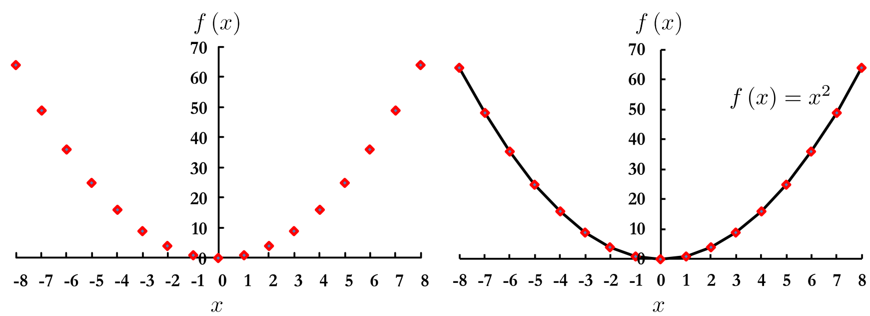


Fig. 5.1: Plot of x^2

5.3 Plot Transformations

Plotting a function is a straightforward procedure when the expression of the function is known. It is interesting to note that one is able to generate a family of plots from the expression of a single parent function by means of simple transformation principles. These are

5.3.1 Horizontal and vertical shifts

A shift is a plane transformation that does not change the shape or size of the graph of the given function. There are two types of shifts. A vertical shift adds (or subtracts) a constant to the y-coordinate while keeping the x-coordinate unchanged. A horizontal shift adds (or subtracts) a constant to the x-coordinate while leaving the y-coordinate unchanged. Very often, we combine vertical and horizontal shifts to get a new graph. Adding a constant to the x-coordinate will result in a horizontal shift and adding a constant to the y-coordinate will result in a vertical shift. The following table explains the details (see also Figure (5.2).

Given the function $y = f(x)$ and a positive constant $c \in R$

- The graph of $y = f(x+c)$ is obtained from the graph of $y = f(x)$ by a horizontal shift of c units to the left.
- The graph of $y = f(x-c)$ is obtained from the graph of $y = f(x)$ by a horizontal shift of c units to the right.
- The graph of $y = f(x)+c$ is obtained from the graph of $y = f(x)$ by a vertical shift of c units upward.
- The graph of $y = f(x)-c$ is obtained from the graph of $y = f(x)$ by a vertical shift of c units downward.

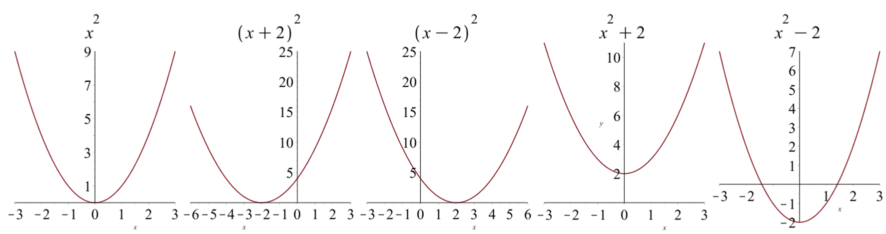


Fig. 5.2: Function Shift

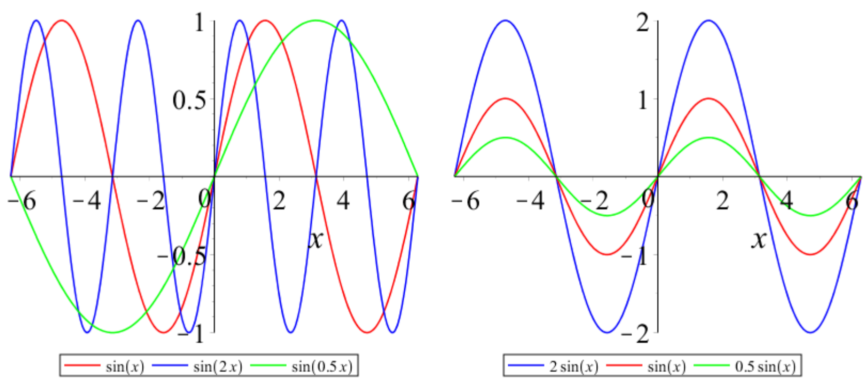
5.3.2 *Horizontal and vertical scaling*

Unlike the horizontal and vertical shifts, a scale is a transformation that usually alters the shape and size of the graph of the function. Given a function $y = f(x)$ and a constant $\alpha \in R$, a horizontal scaling of the graph of f is the graph of the function $f(\alpha x)$ (obtained from the original one by replacing every occurrence of the variable x with αx). A vertical scaling of the graph of f is the graph of the function $\alpha f(x)$.

Replacing x with αx in the function $f(x)$ results in a horizontal stretching or a horizontal compression (depending on the value of α). Similarly, multiplying the function itself with the constant α results in a vertical stretching or a vertical compression (depending on the value of α) of the graph of f . We will see what these terms mean with some specific examples. The following table gives the values of the constant α that correspond to a compression or a stretching of the graph. An example is shown in Figure 5.3.

Given the function $y = f(x)$ and a positive constant $\alpha \in R$

- If $0 < \alpha < 1$, then the graph of $y = f(\alpha x)$ is the graph of $y = f(x)$ stretched horizontally away from the y-axis.
- If $\alpha > 1$, then the graph of $y = f(\alpha x)$ is the graph of $y = f(x)$ compressed horizontally towards the y-axis.
- If $0 < \alpha < 1$, then the graph of $y = \alpha f(x)$ is the graph of $y = f(x)$ compressed vertically towards the x-axis.
- If $\alpha > 1$, then the graph of $y = \alpha f(x)$ is the graph of $y = f(x)$ stretched vertically away from the x-axis.



(a) (b)

Fig. 5.3: Function scaling (a) Horizontal (b) Vertical

5.4 Self-assessment

Question 9: *In Fig. 5.4 below, draw the graph of* $f(x) = \cos x$

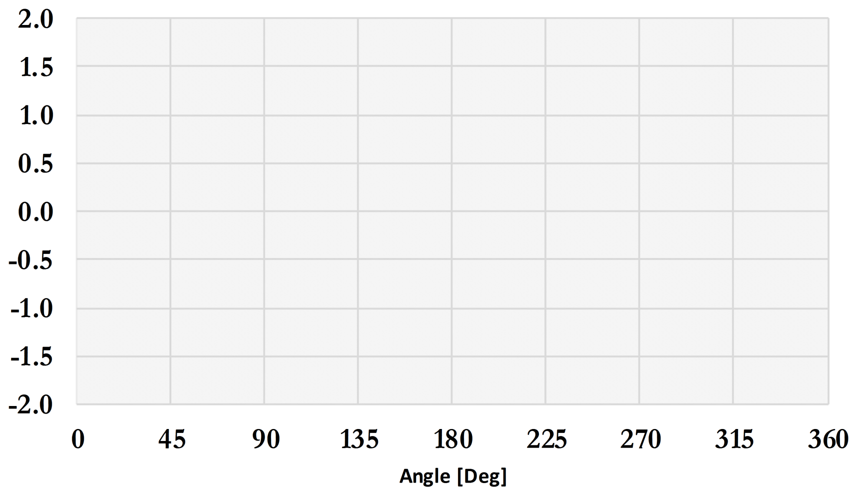


Fig. 5.4: The cos(x) graph

Question 10: *In Fig. 5.5 below, draw the graph of* $f(x) = 2\cos x$

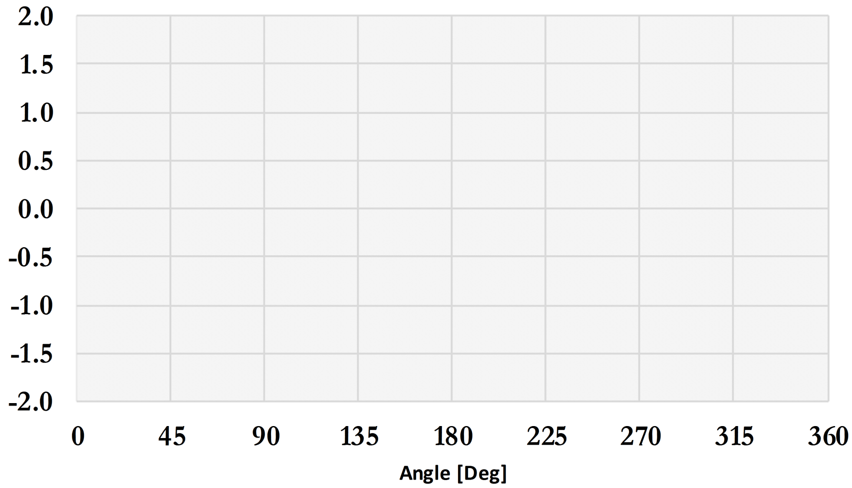


Fig. 5.5: The 2cos(x) graph

Question 11: *In Fig. 5.6 below, draw the graph of* $f(x) = \sin x$

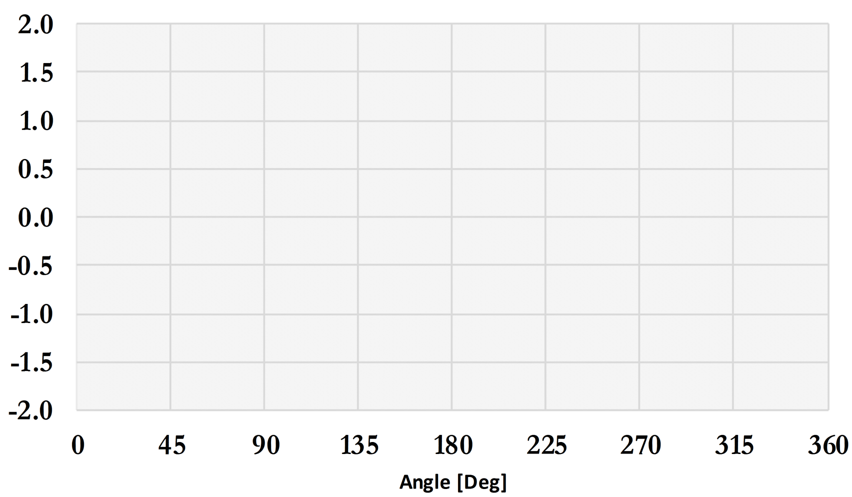


Fig. 5.6: The sin(x) graph

Question 12: *In Fig. 5.7 below, draw the graph of* $f(x) = \sin(x+45)$

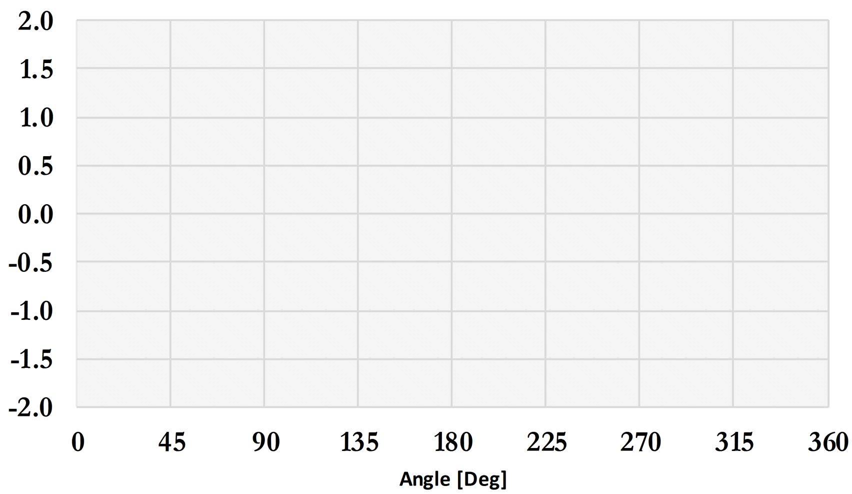


Fig. 5.7: The sin(x+45) graph

Chapter 6
Logarithms

6.1 Definition

Formally, the logarithm of a number x is defined as the exponent to which a fixed value, the base, must be raised to produce x. Based on the definition above, we denote by $\log_\alpha \theta$ the logarithm of θ with respect to the base α.

$$\alpha^x = \theta \Leftrightarrow x = \log_\alpha \theta \tag{6.1}$$

Example 6.1.

$$\begin{aligned}
\log_2 8 &= 3 \quad \text{, because } \ 8 = 2^3 \\
\log_4 2 &= \frac{1}{2} \quad \text{, because } \ 2 = 4^{\frac{1}{2}} \\
\log_{10} 0.001 &= -3 \quad \text{, because } \ 0.001 = 10^{-3} \\
\log_5 0.25 &= 2 \quad \text{, because } \ 0.25 = 0.5^2
\end{aligned}$$

From the above, it follows that if $\alpha > 0$ with $\alpha \neq 1$, then for every $x \in \mathbb{R}$ and every $\theta > 0$:

$$\log_\alpha \alpha^x = x \ \text{ and } \ \alpha^{\log_\alpha \theta} = \theta$$

Also, since $1 = \alpha^0$ and $\alpha = \alpha^1$:

$$\log_\alpha 1 = 0 \ \text{ and } \ \alpha^{\log_\alpha \alpha} = 1$$

6.2 Properties

If $\alpha > 0$ with $\alpha \neq 1$, then for any $\theta_1, \theta_2, \theta > 0$ and $k \in \mathbb{R}$:

- $\log_\alpha(\theta_1\theta_2) = \log_\alpha(\theta_1) + \log_\alpha(\theta_2)$
- $\log_\alpha(\frac{\theta_1}{\theta_2}) = \log_\alpha(\theta_1) - \log_\alpha(\theta_2)$
- $\log_\alpha \theta^k = k \log_\alpha^\theta$

For example, to understand why the first property holds it suffices to assume that:

$$\log_\alpha \theta_1 = x_1 \text{ and } \log_\alpha \theta_2 = x_2 \tag{6.2}$$

and then to notice that:

$$\alpha^{x_1} = \theta_1 \text{ and } \alpha^{x_2} = \theta_2$$

hence:

$$\alpha^{x_1} \cdot \alpha^{x_2} = \theta_1\theta_2 \text{ , i.e. } \alpha^{x_1+x_2} = \theta_1\theta_2$$

where the last equation is equivalent to:

$$\log_\alpha \theta_1\theta_2 = x_1 + x_2$$

which according to 6.2 is equivalent to:

$$\log_\alpha(\theta_1\theta_2) = \log_\alpha(\theta_1) + \log_\alpha(\theta_2)$$

Notice that for every $\theta > 0$ it holds that $\sqrt[n]{\theta} = \theta^{\frac{1}{n}}$, hence

$$\log_\alpha \sqrt[n]{\theta} = \log_\alpha \theta^{\frac{1}{n}} = \frac{1}{n} \log_\alpha \theta$$

Lets see now how these properties can ease the computation of logarithms of positive numbers.

Example 6.2. Find the value of the following expression:

$$A = \frac{1}{2} \log_2 256 + 2 \log_2 3 - \log_2 18$$

Solution 6.1.

$$
\begin{aligned}
A &= \frac{1}{2}\log_2 256 + 2\log_2 3 - \log_2 18 \\
&= \log_2 \sqrt{256} + \log_2 3^2 - \log_2 18 \\
&= \log_2 16 + \log_2 9 - \log_2 18 \\
&= \log_2 \frac{16 \cdot 9}{18} \\
&= \log_2 8 = \log_2 2^3 = 3
\end{aligned}
$$

6.3 Changing the Base of a Logarithm

It is often very practical to change the base of a logarithm.
If α, $\beta > 0$ with $\alpha, \beta \neq 1$, then for every $\theta > 0$:

$$\log_\beta \theta = \frac{\log_\alpha \theta}{\log_\alpha \beta} \tag{6.3}$$

Suppose that $\log_\beta \theta = x$, then $\theta = \beta^x$. Hence,

$$\log_\alpha \theta = \log_\alpha \beta^x = x \cdot \log_\alpha \beta = \log_\beta \theta \cdot \log_\alpha \beta$$

From which it immediately follows that:

$$\log_\beta \theta = \frac{\log_\alpha \theta}{\log_\alpha \beta}$$

6.4 Bases of particular interest

- Logarithms with base 10:

Also called 10-base logarithms, are very handful for manual calculations. We denote the 10-base logarithm of θ simply by $\log \theta$ and not by $\log_{10} \theta$:

$$\log \theta = x \Leftrightarrow 10^x = \theta$$

- Logarithms with base e:

e is the Euler's number named after the mathematician Leonard Euler. It is also often referred to as Napier's constant. It is an important mathematical constant and it is approximately equal to 2.71828.

Logarithms with base e are called natural logarithms. The natural logarithm of a number θ is written as $\ln \theta$ and not as $\log_e \theta$:

$$\ln\theta = x \Leftrightarrow e^x = \theta$$

6.5 Self-assessment

Question 13: *Show that:*

[i] $3\log_{10}2 + \log_{10}5 - \log_{10}4 = 1$

[ii] $2^{\log_2 6 - 2\log_2\sqrt{3}} = 2$

Question 14: *Show that:*

[i] *For every* $x > 0$*:* $\log_a x = \log_{a^2} x^2$

[ii] $\log_\alpha \beta \cdot \log_\beta \alpha = 1$

[iii] $\log_\alpha \beta^2 \cdot \log_\beta \alpha^3 = 6$

[iv] $\log_\alpha \beta \cdot \log_\beta \gamma \cdot \log_\gamma \alpha = 1$

[v] $\log_\alpha \theta + \log_{\frac{1}{\alpha}} \theta = 0$

[vi] $\log_\alpha(\alpha\beta) + \log_\beta(\alpha\beta) = \log_\alpha(\alpha\beta) \cdot \log_\beta(\alpha\beta)$

Chapter 7
Basic Statistics

7.1 Why are Statistics relevant?

The field of statistics deals with the collection, presentation, analysis, and use of data to inform and facilitate decision making and design in Engineering. Fundamental principles of Statistics form an integral part in all aspects of modern Civil Engineering design.

Statistical methods are used to help us describe and understand variability. The mechanical properties of the materials a Civil Engineer utilizes, even the maximum loads a structure has to withstand within its life-span cannot and are not known with absolute certainty. The strength of a concrete block can be identified experimentally; however different experiments, i.e. observations, on a set of concrete blocks taken from the same mix will never result in the same absolute strength value. Statistics enable us to describe this variability and assess, both qualitatively and quantitatively its influence on our structures.

7.2 Basic Definitions

Fundamental statistics are based on a set of *descriptors* that help scientists qualitative describe datasets. These are the **arithmetic mean** , the **median**, the **mode** and the **standard deviation**.

The arithmetic mean or average of a sample is the sum of all sampled values divided by the number of samples. Thus, if we consider that the values $x_1, x_2, \ldots, x_k$ correspond to the height of a group of k people, then the average height $\bar{x}$ corresponds to the value

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_k}{k}$$

The median of a population is the value separating the higher half of the population from the lower half. Thus, if we have the following sample of people height in meters

$$h = \{1.56, 2.1, 1.85, 1.67, 1.52\}$$

to identify the median we would need to sort the sample from the lowest to the height value and pick the one that rests exactly in the middle, i.e. $Md = 1.67m$. Notice that if the number of samples was even then no single middle value exists. The median is then usually defined to be the mean of the two middle values.

The mode of a sample is the sample the element that occurs most often in a population. For example, the mode of the population

$$h = \{1.56, 2.1, 1.85, 1.67, 1.52, 1.52, 1.52, 1.52\}$$

is

$$Mo = 1.52$$

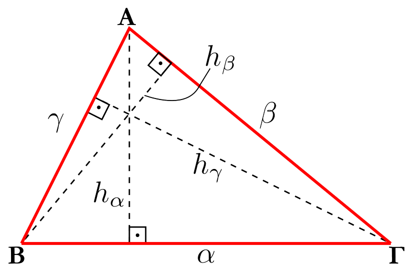
he standard deviation is quantifying the variation or dispersion of a population. A low standard deviation indicates that the data points tend to be close to the mean value of the population of the set, while a high standard deviation indicates that the data points are spread out over a wider range of values. The standard deviation σ of a data set is defined as

$$\sigma = \frac{\sum_{k=1}^{N} (x_k - \bar{x})^2}{N} \tag{7.1}$$

Chapter 8

Basic Geometry

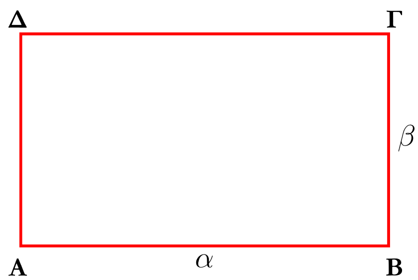
Triangle



Perimeter: $\Pi = \alpha + \beta + \gamma$

Area: $A = \frac{1}{2}\alpha h_\alpha = \frac{1}{2}\beta h_\beta = \frac{1}{2}\gamma h_\gamma$

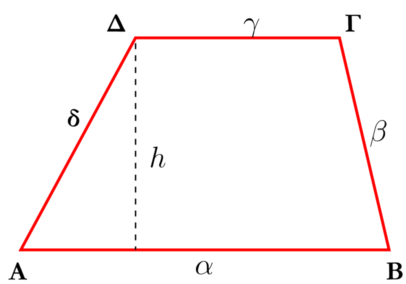
Rectangle



Perimeter: $\Pi = 2\,(\alpha + \beta)$

Area: $A = \alpha\beta$

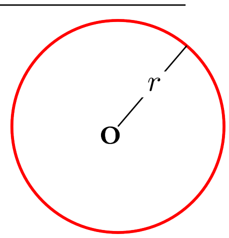
Trapezium



Perimeter: $\Pi = \alpha + \beta + \gamma + \delta$

Area: $A = \frac{1}{2}\,(\alpha + \beta)\,h$

Circle



Perimeter: $\Pi = 2\pi r$

Area: $A = \frac{1}{2}\pi r^2$

Solutions to self-assessment questions

Questions of Chapter 2

Question 1

[i] $x = -\frac{145}{66}$

[ii] $x = \frac{17}{16}$

[iii] $x = -\frac{ln2}{ln5}$

[iv] $x = \frac{1}{5}$

[v] $x = 23$

[vi] $x = 120$

[vii] $x = \frac{4}{3}$

Question 2

[i] no real solutions exist

[ii] $x_1 = 1, x_2 = 1$

[iii] $x_1 = \frac{3}{2} + \frac{1}{10}\sqrt{165}$, $x_2 = \frac{3}{2} - \frac{1}{10}\sqrt{165}$

[iv] $x_1 = \frac{1}{13}\sqrt{273}$, $x_2 = -\frac{1}{13}\sqrt{273}$

[v] $x_1 = -\frac{1}{2}, x_2 = -1$

[vi] $x_1 = 10, x_2 = -5$

[vii] $x_1 = 0, x_2 = 1$

Questions of Chapter 3

Question 3

[i] $\{x = 4, y = 1\}$

[ii] $\{x = -3, y = -2\}$

[iii] $\{x = -5/6, y = 1/2\}$

[iv] $\{x = 0, y = 3\}$

Questions of Chapter 4

Question 4

[i] 180^o

[ii] 60^o

[iii] 120^o

Question 5

[i] $\frac{\pi}{2}$

[ii] $\frac{3\pi}{4}$

[iii] $\frac{\pi}{6}$

Question 6

Angle	cosine	sine
$\pi/3$	$+$	$+$
$\pi/4$	$+$	$+$
$2\pi/3$	$-$	$+$
$4\pi/3$	$-$	$-$
$8\pi/3$	$-$	$+$

Question 7

[i] $\cos(theta)$

[ii] $\sin(theta)$

[iii] $\cos(theta)$

[iv] $-\sin(theta)$

[v] $-\sin(theta)$

[vi] $-\cos(theta)$

[vii] $\sin(theta)$

[viii] $\cos(theta)$

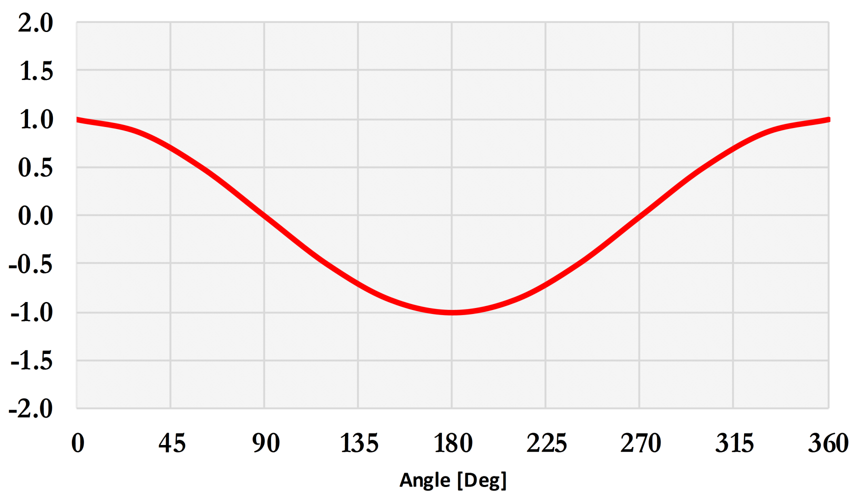
Question 8

[i] $x = \frac{3\pi}{2} + 2\kappa\pi, \kappa = 0, 1, \ldots$

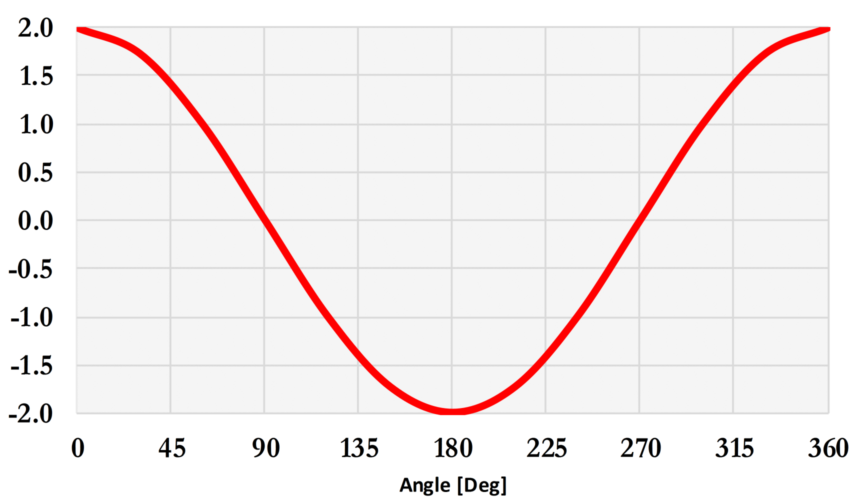
[ii] $x = \frac{2\kappa\pi}{3}, \kappa = 1, 2, \ldots$

Questions of Chapter 5

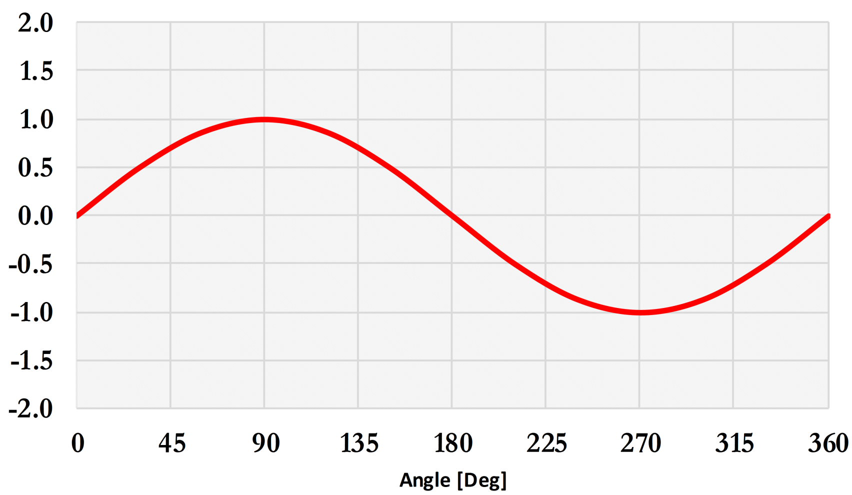
Question 9



Question 10



Question 11



Question 12