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by

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Testing for a Change in Persistence in the Presence of Non-stationary Volatility*

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Abstract

In this paper we consider tests for the null of (trend-) stationarity against the alternative of a change in persistence at some (known or unknown) point in the observed sample, either from $I(0)$ to $I(1)$ behaviour or *vice versa*, of, *inter alia*, Kim (2000). We show that in circumstances where the innovation process displays non-stationary unconditional volatility of a very general form, which includes single and multiple volatility breaks as special cases, the ratio-based statistics used to test for persistence change do not have pivotal limiting null distributions. Numerical evidence suggests that this can cause severe over-sizing in the tests. In practice it may therefore be hard to discriminate between persistence change processes and processes with constant persistence but which display time-varying unconditional volatility. We solve the identified inference problem by proposing wild bootstrap-based implementations of the tests. Monte Carlo evidence suggests that the bootstrap tests perform well in finite samples. An empirical application to a variety of measures of U.S. price inflation data is provided.

Keywords: Persistence change; non-stationary volatility; wild bootstrap.
JEL Classification: C22.

1 Introduction

Recently, both applied economists and econometricians have questioned whether, rather than simply being either $I(1)$ or $I(0)$, series might experience a change in persistence

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between separate $I(1)$ and $I(0)$ regimes. There is now a relatively large body of evidence on changes of this kind in macroeconomic and financial time series; see, *inter alia*, Kim (2000), Busetti and Taylor (2004) [BT], and Leybourne *et al.* (2003), and the citations therein. Commensurately, a number of procedures designed to test against changing persistence have been suggested in the literature. The most popular of these are the ratio-based persistence change tests of, *inter alia*, Kim (2000), Kim, J. *et al.* (2002) and BT, *inter alia*, which we focus on in this paper. These test the null hypothesis that a series is a constant $I(0)$ process against the alternative that it displays a change in persistence from $I(0)$ to $I(1)$, or *vice versa*.

The persistence change tests proposed in the literature are all based on the maintained assumption that, both under the null hypothesis of no change in persistence and the alternatives of $I(0)$-$I(1)$ or $I(1)$-$I(0)$, the time series of interest displays stable (unconditional) volatility. This assumption contrasts with a growing body of recent empirical evidence which documents that many of the main macro-economic and financial variables across developed countries are characterized by the existence of significant non-stationarity in unconditional volatility, in particular, single and multiple (possible smooth transition) breaks in volatility and/or (broken) trending volatility; see, *inter alia*, Busetti and Taylor (2003), Sensier and van Dijk (2004), Kim and Nelson (1999), McConnell and Perez Quiros (2000), and the references therein. Sensier and van Dijk (2004), for example, find that over 80% of the real and price variables in the Stock and Watson (1999) data-set reject the null hypothesis of constant unconditional innovation variance. Considerable evidence against the constancy of unconditional variances in stock market returns and exchange-rate data has also been reported; see, *inter alia*, Loretan and Phillips (1994). Hansen (1995) also notes that empirical applications of autoregressive stochastic volatility [SV] models to financial data generally estimate the dominant root in the SV process to be close to one, such that volatility is non-stationary.

It has recently been demonstrated that both conventional unit root and stationarity tests suffer from potentially large size distortions in the presence of non-stationary unconditional volatility; cf., Kim, T.-H. *et al.* (2002), Busetti and Taylor (2003), Cavaliere (2004a,b) and Cavaliere and Taylor (2005,2006). These findings cast doubt over the reliability of the inferences from persistence change tests when applied to series which are subject to non-stationary volatility effects. For instance, a rejection of the null hypothesis of no change in persistence by these tests might in fact be attributable to a structural break in the unconditional volatility process rather than a true change in persistence, making these events hard to distinguish between in practice. In this paper we address this issue formally by examining the behaviour of persistence tests under a class of non-stationary unconditional volatility processes which includes smooth volatility changes, multiple volatility shifts and trending volatility, among other things.

In Section 2 the model of persistence change which we focus on will be outlined. This model extends that previously considered in the literature by allowing not only for a change in persistence in the series but also for non-stationarity in the unconditional volatility process which may be present under the constant $I(0)$ null hypothesis or under the persistence change alternative. In doing so, rather than assuming a specific parametric model for the volatility dynamics, we do not impose any constraint on the volatility dynamics, apart from the requirement that the (unconditional) variance is bounded, deterministic and displays a countable number of jumps. In Section 3 we provide a brief review of the ratio-based persistence change test statistics of Kim (2000), Kim, J. *et al.* (2002) and BT. In Section 4 we derive the large sample null distributions of these statistics against processes which display non-stationary volatility.

Section 6 uses Monte Carlo methods to explore the effects of a variety of non-stationary volatility processes, including single and multiple breaks in volatility and near-integrated autoregressive stochastic volatility, on the finite sample size and power properties of the persistence change tests. In most of these cases the size properties of the persistence change tests are found to be highly unreliable. Consequently, in Section 5 we propose wild bootstrap-based versions of the tests of Section 3. These are shown to solve the identified inference problem, providing asymptotically pivotal inference under the class of volatility processes considered here, and, in Section 6, to perform well in finite samples. In section 7 we report an application of the persistence change tests of section 3 and their bootstrap counterparts from section 5 to U.S. price inflation rate series from the Stock and Watson (2005) database. Section 8 concludes. Proofs of our main results are placed in a mathematical appendix.

Throughout the paper we will use the notation: $\mathcal{C} := C[0,1]$ to denote the space of continuous processes on $[0,1]$, and $\mathcal{D} := D[0,1]$ the space of right continuous with left limit (càdlàg) processes on $[0,1]$; '$\xrightarrow{w}$' to denote weak convergence in the space $\mathcal{D}$ endowed with the Skorohod metric, '$\xrightarrow{p}$' convergence in probability and '$\xrightarrow{w}_p$' weak convergence in probability (Giné and Zinn, 1990), in each case as the sample size diverges; $\lfloor \cdot \rfloor$ to denote the integer part of its argument; $\mathbb{I}(\cdot)$ to denote the indicator function, and '$x := y$' ('$y =: x$') to mean that x is defined by y. Reference to a variable being $O_p(T^k)$ is taken throughout to hold in its strict sense, meaning that the variable is not $o_p(T^k)$. Finally, given two processes X, Y on $[0,1]$, for any $s \in [a,b] \subseteq [0,1]$ we define $\mathcal{P}_X Y(s;a,b) := \int_a^b Y(r) X(r)' (\int_a^b X(r) X(r)' dr)^{-1} X(s)$, $\mathcal{Q}_X Y(s;a,b) := \int_a^b dY(r) X(r)' \left(\int_a^b X(r) X(r)' dr\right)^{-1} \int_a^s X(r)\ dr$, $\mathcal{P}_X^{\perp} Y(s;a,b) := Y(s) - \mathcal{P}_X Y(s;a,b)$, and $\mathcal{Q}_X^{\perp} Y(s;a,b) := Y(s) - \mathcal{Q}_X Y(s;a,b)$.

2 The Persistence Change Model

Generalising Kim (2000,p.99), *inter alia*, consider the null hypothesis, denoted H_0, that the scalar time-series process y_t is formed as the sum of a purely deterministic component, d_t, and a short memory ($I(0)$) component which displays a time-varying unconditional volatility process; that is,

$$y_t = d_t + z_{t,0}\ , \quad t = 1, ..., T \tag{1}$$

$$d_t = \mathbf{x}_t' \beta \tag{2}$$

$$z_{t,0} = \sigma_t \varepsilon_t \tag{3}$$

This DGP generalizes that of Kim (2000,p.99), reducing to Kim's model only where the process displays constant unconditional volatility; that is, $\sigma_t = \sigma$, $t = 1, ..., T$. In what follows we will assume that the following conditions hold on σ_t, ε_t and d_t in (1)-(3):

Assumption $\mathcal{V}$. The term $\{\sigma_t\}$ satisfies $\sigma_{\lfloor sT \rfloor} = \omega(s)$, where $\omega(\cdot) \in \mathcal{D}$ is a non-stochastic function with a finite number of points of discontinuity; moreover, $\omega(\cdot) > 0$ and satisfies a (uniform) first-order Lipschitz condition except at the points of discontinuity.

Assumption $\mathcal{E}$. $\{\varepsilon_t\}$ is a zero-mean, unit variance, strictly stationary mixing process with $E|\varepsilon_t|^p < \infty$ for some $p > 2$ and with mixing coefficients $\{\alpha_m\}$ satisfying $\sum_{m=0}^{\infty} \alpha_m^{2(1/r-1/p)} < \infty$ for some $r \in (2, 4]$, $r \leq p$. The long run variance $\lambda_\varepsilon^2 := \sum_{k=-\infty}^{\infty} E(\varepsilon_t \varepsilon_{t+k})$ is strictly positive. As is standard, we refer to $\{\varepsilon_t\}$ as an $I(0)$ process.

Assumption $\mathcal{X}$. $\mathbf{x}_t$ is a $(k+1) \times 1$ deterministic vector with $\mathbf{x}_{1t} = 1$, all t, and satisfying the condition that there exists a scaling matrix δ_T and a bounded piecewise continuous function $F(\cdot)$ on $[0, 1]$ such that $\delta_T \mathbf{x}_{\lfloor \cdot T \rfloor} \to \mathbf{x}(\cdot)$ uniformly on $[0, 1]$, and where, for all $\tau \in \Lambda$, $\Lambda = [\tau_l, \tau_u]$ the compact subset of $[0, 1]$ used in section 3 below, $\int_0^\tau \mathbf{x}(s)\mathbf{x}(s)' ds$ and $\int_\tau^1 \mathbf{x}(s)\mathbf{x}(s)' ds$ are both positive definite.

Under Assumption $\mathcal{V}$, $z_{t,0} := \sigma_t \varepsilon_t$ is heteroskedastic; however, $z_{t,0}$ is still short memory in the sense that its scaled partial sums admit a functional central limit theorem (see the proof of Lemma 1) and we shall therefore refer to such processes as $I(0)$ throughout the paper. Observe, that $\{y_t\}$ in (1) is therefore also $I(0)$ and heteroskedastic. Assumption $\mathcal{V}$ requires the variance process only to be non-stochastic, bounded and to display a countable number of jumps and therefore allows for an extremely wide class of possible volatility processes. Models of single or multiple variance shifts satisfy Assumption $\mathcal{V}$ with $\omega(\cdot)$ piecewise constant. For example, the function $\omega(s) := \sigma_0 + (\sigma_1 - \sigma_0)\mathbb{I}(s \geq m)$ gives the single break model with a variance shift at time $\lfloor mT \rfloor$, $0 < m < 1$. If $\omega(\cdot)^2$ is an affine function, then the unconditional variance of the errors displays a linear trend. Piecewise affine functions are also permitted, allowing for variances which follow a broken trend. Moreover, smooth transition variance shifts also satisfy Assumption $\mathcal{V}$: e.g., the function $\omega(s)^2 := \sigma_0^2 + (\sigma_1^2 - \sigma_0^2)\mathbb{S}(s)$, $\mathbb{S}(s) = (1 + \exp(-\gamma(s - m)))^{-1}$, which corresponds to a smooth (logistic) transition from σ_0^2 to σ_1^2 with transition midpoint $\lfloor mT \rfloor$ and speed of transition controlled by γ. The case of constant unconditional volatility where $\sigma_t = \sigma$, for all t, clearly satisfies Assumption $\mathcal{V}$ with $\omega(s) = \sigma$.

Remark 1. The assumption that the volatility function $\omega(\cdot)$ is non-stochastic allows for a considerable simplification of the theoretical set-up. However, we conjecture that, under suitable memory and moment conditions, this assumption can be weakened to allow for cases where the innovations $\{e_t\}$ and $\omega(\cdot)$ are stochastically independent. Indeed, in such cases if the (exogenous) volatility process $\omega(\cdot)$ has sample paths satisfying Assumption $\mathcal{A}_3$, then the results presented in the paper can be thought of as holding *conditional* on a given realization of $\omega(\cdot)$. The conditioning argument used here in the context of the volatility function serves the same purpose as the exogeneity assumption used by Perron (1989,pp.1387-8) to permit stochastic changes in the trend function. In the (exogenous) SV framework, Markov-switching variances obtain by assuming that $\omega(\cdot)$ is a strictly positive, continuous-time Markov chain with a finite number of states, while non-stationary autoregressive SV models obtain for $\omega(s) = h(J(s))$, $J(\cdot)$ a diffusion process in $\mathcal{D}$ and $h(\cdot)$ a strictly positive continuous function; see Hansen (1995).

Remark 2. Assumption $\mathcal{E}$ imposes the familiar strong mixing conditions of, *inter alia*, Phillips and Perron (1988, p.336). If $\omega(\cdot)$ is non-constant then $\{z_{t,0}\}$ is an unconditionally heteroskedastic process. Conditional heteroskedasticity is also permitted through Assumption $\mathcal{E}$; see, e.g., Hansen (1992). The strict stationarity assumption is made without loss of generality and may be weakened to allow for weak heterogeneity of the errors, as in, e.g., Phillips (1987), although here one would need to explicitly assume that λ_ε is strictly positive. Moreover, the results presented in this paper are not wedded to the mixing aspect of Assumption $\mathcal{E}$, and remain valid provided the partial sum processes involved in the construction of the statistics admit a functional central limit theorem. An important further example satisfying this condition is the linear process assumption of, *inter alia*, Phillips and Solo (1992).

Remark 3. The conditions placed on the vector $\mathbf{x}_t$ in Assumption $\mathcal{X}$ are based on the mild regularity conditions of Phillips and Xiao (1998). A leading example satisfying these conditions is given by the k-th order polynomial trend, $\mathbf{x}_t = (1, t, ..., t^k)'$. Furthermore, the broken intercept and broken intercept and trend functions considered in, for example, Busetti and Harvey (2001) are also permitted. Notice that, since the first element of $\mathbf{x}_t$ is fixed at unity throughout, model (1) always contains an intercept. □

Following Kim (2000) we consider two alternative hypotheses: the first, denoted H_{01}, is that y_t displays a change in persistence from $I(0)$ to $I(1)$ behaviour[1] at time $t = \lfloor \tau^* T \rfloor$, while the second, H_{10}, is that there is a change in persistence from $I(1)$ to $I(0)$ behaviour at time $t = \lfloor \tau^* T \rfloor$. Both may be expressed conveniently within a generalization of the persistence change data generating process (DGP) of Kim (2000,p.100)

$$y_t = d_t + z_{t,1}, \quad t = 1, ..., \lfloor \tau^* T \rfloor, \ \tau^* \in (0,1) \tag{4}$$

$$y_t = d_t + z_{t,2}, \quad t = \lfloor \tau^* T \rfloor + 1, ..., T. \tag{5}$$

The $I(0)$-$I(1)$ persistence change alternative is obtained under the alternative

$$\begin{aligned} \mathsf{H}_{01} \ : \quad & z_{t,2} = z_{t-1,2} + \sigma_t \varepsilon_t \\ & z_{t,1} = \sigma_t u_t \\ & z_{\lfloor \tau^* T \rfloor,2} = z_{\lfloor \tau^* T \rfloor,1} \end{aligned} \tag{6}$$

while the $I(1)$-$I(0)$ alternative is given by

$$\begin{aligned} \mathsf{H}_{10} \ : \quad & z_{t,1} = z_{t-1,1} + \sigma_t \varepsilon_t \\ & z_{t,2} = \sigma_t u_t + z_{\lfloor \tau^* T \rfloor,1}. \end{aligned} \tag{7}$$

[1] An $I(1)$ series is defined to be one formed from the accumulation of an $I(0)$ series.

Both (6) and (7) embody end-effect corrections, as are also used in Banerjee *et al.* (1992,p.278) and BT, which ensure that a given realization of the process will not display a spurious sharp jump in level at the break point. Under both H_{01} and H_{10} we require Assumptions $\mathcal{V}$ and $\mathcal{X}$ to hold on σ_t and $\mathbf{x}_t$, respectively. Furthermore, we require that both ε_t and u_t are $I(0)$, as stated in the following assumption.

Assumption $\mathcal{E}'$. Both $\{\varepsilon_t\}$ and $\{u_t\}$ satisfy Assumption $\mathcal{E}$ with strictly positive long-run variances, denoted by λ_ε^2 and λ_u^2, respectively.

Remark 4. Again, notice that under both H_{01} and H_{10}, (4)-(5) reduces to the corresponding persistence change model in Kim (2000) only where $\sigma_t = \sigma$, $t = 1, ...T$.

3 Persistence Change Tests

Kim (2000), Kim, J. *et al.* (2002) and BT, develop tests which reject the constant $I(0)$ null (H_0) in favour of the $I(0)$-$I(1)$ change alternative (H_{01}), based on the ratio statistic

$$\mathcal{K}(\tau) := \frac{(T - \lfloor \tau T \rfloor)^{-2} \sum_{t=\lfloor \tau T \rfloor + 1}^{T} (\breve{S}_t(\tau))^2}{\lfloor \tau T \rfloor^{-2} \sum_{t=1}^{\lfloor \tau T \rfloor} (\hat{S}_t(\tau))^2} \tag{8}$$

where

$$\breve{S}_t(\tau) := \sum_{i=\lfloor \tau T \rfloor + 1}^{t} \breve{\varepsilon}_{i,\tau}, \quad \hat{S}_t(\tau) := \sum_{i=1}^{t} \hat{\varepsilon}_{i,\tau} \tag{9}$$

where, in order to obtain exact invariance to β (the vector of parameters characterising d_t), $\hat{\varepsilon}_{t,\tau}$ are the residuals from the OLS regression of y_t on $\mathbf{x}_t$, for $t = 1, \ldots, \lfloor \tau T \rfloor$. Similarly, $\breve{\varepsilon}_{t,\tau}$ are the OLS residuals from regressing y_t on $\mathbf{x}_t$ for $t = \lfloor \tau T \rfloor + 1, ..., T$.[2]

Where the true (potential) changepoint, τ^*, is known the null of no persistence change is rejected for large values of $\mathcal{K}(\tau^*)$. However, in the more realistic case where τ^* is unknown, Kim (2000), Kim, J. *et al.* (2002) and BT consider three statistics based on the sequence of statistics $\{\mathcal{K}(\tau),\ \tau \in \Lambda\}$, where $\Lambda = [\tau_l, \tau_u]$ is a closed subset of $(0, 1)$. These are:

$$\begin{aligned} \mathcal{K}_1 &:= \max_{s \in \{\lfloor \tau_l T \rfloor, \ldots, \lfloor \tau_u T \rfloor\}} \mathcal{K}(s/T) \\ \mathcal{K}_2 &:= T_*^{-1} \sum_{s=\lfloor \tau_l T \rfloor}^{\lfloor \tau_u T \rfloor} \mathcal{K}(s/T) \\ \mathcal{K}_3 &:= \ln \left\{ T_*^{-1} \sum_{s=\lfloor \tau_l T \rfloor}^{\lfloor \tau_u T \rfloor} \exp\left(\frac{1}{2}\mathcal{K}(s/T)\right) \right\}, \end{aligned} \tag{10}$$

[2]When constructing the sub-sample residuals, $\hat{\varepsilon}_{t,\tau}$ and $\breve{\varepsilon}_{t,\tau}$, if any of the elements of $\mathbf{x}_t$, other than the first, are constant throughout the sub-sample they must be omitted from $\mathbf{x}_t$, in accordance with the requirement that both $\int_0^\tau \mathbf{x}(s)\mathbf{x}(s)' ds$ and $\int_\tau^1 \mathbf{x}(s)\mathbf{x}(s)' ds$ must be positive definite.

where $T_* \equiv \lfloor \tau_u T \rfloor - \lfloor \tau_l T \rfloor + 1$. The first of these, after Andrews (1993), takes the maximum over the sequence, the second uses Hansen's (1991) mean score statistic, and the third Andrews and Ploberger's (1994) mean-exponential statistic. In each case the null is rejected for large values of these statistics.

In order to test H_0 against the $I(1)$-$I(0)$ change DGP (H_{10}), BT propose further tests based on the sequence of *reciprocals* of $\mathcal{K}(\tau)$, $\tau \in \Lambda$; precisely,

$$
\begin{aligned}
\mathcal{K}_1' &:= \max_{s \in \{\lfloor \tau_l T \rfloor, \dots, \lfloor \tau_u T \rfloor\}} \mathcal{K}(s/T)^{-1} \\
\mathcal{K}_2' &:= T_*^{-1} \sum_{s=\lfloor \tau_l T \rfloor}^{\lfloor \tau_u T \rfloor} \mathcal{K}(s/T)^{-1} \\
\mathcal{K}_3' &:= \ln \left\{ T_*^{-1} \sum_{s=\lfloor \tau_l T \rfloor}^{\lfloor \tau_u T \rfloor} \exp(\frac{1}{2}\mathcal{K}(s/T)^{-1}) \right\},
\end{aligned}
\tag{11}
$$

and, in order to test against an unknown direction of change (that is, either a change from $I(0)$ to $I(1)$ or *vice versa*), they also propose

$$
\mathcal{K}_4 := \max(\mathcal{K}_1, \mathcal{K}_1'), \quad \mathcal{K}_5 := \max(\mathcal{K}_2, \mathcal{K}_2'), \quad \mathcal{K}_6 := \max(\mathcal{K}_3, \mathcal{K}_3') .
$$

Representations for and critical values from the limiting distributions of the foregoing statistics under the null hypothesis (1)-(3) in the constant unconditional volatility case, $\sigma_t = \sigma$, for all t, are given in Kim, J. *et al.* (2002) and BT. Crucially, they show that these representations do not depend on the long run variance of $\{\varepsilon_t\}$, λ_ε^2, even though neither the numerator nor the denominator of $\mathcal{K}(\tau)$ of (8) is scaled by a long run variance estimator.

Although the original ratio-based tests of Kim (2000), Kim, J. *et al.* (2002) and BT are based on statistics where no variance estimator is employed, Leybourne and Taylor (2004) have recently discussed tests based on statistics where the numerator and denominator of (8) are scaled by appropriate sub-sample long run variance estimators. Precisely, they consider replacing $\mathcal{K}(\tau)$ of (8), for each $\tau \in \Lambda$, by the modified (standardized) statistic

$$
\mathcal{K}^*(\tau) := \frac{\hat{\lambda}^2_{m_T, \lfloor \tau T \rfloor}}{\breve{\lambda}^2_{m_T, \lfloor \tau T \rfloor}} \mathcal{K}(\tau) \tag{12}
$$

where, following Kwiatkowski et al. (1992) [KPSS],

$$
\begin{aligned}
\hat{\lambda}^2_{m_T, \lfloor \tau T \rfloor} \ &: \ = \frac{1}{\lfloor \tau T \rfloor} \sum_{t=1}^{\lfloor \tau T \rfloor} \hat{\varepsilon}^2_{t,\tau} + \frac{2}{\lfloor \tau T \rfloor} \sum_{j=1}^{\lfloor \tau T \rfloor - 1} k\left(j/m_T\right) \sum_{t=j+1}^{\lfloor \tau T \rfloor} \hat{\varepsilon}_{t,\tau} \hat{\varepsilon}_{t-j,\tau} \\
\breve{\lambda}^2_{m_T, \lfloor \tau T \rfloor} \ &: \ = \frac{1}{T - \lfloor \tau T \rfloor} \sum_{t=\lfloor \tau T \rfloor + 1}^{T} \breve{\varepsilon}^2_{t,\tau} + \frac{2}{T - \lfloor \tau T \rfloor} \sum_{j=1}^{T - \lfloor \tau T \rfloor - 1} k\left(j/m_T\right) \sum_{t=j+\lfloor \tau T \rfloor + 1}^{T} \breve{\varepsilon}_{t,\tau} \breve{\varepsilon}_{t-j,\tau},
\end{aligned}
$$

with $k(\cdot)$ any suitable kernel function (see Assumption $\mathcal{K}$ below), are long run variance estimators applied to the first $\lfloor \tau T \rfloor$ and last $T - \lfloor \tau T \rfloor$ sample observations respectively. The various tests for a change in persistence occurring at an unknown date are then constructed as above, replacing $\mathcal{K}(\tau)$ by $\mathcal{K}^*(\tau)$ throughout. With an obvious notation we denote these statistics as $\mathcal{K}_j^*$, $j = 1, ..., 6$ and $\mathcal{K}_j'^*$, $j = 1, ..., 3$. The limiting null distribution of each of these statistics coincides with that of the corresponding un-standardized statistic. Using the Bartlett kernel function

$$k\left(j/m_T\right) = \omega_B\left(j/m_T\right), \omega_B\left(x\right) := \left(1 - x\right) I\left(x \leq 1\right)$$

Leybourne and Taylor (2004) find significant improvements in the finite sample size properties of the tests based on $\mathcal{K}^*(\tau)$ in the presence of weak dependence in $\{\varepsilon_t\}$. The bandwidth parameter m_T used in $\mathcal{K}^*(\tau)$ is not required to grow to infinity as the sample size diverges to obtain pivotal limiting distributions. Indeed, Leybourne and Taylor (2004) find that setting $m_T = 1$ or $m_T = 2$ (i.e., the sub-sample long run variance estimators contain either zero or one lagged covariance terms) provides a useful pragmatic balance between re-dressing the finite size problems of the tests under weakly dependence yet keeping power losses, relative to the un-standardized tests, when there is persistence change relatively small.

4 The Effects of Non-stationary Volatility

In this section we derive the asymptotic distribution of existing persistence change tests of section 3 in the presence of time-varying unconditional variances satisfying Assumption $\mathcal{V}$. First, in section 4.1, in order to assess whether the presence of non-stationary volatility might be confused with a change in persistence, we derive representations for the asymptotic (null) distributions of the persistence change tests under H_0. Second, in section 4.2, we analyze to what extent non-stationary volatility affects the power properties (consistency) of the tests by analysing their large sample behaviour under H_{01} and H_{10}.

In what follows, two key processes will play a fundamental role. The first is given by the following function in $\mathcal{C}$:

$$\eta\left(s\right) := \left(\int_0^1 \omega(r)^2 dr\right)^{-1} \int_0^s \omega(r)^2 dr \; ; \tag{13}$$

which we term the *variance profile*. The second is the process

$$B_\omega\left(s\right) := \frac{\int_0^s \omega(r) dB(r)}{\left(\int_0^1 \omega(r)^2 dr\right)^{1/2}}$$

which, up to a scaling factor, is the diffusion solving the stochastic differential equation, $dB_\omega\left(s\right) = \omega\left(r\right) dB\left(r\right)$, $B(\cdot)$ a standard Brownian motion.

Remark 5. The variance profile satisfies $\eta(s) = s$ under constant unconditional volatility, while it deviates from s if σ_t is non-constant. Under Assumption $\mathcal{V}$, the square of the denominator of (13), say $\bar{\omega}^2 := \int_0^1 \omega(r)^2 dr$, is the limit of $T^{-1}\sum_{t=1}^T \sigma_t^2$, which may therefore be interpreted as the (asymptotic) average (unconditional) variance.

Remark 6. Since B_ω is Gaussian, has independent increments and unconditional variance $E(B_\omega(s)^2) = \eta(s)$, B_ω is a time-change Brownian motion; see Cavaliere (2004b) and Cavaliere and Taylor (2006) for further discussion on such process.

4.1 Asymptotic Size

Theorem 1 provides representations for the limiting null distributions of the persistence change tests of Kim (2000), Kim, J. *et al.* (2002) and BT under non-stationary volatility satisfying Assumption $\mathcal{V}$. Initially, we assume that the potential persistence change date τ is specified *a priori.*

Theorem 1 *Suppose that $\{y_t\}$ is generated according to the DGP (1)–(3) under Assumptions $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{X}$. Then, for any $\tau \in \Lambda$, $\mathcal{K}(\tau)$ of (8) satisfies*

$$\mathcal{K}(\tau) \xrightarrow{w} L_\omega(\tau) := \frac{(1-\tau)^{-2}\int_\tau^1 \breve{B}_\omega(s,\tau)^2 ds}{\tau^{-2}\int_0^\tau \hat{B}_\omega(s,\tau)^2 ds}$$

where $\breve{B}_\omega(s,\tau) := \mathcal{Q}_X^\perp B_\omega(s;\tau,1) - B_\omega(\tau)$ and $\hat{B}_\omega(s,\tau) := \mathcal{Q}_X^\perp B_\omega(s;0,\tau)$ are the residuals from the non-orthogonal Hilbert projections of $B_\omega(s)$ on the space spanned by $\mathbf{x}(s)$, $s \in [\tau,1]$ and $s \in [0,\tau]$, respectively.

Remark 7. The key implication of Theorem 1 is that under non-stationary volatility the persistence change tests of section 3 do not have their usual asymptotic null distributions. Rather, their distributions depend on the sample path of the volatility process, $\omega(\cdot)$.

Remark 8. In the special case $\omega(\cdot) = \sigma$, $B_\omega(\cdot)$ reduces to a standard Brownian motion and the above asymptotic distributions reduce to those given in Kim (2000), Kim, J. *et al.* (2002) and BT. □

We now derive the asymptotic null distributions of the tests when the variance standardization of Leybourne and Taylor (2004) is employed. To that end, we make the following assumption regarding the bandwidth, m_T, and kernel function, $k(\cdot)$.

Assumption $\mathcal{K}$ (de Jong, 2000). (K_1) For all $x \in \mathbb{R}$, $|k(x)| \leq 1$ and $k(x) = k(-x)$; $k(0) = 1$; $k(x)$ is continuous at 0 and for almost all $x \in \mathbb{R}$; $\int_{-\infty}^{\infty} |k(x)| dx < \infty$; $|k(x)| \leq l(x)$, where $l(x)$ is a non-increasing function such that $\int_{-\infty}^{\infty} |x| |l(x)| dx < \infty$; (K_2) $m_T \to \infty$ as $T \to \infty$, and $m_T = o(T^\gamma)$, $\gamma \leq 1/2 - 1/r$, where r is given in $\mathcal{E}$.

Remark 9. Notice that under Assumption $\mathcal{K}$, the bandwidth parameter, m_T, is assumed to increase as the sample size increases. This requirement is, however, not strictly necessary and most of the results given in this paper continue to hold if $m_T = O(1)$. In such cases, $\hat{\lambda}^2_{m_T,\lfloor \tau T \rfloor}$ and $\breve{\lambda}^2_{m_T,\lfloor \tau T \rfloor}$ no longer consistently estimate the long run variance, even in the homoskedastic case. Consistent estimation of the long run variance is, however, not required to obtain (asymptotically) similar tests under H_0. □

Theorem 2 *Under the conditions of Theorem 1 and provided that Assumption $\mathcal{K}$ also holds, then for any $\tau \in \Lambda$, $\mathcal{K}^*(\tau)$ of (12) satisfies*

$$\mathcal{K}^*(\tau) \xrightarrow{w} \kappa_\omega(\tau) L_\omega(\tau) =: L^*_\omega(\tau) \tag{14}$$

where $\kappa_\omega(\tau) := \frac{1-\tau}{\tau}[\eta(\tau)/(1-\eta(\tau))]$ is the ratio of the asymptotic average volatilities in the first and second sub-samples.

Remark 10. As Theorem 2 demonstrates, the standardization suggested in Leybourne and Taylor (2004) introduces the additional term $\kappa_\omega(\tau)$ into the asymptotic null distributions of the statistics, relative to those for the un-standardized statistics in Theorem 1. This term depends on the time-path of the volatility process, and equals unity if and only if the asymptotic average volatilities are equal in the first and second sub-samples. Notice, however, that $\kappa_\omega(\cdot)$ does not depend on the long run variance λ^2_ε.

Remark 11. As in Remark 8, in the special case where $\omega(\cdot) = \sigma$, $B_\omega(\cdot)$ reduces to the standard Brownian motion $B(\cdot)$ and $\kappa_\omega(\tau) = 1$, and, hence, the representation in (14) reduces to that given in Kim (2000), Kim, J. *et al.* (2002) and BT.

Remark 12. Interestingly, in the special case of a single break in volatility occurring at time $\lfloor \tau_\varepsilon T \rfloor$, it can be shown that $\mathcal{K}^*(\tau_\varepsilon) \xrightarrow{w} L(\tau_\varepsilon)$, which is therefore independent of the break in volatility. Hence, under these circumstances, a test based on $\mathcal{K}^*(\tau_\varepsilon)$ would be correctly sized in the limit. □

In Theorem 3 we now generalize the foregoing results to the case of an unspecified persistence change date.

Theorem 3 *Under the conditions of Theorem 1, and defining $a := (\tau_u - \tau_l)^{-1}$,*

$$\begin{aligned}
\mathcal{K}_1 &\xrightarrow{w} \sup_{\tau\in\Lambda} L_\omega(\tau) =: \mathcal{K}_{1,\infty}, & \mathcal{K}'_1 &\xrightarrow{w} \sup_{\tau\in\Lambda} L_\omega(\tau)^{-1} =: \mathcal{K}'_{1,\infty} \\
\mathcal{K}_2 &\xrightarrow{w} a\int_{\tau_l}^{\tau_u} L_\omega(\tau)\, d\tau =: \mathcal{K}_{2,\infty}, & \mathcal{K}'_2 &\xrightarrow{w} a\int_{\tau_l}^{\tau_u} L_\omega(\tau)^{-1}\, d\tau =: \mathcal{K}'_{2,\infty} \\
\mathcal{K}_3 &\xrightarrow{w} \ln\left\{a\int_{\tau_l}^{\tau_u} \exp(\tfrac{1}{2}L_\omega(\tau))d\tau\right\} =: \mathcal{K}_{3,\infty}, & \mathcal{K}'_3 &\xrightarrow{w} \ln\left\{a\int_{\tau_l}^{\tau_u} \exp(\tfrac{1}{2}L_\omega(\tau))d\tau\right\} =: \mathcal{K}'_{3,\infty}
\end{aligned}$$

while $\mathcal{K}_4 \xrightarrow{w} \max(\mathcal{K}_{1,\infty}, \mathcal{K}'_{1,\infty})$, $\mathcal{K}_5 \xrightarrow{w} \max(\mathcal{K}_{2,\infty}, \mathcal{K}'_{2,\infty})$, and $\mathcal{K}_6 \xrightarrow{w} \max(\mathcal{K}_{3,\infty}, \mathcal{K}'_{3,\infty})$. Moreover, if Assumption $\mathcal{K}$ also holds,

$$\begin{aligned}
\mathcal{K}^*_1 &\xrightarrow{w} \sup_{\tau\in\Lambda} L^*_\omega(\tau) =: \mathcal{K}^*_{1,\infty}, & \mathcal{K}'^*_1 &\xrightarrow{w} \sup_{\tau\in\Lambda} L^*_\omega(\tau)^{-1} =: \mathcal{K}'^*_{1,\infty} \\
\mathcal{K}^*_2 &\xrightarrow{w} a\int_{\tau_l}^{\tau_u} L^*_\omega(\tau)\, d\tau =: \mathcal{K}^*_{2,\infty}, & \mathcal{K}'^*_2 &\xrightarrow{w} a\int_{\tau_l}^{\tau_u} L^*_\omega(\tau)^{-1}\, d\tau =: \mathcal{K}'^*_{2,\infty} \\
\mathcal{K}^*_3 &\xrightarrow{w} \ln\left\{a\int_{\tau_l}^{\tau_u} \exp(\tfrac{1}{2}L^*_\omega(\tau))d\tau\right\} =: \mathcal{K}^*_{3,\infty}, & \mathcal{K}'^*_3 &\xrightarrow{w} \ln\left\{a\int_{\tau_l}^{\tau_u} \exp(\tfrac{1}{2}L^*_\omega(\tau))d\tau\right\} =: \mathcal{K}'^*_{3,\infty}
\end{aligned}$$

while $\mathcal{K}_4^* \xrightarrow{w} \max(\mathcal{K}_{1,\infty}^*, \mathcal{K}_{1,\infty}'^*)$, $\mathcal{K}_5^* \xrightarrow{w} \max(\mathcal{K}_{2,\infty}^*, \mathcal{K}_{2,\infty}'^*)$, *and* $\mathcal{K}_6^* \xrightarrow{w} \max(\mathcal{K}_{3,\infty}^*, \mathcal{K}_{3,\infty}'^*)$.

Remark 13. Notice that, even under the conditions of Remark 12, $\mathcal{K}_j^*$, $j = 1, ..., 6$, and $\mathcal{K}_i'^*$, $i = 1, ..., 3$, will not have pivotal limiting null distributions because the (asymptotic) invariance to the break in that case occurs only at $\tau = \tau_\varepsilon$.

4.2 Consistency

We now turn to an analysis of the consistency properties of the persistence change tests of section 3 under non-stationary volatility satisfying Assumption $\mathcal{V}$. In sections 4.2.1 and 4.2.2 we derive the large sample distributions of the basic and standardized statistics, respectively, of section 3, together with the consistency rates of the associated tests, under the persistence change model H_{01}; recall from Section 2 that this model corresponds to a change in persistence from $I(0)$ to $I(1)$ behaviour occurring at time $\lfloor \tau^* T \rfloor$ for some $\tau^* \in (0, 1)$. Results for the tests under H_{10} are briefly discussed in section 4.2.3.

4.2.1 H_{01}: ratio tests

We first analyze the behaviour of a test based on $\mathcal{K}(\tau)$ in the following theorem, where the following notation is used: $B_\omega^*(\cdot) := B_\omega(\cdot)\mathbb{I}(\cdot \geq \tau^*)$, $\mathbb{B}_\omega(\cdot) := \int_0^{\cdot} B_\omega$ and $\mathbb{B}_\omega^*(\cdot) := \int_0^{\cdot} B_\omega^*$.

Theorem 4 *Suppose that* $\{y_t\}$ *is generated according to the DGP (4)-(5) under* H_{01} *of (6) and Assumptions* $\mathcal{V}$, $\mathcal{E}'$ *and* $\mathcal{X}$. *Then, for* $0 < \tau^* < \tau < 1$, $\mathcal{K}(\tau)$ *of (8) satisfies*

$$\mathcal{K}(\tau) \xrightarrow{w} \frac{\tau^2 \int_\tau^1 \left(\mathcal{Q}_{\mathbf{x}}^{\perp} \mathbb{B}_\omega(s; \tau, 1) - \mathbb{B}_\omega(\tau)\right)^2 ds}{(1-\tau)^2 \int_0^\tau \left(\mathcal{Q}_{\mathbf{x}}^{\perp} \mathbb{B}_\omega^*(s; 0, \tau)\right)^2 ds} \tag{15}$$

while, for $0 < \tau \leq \tau^* < 1$,

$$T^{-2}\mathcal{K}(\tau) \xrightarrow{w} \frac{\tau^2 \lambda_\varepsilon^2 \int_\tau^1 \left(\mathcal{Q}_{\mathbf{x}}^{\perp} \mathbb{B}_\omega^*(s; \tau, 1) - \mathbb{B}_\omega^*(\tau)\right)^2 ds}{(1-\tau)^2 \lambda_u^2 \int_0^\tau \hat{B}_\omega(s, \tau)^2 ds} \ . \tag{16}$$

For the tests based on an unknown persistence change date, we have the following corollary of Theorem 4:

Corollary 1 *Under the conditions of Theorem 4, provided* $[0, \tau^*] \cap [\tau_l, \tau_u] \neq \varnothing$, $\mathcal{K}_i$, $i = 1, ..., 6$, *are of* $O_p(T^2)$. *Conversely* $\mathcal{K}_i'$, $i = 1, ..., 3$, *are of* $O_p(1)$.

As can be seen from the results in Theorem 4, a persistence change test based on $\mathcal{K}(\tau)$ will be consistent at rate $O_p(T^2)$ provided $\tau \leq \tau^*$, as will the tests based on the $\mathcal{K}_i$, $i = 1, ..., 6$, statistics provided $\tau^* \in \Lambda$ (i.e. provided the persistence changepoint is included in the search set). These are the same rates of consistency as hold for these tests in the constant unconditional volatility case; see BT. However, since all of these statistics (scaled by T^{-2}) have distributions which depend upon the dynamics of the volatility process, it is anticipated that the finite sample power of the associated tests will depend on the time-series behaviour of the underlying volatility process. Notice also that although not consistent under H_{01}, the behaviour of tests based on the $\mathcal{K}(\tau)$ statistic for $\tau > \tau^*$ and on $\mathcal{K}_i'$, $i = 1, ..., 3$, will also depend on the volatility process.

4.2.2 H_{01}: standardized ratio tests

We now derive the large sample properties of the standardized persistence change tests of Leybourne and Taylor (2004) under H_{01}. As discussed in section 3, these require a choice of the bandwidth parameter, m_T, which, as would be expected, affects the consistency rate under the alternative. This result is formalized in Theorem 5.

Theorem 5 *Let the conditions of Theorem 4 hold and let Assumption $\mathcal{K}$ hold. Then, for $0 < \tau^* < \tau < 1$, $\mathcal{K}^*(\tau)$ of (12) satisfies*

$$\mathcal{K}^*(\tau) \xrightarrow{w} \frac{\tau}{1-\tau} \frac{\int_0^\tau \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_\omega^*(s;0,\tau)\right)^2 ds}{\int_\tau^1 \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_\omega(s;\tau,1)\right)^2 ds} \frac{\int_\tau^1 \left(\mathcal{Q}_{\mathbf{x}}^{\perp} \mathbb{B}_\omega(s;\tau,1) - \mathbb{B}_\omega(\tau)\right)^2 ds}{\int_0^\tau \left(\mathcal{Q}_{\mathbf{x}}^{\perp} \mathbb{B}_\omega^*(s;0,\tau)\right)^2 ds} \tag{17}$$

while, for $0 < \tau \leq \tau^* < 1$,

$$\frac{m_T}{T}\mathcal{K}^*(\tau) \xrightarrow{w} \frac{\tau\eta(\tau)}{(1-\tau)\int_{-\infty}^{+\infty} k} \frac{\int_\tau^1 \left(\mathcal{Q}_{\mathbf{x}}^{\perp} \mathbb{B}_\omega^*(s;\tau,1) - \mathbb{B}_\omega^*(\tau)\right)^2 ds}{\int_\tau^1 \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_\omega^*(s;\tau,1)\right)^2 ds \int_0^\tau \hat{B}_\omega(s,\tau)^2 ds} \tag{18}$$

where $k(\cdot)$ is the kernel function defined in Assumption $\mathcal{K}$.

For the case of an unknown persistence change date, we therefore have the following corollary:

Corollary 2 *Under the conditions of Theorem 5, provided $[0,\tau^*] \cap [\tau_l, \tau_u] \neq \varnothing$, $\mathcal{K}_i^*$, $i = 1, ..., 6$, are of $O_p(T/m_T)$ under Assumption $\mathcal{K}$. Conversely $\mathcal{K}_i'^*$, $i = 1, ..., 3$, are of $O_p(1)$.*

As with the results in section 4.2.1, the standardized persistence change statistics have limiting distributions which depend on the underlying volatility process, so that again the volatility process is anticipated to impact on the finite sample behaviour of the tests. Moreover, the rate of consistency of tests based on $\mathcal{K}^*(\tau)$ is also slowed down, relative to those based on $\mathcal{K}(\tau)$, since, under H_{01}, $\mathcal{K}^*(\tau)$ is of $O_p(T/m_T)$, provided $\tau \leq \tau^*$. Again, these are the same rates of consistency as apply to these tests in the constant unconditional volatility case; see Leybourne and Taylor (2004).

Remark 14. It can be shown that Leybourne and Taylor's (2004) suggestion of $m_T = 1$ yields tests, $\mathcal{K}_i^*$, $i = 1, ..., 6$, which are consistent at rate $\mathrm{O}_p(T)$, provided $\tau^* \in \Lambda$. This result holds for any finite integer value of m_T.

4.2.3 Results under H_{10}

Under the alternative of a change from $I(1)$ to $I(0)$ behaviour at time $\lfloor T\tau^*\rfloor$, that is H_{10}, a very similar analysis (omitted in the interests of brevity) to that given above under H_{01} shows that for $\tau \geq \tau^*$, $\mathcal{K}(\tau)^{-1}$ $[\mathcal{K}^*(\tau)^{-1}]$ is of $O_p(T^2)$ $[O_p(T/m_T)]$, while for $\tau < \tau^*$, $\mathcal{K}(\tau)^{-1}$ and $\mathcal{K}^*(\tau)^{-1}$ are both of $O_p(1)$. Consequently, if the intersection of the intervals $[\tau^*, 1]$ and Λ is non-empty then $\mathcal{K}_j'$, $j = 1, ..., 3$, and $\mathcal{K}_k$, $k = 4, ..., 6$, $[\mathcal{K}_j'^*$, $j = 1, ..., 3$, and $\mathcal{K}_k^*$, $k = 4, ..., 6]$ are each of $O_p(T^2)$ $[O_p(T/m_T)]$, but are otherwise of $O_p(1)$, while the $\mathcal{K}_j$ and $\mathcal{K}_j^*$ $j = 1, ..., 3$, are each of $O_p(1)$ for all $\tau \in \Lambda$. As with the results under H_{01}, the limiting distributions of all of these statistics (scaled where appropriate) can be shown to depend on the dynamics of the underlying volatility process.

5 Bootstrap Persistence Change Tests

In order to overcome the inference problems identified above with the persistence change tests of section 3, in this section we propose bootstrap versions of these tests. We demonstrate that in the presence of volatility satisfying Assumption $\mathcal{V}$ the bootstrap tests provide asymptotically pivotal inference under H_0. We also derive their consistency properties under H_{01} and H_{01}. In order to account for $\mathbf{x}_t$, the test builds on Hansen's (2000) heteroskedastic fixed regressor (wild) bootstrap; see also Cavaliere and Taylor (2005). Our bootstrap tests for both the known and unknown changepoint cases are outlined in section 5.1. Their large sample size and power properties are established in sections 5.2 and 5.3 respectively.

5.1 The Bootstrap Algorithm

The first stage of the bootstrap algorithm is to compute the full sample residuals, say $\tilde{\varepsilon}_t$, obtained by regressing y_t on $\mathbf{x}_t$ for $t = 1, ..., T$. A bootstrap sample is then generated as

$$y_t^b := \tilde{\varepsilon}_t w_t \ , \ t = 1, ..., T, \tag{19}$$

with $\{w_t\}_{t=1}^T$ an independent $N(0, 1)$ sequence. Now, let $\breve{\varepsilon}_{t,\tau}^b$ be defined as the residuals obtained from the OLS projection of y_t^b on $\mathbf{x}_t$ for $t = \lfloor T\tau\rfloor + 1, ..., T$; similarly, let $\hat{\varepsilon}_{t,\tau}^b$ be defined as the residuals obtained from the OLS projection of y_t^b on $\mathbf{x}_t$ for $t = 1, ..., \lfloor T\tau\rfloor$.

The bootstrap analogue of $\mathcal{K}(\tau)$ of (8) is then given by the statistic

$$\mathcal{K}^b(\tau) := \frac{(T - \lfloor \tau T\rfloor)^{-2} \sum_{t=\lfloor \tau T\rfloor+1}^{T} \left(\sum_{i=\lfloor \tau T\rfloor+1}^{t} \breve{\varepsilon}_{i,\tau}^b\right)^2}{\lfloor \tau T\rfloor^{-2} \sum_{t=1}^{\lfloor \tau T\rfloor} \left(\sum_{i=1}^{t} \hat{\varepsilon}_{i,\tau}^b\right)^2} \tag{20}$$

which corresponds to the statistic in (8) except that it is constructed from the pseudo-residuals $\breve{\varepsilon}_{t,\tau}^b$ and $\hat{\varepsilon}_{t,\tau}^b$ rather than the residuals based on the original time series, $\breve{\varepsilon}_{t,\tau}$ and $\hat{\varepsilon}_{t,\tau}$, respectively. The associated bootstrap p-value is then given by $p_T^b(\tau) :=$

$1 - G_T^b(\mathcal{K}(\tau);\tau)$, where $G_T^b(\cdot;\tau)$ denotes the cumulative distribution function (cdf) of $\mathcal{K}^b(\tau)$. In the case of Leybourne and Taylor's (2004) standardized version of $\mathcal{K}(\tau)$, $\mathcal{K}^*(\tau)$ of (12), the bootstrap p-value is given by $p_T^{*b}(\tau) := 1 - G_T^{*b}(\mathcal{K}^*(\tau);\tau)$, where $G_T^{*b}(\cdot;\tau)$ denotes the cdf of the bootstrap statistic

$$\mathcal{K}^{*b}(\tau) = \frac{\hat{\lambda}^{b\,2}_{m_T^b,\lfloor\tau T\rfloor}}{\check{\lambda}^{b\,2}_{m_T^b,\lfloor\tau T\rfloor}}\mathcal{K}^b(\tau)$$

where $\hat{\lambda}^{b\,2}_{m_T^b,\lfloor\tau T\rfloor}$ and $\check{\lambda}^{b\,2}_{m_T^b,\lfloor\tau T\rfloor}$ are long run variance estimators of the same form as used in (12), with bandwidth m_T^b, applied to, respectively, the first $\lfloor\tau T\rfloor$ and last $T - \lfloor\tau T\rfloor$ observations from the bootstrap sample, y_t^b, $t = 1, ..., T$.

Where the (potential) changepoint τ^* is known, the foregoing quantities are evaluated at $\tau = \tau^*$. Where the potential persistence change point is not specified *a priori* we form the corresponding bootstrap equivalents of the $\mathcal{K}_j$ and $\mathcal{K}_j^*$, $j = 1, ..., 6$, and $\mathcal{K}_j'$ and $\mathcal{K}_j'^*$, $j = 1, ..., 3$, tests of section 3. For brevity, but without loss of generality, we shall confine our discussion to the $\mathcal{K}_1$ and $\mathcal{K}_1^*$ tests. Exactly the same reasoning extends straightforwardly to the other tests in an obvious way. The bootstrap analogue of $\mathcal{K}_1$ of (10) is constructed as

$$\mathcal{K}_1^b := \max_{s\in\{\lfloor\tau_l T\rfloor,...,\lfloor\tau_u T\rfloor\}} \mathcal{K}^b(s/T),$$

with the associated bootstrap p-value given by $p_{1,T}^b := 1 - G_{1,T}^b(\mathcal{K}_1)$, where $G_{1,T}^b(\cdot)$ denotes the cdf of $\mathcal{K}_1^b$. The bootstrap version of the $\mathcal{K}_1^*$ test is constructed in a similar manner. Specifically, the bootstrap analogue of $\mathcal{K}_1^*$ of (11) is given by

$$\mathcal{K}_1^{*b} := \max_{s\in\{\lfloor\tau_l T\rfloor,...,\lfloor\tau_u T\rfloor\}} \mathcal{K}^{*b}(s/T)$$

with associated p-value $p_{1,T}^{*b} := 1 - G_{1,T}^{*b}(\mathcal{K}_1^*)$, where $G_{1,T}^{*b}(\cdot)$ denotes the cdf of $\mathcal{K}_1^{*b}$. The bootstrap analogues of the $\mathcal{K}_j$ and $\mathcal{K}_j^*$, $j = 2, ..., 6$, and $\mathcal{K}_j'$ and $\mathcal{K}_j'^*$, $j = 1, ..., 3$, statistics will be denoted similarly as $\mathcal{K}_j^b$ and $\mathcal{K}_j^{*b}$, $j = 2, ..., 6$, and $\mathcal{K}_j'^b$ and $\mathcal{K}_j'^{*b}$, $j = 1, ..., 3$, respectively.

Remark 15. In practice the cdfs $G_T^b(\cdot;\tau)$, $G_T^{*b}(\cdot;\tau)$, $G_{1,T}^b(\cdot)$ and $G_{1,T}^{*b}(\cdot)$ will be unknown. However, they can be approximated in the usual way. Taking the $\mathcal{K}_1$ statistic to illustrate the procedure is as follows. Generate N conditionally independent bootstrap statistics, $\mathcal{K}_{1,i}^b$, $i = 1, ..., N$, computed as above but from $y_{i,t}^b := \tilde{\varepsilon}_t w_{i,t}$, $t = 1, ..., T$ with $\{\{w_{i,t}\}_{t=1}^T\}_{i=1}^N$ a doubly independent $N(0,1)$ sequence. The simulated bootstrap p-value is then given by $\tilde{p}_{1,T}^b := N^{-1}\sum_{i=1}^N \mathbb{I}\left(\mathcal{K}_{1,i}^b \geq \mathcal{K}_1\right)$. By standard arguments, see e.g. Hansen (1996), $\tilde{p}_{1,T}^b$ is consistent for $p_{1,T}^b$ as $N \to \infty$.

Remark 16. As is well known in the wild bootstrap literature (see Davidson and Flachaire, 2001, for a review) in certain cases better bootstrap accuracy can be obtained by replacing the Gaussian distribution used for generating the pseudo-data w_t in (19) by an asymmetric distribution with $E(w_t) = 0$, $E(w_t^2) = 1$ and $E(w_t^3) = 1$ (Liu, 1988). A well known example of this is Mammen's (1993) two-point distribution: $P(w_t = -0.5(\sqrt{5}-1)) = 0.5(\sqrt{5}+1)/\sqrt{5} =: \pi$, $P(w_t = 0.5(\sqrt{5}+1)) = 1-\pi$. We found no discernible differences between the finite sample properties of the bootstrap persistence tests based on the Gaussian or Mammen's distribution. □

5.1.1 Asymptotic Size

The next two theorems establish that in the presence of volatility satisfying Assumption $\mathcal{V}$, the bootstrap p-values defined above are asymptotically pivotal and uniformly distributed and, hence, that the associated bootstrap tests are correctly sized for samples of sufficiently large dimension.

Theorem 6 *Under the conditions of Theorem 1: (i) for all* $\tau \in \Lambda$, $\mathcal{K}^b(\tau) \xrightarrow{w}_p L_\omega(\tau)$, *and* $p_T^b(\tau) \xrightarrow{w} U[0,1]$, *a uniform distribution on* $[0,1]$*; (ii)* $\mathcal{K}_1^b \xrightarrow{w}_p \mathcal{K}_{1,\infty}$ *and* $p_{1,T}^b \xrightarrow{w} U[0,1]$.

Turning to the studentized bootstrap statistics, $\mathcal{K}^{*b}(\tau)$ and $\mathcal{K}_1^{*b}$, provided we make the additional assumption that ε_t has finite fourth moments, the following results hold under H_0.

Theorem 7 *Under the conditions of Theorem 2, and if* $E(\varepsilon_t^4) < \infty$ *and* $m_T^b/T^{1/2} \to 0$ *as* $T \to \infty$*, then: (i) for all* $\tau \in \Lambda$, $\mathcal{K}^{*b}(\tau) \xrightarrow{w}_p L_\omega^*(\tau)$ *and* $p_T^{*b}(\tau) \xrightarrow{w} U[0,1]$*, and (ii)* $\mathcal{K}_1^{*b} \xrightarrow{w}_p \mathcal{K}_{1,\infty}^*$ *and* $p_{1,T}^{*b} \xrightarrow{w} U[0,1]$.

Remark 17. Theorems 6-7 show that as the sample size diverges, the bootstrap statistics, $\mathcal{K}^b(\tau)$, $\mathcal{K}_1^b$, $\mathcal{K}^{*b}(\tau)$ and $\mathcal{K}_1^{*b}$, have the same null distribution as those of the original statistics, $\mathcal{K}(\tau)$, $\mathcal{K}_1$, $\mathcal{K}^*(\tau)$ and $\mathcal{K}_1^*$, respectively, and, hence, that the associated bootstrap p-values are uniformly distributed under the null hypothesis, leading to tests with (asymptotically) correct size. These results hold for any volatility process satisfying Assumption $\mathcal{V}$.

Remark 18. In relation to the bootstrap $\mathcal{K}^{*b}(\tau)$ and $\mathcal{K}_1^{*b}(\tau)$ statistics, it is worth noting that m_T^b can either be fixed or diverge at rate $o(T^{1/2})$. Moreover, m_T^b needs not equal the bandwidth parameter, m_T, used to compute the original statistic, $\mathcal{K}^*(\tau)$. □

5.1.2 Consistency Rates

We now consider the behaviour of the bootstrap tests of section 5.1 under the $I(0)$-$I(1)$ persistence change alternative, H_{01}. We will demonstrate that the bootstrap tests attain exactly the same rates of consistency as given for the corresponding standard test from section 4.2. Formal statements of the asymptotic distribution of the bootstrap statistics under H_{01} are provided in the appendix.

Theorem 8 *Under the conditions of Theorem 4, for* $0 < \tau < 1$, $\mathcal{K}^b(\tau) = O_p(1)$ *and* $\mathcal{K}_1^b(\tau) = O_p(1)$. *Consequently, provided* $\tau \leq \tau^*$, $p_T^b(\tau^*) \xrightarrow{p} 0$. *Moreover, provided* $[0, \tau^*] \cap [\tau_l, \tau_u] \neq \varnothing$, $p_{1,T}^b \xrightarrow{p} 0$.

Theorem 9 *Under the conditions of Theorem 5, and if* $E(\varepsilon_t^4) < \infty$ *and* $m_T^b / T^{1/2} \to 0$ *as* $T \to \infty$, *then for* $0 < \tau < 1$, $\mathcal{K}^{*b}(\tau) = O_p(1)$ *and* $\mathcal{K}_1^{*b}(\tau) = O_p(1)$. *Consequently, provided* $\tau \leq \tau^*$, $p_T^{*b}(\tau^*) \xrightarrow{p} 0$*; furthermore, provided* $[0, \tau^*] \cap [\tau_l, \tau_u] \neq \varnothing$, $p_{1,T}^{*b} \xrightarrow{p} 0$.

Remark 19. An important consequence of the results in Theorems 8 and 9 is that, as with their standard counterparts, the bootstrap $\mathcal{K}^b(\tau)$ and $\mathcal{K}^{*b}(\tau)$ tests are consistent at rates $O_p(T^2)$ and $O_p(T/m_T)$, respectively, provided $\tau \leq \tau^*$. This is the case because while the bootstrap $\mathcal{K}^b(\tau)$ and $\mathcal{K}_1^b(\tau)$ statistics are both of $O_p(1)$ for all τ, the $\mathcal{K}(\tau)$ and $\mathcal{K}^*(\tau)$ statistics diverge at rates $O_p(T^2)$ and $O_p(T/m_T)$, respectively, provided $\tau \leq \tau^*$; cf. Theorems 4 and 5. Similarly, the bootstrap $\mathcal{K}_i^b$ and $\mathcal{K}_i^{*b}$, $i = 1, ..., 6$, tests are also consistent at rates $O_p(T^2)$ and $O_p(T/m_T)$, respectively, provided $\tau^* \in \Lambda$. Notice, moreover, that these results hold irrespective of the choice of the bootstrap bandwidth parameter, m_T^b.

Remark 20. Observe from (A.14) and (A.17) in the proof of Theorems 8 and 9 that the limiting distributions of the bootstrap statistics under H_{01} again depend on the behaviour of the underlying volatility process through $\omega(\cdot)$ of (13). However, it is important to note that these distributions are not the same as those obtained under H_0 (cf. Theorems 6 and 7) nor do they coincide with those of the (scaled) standard tests under H_{01} (cf. Theorems 4 and 5). The asymptotic theory therefore predicts that the finite sample power properties of the standard and corresponding bootstrap tests will not, in general, coincide.

Remark 21. Under H_{10} the bootstrap statistics all remain of $O_p(1)$ for all τ and, hence, the bootstrap tests will all have same rates of consistency as noted in section 4.2.3. For example, bootstrap implementations of the $\mathcal{K}_j'$, $j = 1, 2, 3$ and $\mathcal{K}_i$, $i = 4, 5, 6$ tests will therefore all be consistent at rate $O_p(T^2)$. Full details are omitted in the interests of brevity but are available on request. □

6 Numerical Results

In this section we use Monte Carlo simulation methods to compare the finite sample size and power properties of the $\mathcal{K}_1$, $\mathcal{K}_1'$, $\mathcal{K}_4$, $\mathcal{K}_1^*$, $\mathcal{K}_1'^*$ and $\mathcal{K}_4^*$ persistence change tests of section 3, the tests being run at the nominal (asymptotic) 5% level using the critical values from BT (Table 1, p.38), with their bootstrap counterparts of section 5, based on de-meaned ($\mathbf{x}_t = 1$) data, for a variety of volatility processes.[3] The finite sample size and power properties of the tests are discussed in sections 6.1 and 6.2 respectively.

[3]Results for the other persistence change tests discussed in this paper and for tests based on de-trended data are qualitatively similar and are available on request.

As is typical we take the search set Λ to be $[0.2, 0.8]$. Results are reported for samples of size $T = 100$ and 200, with all experiments conducted using $10,000$ replications and the `rndKMn` random number generator of Gauss 5.0. All bootstrap tests used $N = 400$ bootstrap replications; cf. Remark 15. For the standardized ratio tests we set $m_T = 1$ (thereby yielding OLS sub-sample variance estimators), as suggested by Leybourne and Taylor (2004), and, accordingly, we also set a bandwidth of $m_T^b = 1$ in their bootstrap counterparts.

Results are reported for the following models for σ_t:

Model 1. (Single volatility shift): $\sigma_t = \sigma_0^* + (\sigma_1^* - \sigma_0^*)\mathbb{I}\,(t \geq \tau_\varepsilon T)$, with $\tau_\varepsilon = 0.5$.

Model 2. (Trending volatility): Volatility follows a linear trend, between σ_0^* for $t = 1$ and σ_1^* for $t = T$; that is, $\sigma_t = \sigma_0^* + (\sigma_1^* - \sigma_0^*)\left(\frac{t-1}{T-1}\right)$, $t = 1, ..., T$.

Model 3. (Exponential (near-) integrated stochastic volatility): Following Hansen (1995, p.1116), the volatility process is generated as $\sigma_t = \sigma_0^* \exp(\frac{1}{2}\nu b_t/\sqrt{T})$ where b_t is generated according to the first-order autoregression, $b_t = (1-c/T)b_{t-1} + k_t$, $t = 1, ..., T$, with $k_t \sim NIID(0, 1)$ and $b_0 = 0$.

Without loss of generality, we set $\sigma_0^* = 1$ in all cases. For Model 1 we let the ratio $\delta := \sigma_0^*/\sigma_1^*$ vary among $\{1, 1/3, 3\}$ (notice that $\delta = 1$ yields a benchmark constant volatility process) so that both positive ($\delta < 1$) and negative ($\delta > 1$) breaks in volatility are allowed. For Model 2 we let $\delta := \sigma_0^*/\sigma_1^*$ take values among $\{1/3, 3\}$ so that both positively and negatively trending volatilities are generated. For Model 3 we consider $\nu = 5$ and vary c among $\{0, 10\}$.[4]

6.1 Size Properties

Table 1 reports the empirical rejection frequencies (sizes), for the $\mathcal{K}_1$, $\mathcal{K}_1'$, $\mathcal{K}_4$, $\mathcal{K}_1^*$, $\mathcal{K}_1'^*$ and $\mathcal{K}_4^*$ tests for data generated according to the null DGP (no persistence change) (1)-(3) with $\beta = 0$ (without loss of generality) and σ_t generated according to the models detailed above. The innovation process $\{\varepsilon_t\}$ was generated according to the $ARMA(1, 1)$ design,

$$\varepsilon_t = \phi\,\varepsilon_{t-1} + v_t - \theta v_{t-1}, \quad v_t \sim NIID(0, 1)$$

with $(\phi, \theta) \in \{(0, 0), (0.5, 0), (0, 0.5)\}$, thereby allowing for IID, $AR(1)$ and $MA(1)$ innovations. Corresponding size results for the bootstrap counterpart tests are reported in Table 2.

Consider first Model 1, the case of a single break in volatility. Where $\delta \neq 1$ the results in Table 1 highlight the presence of large size distortions in the basic persistence change tests. For $\delta = 1/3$ the $\mathcal{K}_1$ test for a change in persistence from $I(0)$ to $I(1)$ is severely over-sized when $\delta = 1/3$ and severely under-sized when $\delta = 3$. The reverse

[4]For each model other combinations of parameter values were also considered, but these qualitatively add little to the reported results.

pattern holds for the $\mathcal{K}_1'$ test for a change from $I(1)$ to $I(0)$. The $\mathcal{K}_4$ test for either direction of change is severely over-sized for both $\delta = 1/3$ and $\delta = 3$. These size distortions vary slightly with ϕ and θ, with sizes increased (decreased), relative to $\phi = \theta = 0$, when $\phi > 0$ ($\theta > 0$): this pattern is also observed under Models 2 and 3. The studentized $\mathcal{K}_1^*$, $\mathcal{K}_1'^*$ and $\mathcal{K}_4^*$ tests appear much better behaved avoiding the large over-size problems that are seen with the basic tests when $\delta \neq 1$. It should, of course, be stressed that these statistics do not have pivotal limiting null distributions (cf. Theorems 2 and 3) and so while the distortions are modest for the models considered here this should not be expected to necessarily hold in general. The studentised tests also appear somewhat less dependent on ϕ and θ than the basic tests. Turning to Table 2, it is seen that the bootstrap tests also generally avoid the size distortions seen in the basic tests under the non-stationary volatility models considered and appear to deliver further improvements relative to the size properties of studentized tests, as should be expected; cf. Theorems 6 and 7.

Tables 1 − 5 about here.

The results for Model 2 in Tables 1 and 2 suggest that in general, linear trending volatility has a lower impact on the size of the standard tests than abrupt changes, for a given value of δ, although where under-sizing occurs it tends to be slightly worse than under Model 1. The basic conclusions drawn for the relative performance of the various tests for Model 1 above appears germane here also. Finally for Model 3, severe over-sizing is again seen in the basic persistence change tests which is greatest, other things equal, for the case of $c = 0$. The studentized tests again behave better but still significantly over-sized for $c = 0$. The bootstrap tests again appear to deliver a further improvement overall.

Overall, across the volatility models considered, the bootstrap $\mathcal{K}_1^{*b}$, $\mathcal{K}_1'^{*b}$ and $\mathcal{K}_4^{*b}$ tests deliver the best size control among the tests considered in the presence of both non-stationary volatility and serially correlated innovations.

6.2 Power Properties

Tables 3 and 4 report the empirical rejection frequencies (powers) and size-adjusted powers respectively, for the $\mathcal{K}_1$, $\mathcal{K}_1'$, $\mathcal{K}_4$, $\mathcal{K}_1^*$, $\mathcal{K}_1'^*$ and $\mathcal{K}_4^*$ tests for data generated according to the $I(0)$ to $I(1)$ switching $AR(1)$ DGP,

$$\begin{aligned} y_t &= \rho_t y_{t-1} + z_{t,0}, \quad t = 1, ..., T \\ z_{t,0} &= \sigma_t \varepsilon_t, \quad \varepsilon_t \sim NIID(0,1) \end{aligned}$$

where

$$\rho_t = \begin{cases} 0.8, & t = -100, ..., \lfloor \tau^* T \rfloor \\ 1.0, & t = \lfloor \tau^* T \rfloor + 1, ..., T \end{cases}$$

The persistence change-point is varied among $\tau^* \in \{0.25, 0.50, 0.75\}$, for the same set of models for σ_t as considered in section 6.1. Results for the corresponding bootstrap tests are reported in Table 5. Recall that under H_{01} the $\mathcal{K}_1'$ and $\mathcal{K}_1'^*$ tests and their bootstrap analogues are not consistent.[5]

For the case of homoskedastic errors, that is Model 1 with $\delta = 1$, there tends to be a drop in empirical power in using the bootstrap analogues of the basic $\mathcal{K}_1$ and $\mathcal{K}_4$ tests, although in all but the case of $\tau^* = 0.75$ these losses are generally quite modest. Consequently, in general, our bootstrap procedure does not seem to cause significant power losses when unnecessary. In contrast, significant power losses are seen throughout in using the studentized $\mathcal{K}_1^*$ and $\mathcal{K}_4^*$ tests and their bootstrap analogues, which display considerably lower power than both the basic and bootstrapped basic tests under homoskedasticity. This ranking also holds true, in general, under the non-stationary volatility models considered.

The effect of non-stationary volatility on power is mixed and depends on whether we consider raw or size-adjusted power for the basic tests, recalling that in some scenarios these were heavily over-sized and in others badly under-sized. Different volatility models also have different impacts on the power rankings between the various tests, as predicted by the large sample theory; cf. Theorems 4 and 5 and Remark 20. For example, under Model 3 the size-adjusted power of the basic tests is much lower than for their bootstrap equivalents, while under Models 1 and 2 the opposite tends to be the case.

Taking both size and power results into consideration, we recommend the use of the bootstrap $\mathcal{K}_1^b$, $\mathcal{K}_1'^b$ and $\mathcal{K}_4^b$ tests. Although these do not control size quite as well as the bootstrap studentised $\mathcal{K}_1^{*b}$, $\mathcal{K}_1'^{*b}$ and $\mathcal{K}_4^{*b}$ tests in the presence of serially correlated innovations, they do not suffer the large power losses associated with the latter and do not require the additional assumption of finite fourth moments in $\{\varepsilon_t\}$; cf. Theorem 7.

7 Application to U.S. Inflation Data

In this section we apply, for $\Lambda = [0.2, 0.8]$ as in section 6, the $\mathcal{K}_i$, $\mathcal{K}_j'$, $i = 1, ..., 6$, $j = 1, ..., 3$, and the studentised ratio tests $\mathcal{K}_i^*$, $\mathcal{K}_j'^*$, $i = 1, ..., 6$, $j = 1, ..., 3$, together with the corresponding bootstrap tests to the monthly U.S. price inflation series from Stock and Watson (2005). Specifically, we consider twenty series of inflation rates, measured as the first difference of the logarithm of the relevant monthly (seasonally adjusted) price indices/deflators. The data are identified by the same reference codes as given on page 47 of Appendix A of Stock and Watson (2005). The sample period used for all series was 1967:1-2003:12.

The series are graphed in Figure 1. In order to assess the time-series behaviour of volatility in these series we also graph in Figure 2 Cavaliere and Taylor's (2006, section 4.1) estimate of the variance profile, $\omega(s)$ of (13), for each series. For almost all of the series the estimated variance profile shows substantial deviations from the 45° line which pertains to a constant variance process; cf. Remark 5. Typically these patterns

[5]Results for a corresponding $I(1)$-$I(0)$ switching $AR(1)$ DGP were also computed and gave qualitatively similar conclusions. These results are available on request.

are consistent with the presence of multiple breaks in variance. For some series the breaks appear to follow relatively abrupt transition paths (e.g. PWIMSA and PU83), while for others (e.g. PSM99Q and PUCD) the transition path tends to be slower, consistent with smooth-transition breaks. The estimated variance profile for PU85 follows a relatively smooth arc above the 45° line, consistent with negatively trending volatility, or possibly a single (relatively slow) smooth-transition variance break.

Figures 1 – 2 and Tables 6 – 7 about here

Tables 6 and 7 reports the outcome of the persistence change statistics for these data. All of the statistics were computed on de-meaned data. For each outcome two bootstrap p-values are reported. The first, denoted p_{hom}, is obtained from a standard bootstrap and the second, denoted p_{het}, from using the wild bootstrap method of section 5. The standard bootstrap was implemented exactly as detailed in section 5 except that the bootstrap sample in (19) was replaced by $y_t^b := w_t$, $t = 1, ..., T$. Finally, for the standardized ratio tests we set $m_T = m_T^b = 1$, as in section 6.

Consider first the results for the $\mathcal{K}_4$ statistic (which does not assume a known direction of persistence change *a priori*), in Table 6. Using the homoskedastic bootstrap p values it is seen that the null hypothesis of no persistence change can be rejected for 15 of the 20 series at the 1% level and for 18 of the 20 series at the 5% level. However, using the heteroskedastic bootstrap p values reduces this to 10 out of 20 significant at the 1% level and 15 out of 20 significant at the 5% level. In no case is the estimated p-value smaller for the heteroskedastic bootstrap-based tests. The most striking difference between the homoskedastic and heteroskedastic-based bootstraps is for PU84 where the former yields a significant outcome at the 5% level while the latter is only just significant at the 15% level. Of the series where the $\mathcal{K}_4^b$ test rejects the null hypothesis at the 1% level, namely PUNEW, PU83, PUCD, PUS, PUXHS, PUXM, GMDC, GMDCD, GMDCN and GMDCS, a comparison of the (heteroskedastic) p-values for the outcomes of the $\mathcal{K}_1$ and $\mathcal{K}_1'$ statistics are suggestive of $I(0)$-$I(1)$ changes for PU83, PUCD and GMDC, and $I(1)$-$I(0)$ changes for PUNEW, PUXHS, PUXM, GMDCD, GMDCN and GMDCS.

The results for the tests based on $\mathcal{K}_6$ are very similar to those discussed above for $\mathcal{K}_4$, while those for $\mathcal{K}_5$ are again similar although both tests tend to be less significant in general. Specifically, $\mathcal{K}_6$ yields 15 and 17 (11 and 15) out of 20 significant rejections at the 1% and 5% levels level, respectively, based on the homoskedastic (heteroskedastic) bootstrap, while $\mathcal{K}_5$ yields 11 and 12 (7 and 12) out of 20 significant rejections at the 1% and 5% levels level, respectively, using the homoskedastic (heteroskedastic) bootstrap.

Turning to the results for the studentised statistic in Table 7, a further reduction in the number of significant outcomes relative to those in Table 6 is seen. For example, the $\mathcal{K}_4^*$ statistic yields 13 and 16 (7 and 13) out of 20 significant rejections at the 1% and 5% levels level, respectively, based on the homoskedastic (heteroskedastic) bootstrap. The studentised $\mathcal{K}_4^{b*}$ test rejects the null hypothesis at the 1% level for PU85, PUCD, GMDC, GMDCD, GMDCN and GMDCS, and a comparison of the (heteroskedastic) p-values for the outcomes of the $\mathcal{K}_1^*$ and $\mathcal{K}_1'^*$ statistics are suggestive of $I(0)$-$I(1)$ changes for PU85, PUCD and GMDC, and $I(1)$-$I(0)$ changes for GMDCD, GMDCN and GMDCS. The same conclusions are drawn using the $\mathcal{K}_3^*$, $\mathcal{K}_3'^*$ and $\mathcal{K}_6^*$ except that PU83 also provides significant evidence at the 1% level of an $I(0)$-$I(1)$ change.

8 Conclusions

In this paper we have analyzed the behaviour of tests for the null of trend stationarity against the alternative of a change in persistence in circumstances where the innovation process displays non-stationary volatility. We have shown that, under the null hypothesis of no change in persistence, non-stationary volatility modifies the limiting distributions of these test statistics, relative to the case of stationary volatility, with these no longer being pivotal. Monte Carlo evidence suggests that for a range of relevant volatility processes this often results in a considerable degree of over-size in the tests. As a consequence, it is likely to be hard for practitioners to discriminate between true persistence change processes and constant persistence processes which display non-stationary volatility on the basis of these tests. In order to solve the identified inference problem we have proposed bootstrap-based implementations of the persistence change tests using a fixed regressor (wild) bootstrap algorithm. Our proposed bootstrap tests were shown to deliver correctly sized inference in the limit, within the class of non-stationary volatility processes considered. Monte Carlo evidence presented suggests that our proposed bootstrap tests work well in finite samples being approximately correctly sized in the presence of a range of time-varying volatility processes, yet not losing a significant degree of power relative to the standard tests under persistence changes. An empirical application of the tests discussed in this paper to the price inflation data series from the Stock and Watson (2005) database was also reported. Although fewer rejections were found overall when using our bootstrap tests, which control for the possibility of spurious rejections due to non-stationary volatility, there still remained significant evidence of persistence change in a number of the series analysed.

A Mathematical Appendix

All the statistics discussed in the paper are exact invariant to β and we therefore set $\beta = 0$ in what follows with no loss of generality.

Proof of Theorem 1. Let $S_T^0(\cdot) := T^{-1/2}\sum_{t=1}^{\lfloor \cdot T\rfloor} z_{t,0}$. Under the stated conditions we may apply Lemma 1 of Cavaliere (2004b) to obtain the weak convergence to heteroskedastic Brownian motion result, $S_T^0 \xrightarrow{w} \lambda_\varepsilon \bar{\omega} B_\omega$, where $\bar{\omega}^2 := \int_0^1 \omega^2$. Moreover, $S_T^{0x}(\cdot) := T^{-1/2}\sum_{t=1}^{\lfloor \cdot T\rfloor} z_{t,0}\mathbf{x}_t'\delta_T \xrightarrow{w} \lambda_\varepsilon \overline{\omega} \int_0^{\cdot} dB_\omega(r)\,\mathbf{x}(r)'$; see also Cavaliere and Taylor (2005, Th.1). After some algebra we obtain from the above results and applications of the continuous mapping theorem [CMT] that,

$$\frac{1}{T^{1/2}}\breve{S}_{\lfloor \cdot T\rfloor}(\tau) := \frac{1}{T^{1/2}}\sum_{t=\lfloor T\tau\rfloor+1}^{\lfloor \cdot T\rfloor}\breve{\varepsilon}_t \xrightarrow{w} \lambda_\varepsilon\bar{\omega}\left(\mathcal{Q}_X^{\perp}B_\omega(\cdot;\tau,1) - \mathcal{Q}_X^{\perp}B_\omega(\tau;\tau,1)\right)$$

$$= \lambda_\varepsilon\bar{\omega}\left(\mathcal{Q}_X^{\perp}B_\omega(\cdot;\tau,1) - B_\omega(\tau)\right) = \lambda_\varepsilon\bar{\omega}\breve{B}_\omega(\cdot,\tau) \ .$$

Similarly, $T^{-1/2}\hat{S}_{\lfloor \cdot T\rfloor}(\tau) := T^{-1/2}\sum_{t=1}^{\lfloor \cdot T\rfloor}\hat{\varepsilon}_t \xrightarrow{w} \lambda_\varepsilon\bar{\omega}\mathcal{Q}_X^{\perp}B_\omega(\cdot;0,\tau) = \lambda_\varepsilon\bar{\omega}\hat{B}_\omega(\cdot,\tau)$. A standard application of the CMT then delivers that

$$\mathcal{K}(\tau) := \frac{(T-\lfloor\tau T\rfloor)^{-2}\sum_{t=\lfloor\tau T\rfloor+1}^{T}(\breve{S}_t(\tau))^2}{\lfloor\tau T\rfloor^{-2}\sum_{t=1}^{\lfloor\tau T\rfloor}(\hat{S}_t(\tau))^2} \xrightarrow{w} \frac{\lambda_\varepsilon^2\bar{\omega}^2(1-\tau)^{-2}\int_\tau^1\hat{B}_\omega(s,\tau)\,ds}{\lambda_\varepsilon^2\bar{\omega}^2\,\tau^{-2}\int_0^\tau\hat{B}_\omega(s,\tau)\,ds} = L_\omega(\tau) \ .$$

Proof of Theorem 2. Let Assumption $\mathcal{K}$ hold. A direct application of Theorem 4 of Cavaliere (2004b) to the $[0,\tau]$ interval yields the result that $(\lfloor\tau T\rfloor/T)\hat{\lambda}^2_{m_T,\lfloor\tau T\rfloor} \xrightarrow{p} \lambda_\varepsilon^2\int_0^\tau\omega^2$; similarly, on the $[\tau,1]$ interval it yields that $((T-\lfloor\tau T\rfloor)/T)\breve{\lambda}^2_{m_T,\lfloor\tau T\rfloor} \xrightarrow{p} \lambda_\varepsilon^2\int_\tau^1\omega^2$. Consequently,

$$\frac{\hat{\lambda}^2_{m_T,\lfloor\tau T\rfloor}}{\breve{\lambda}^2_{m_T,\lfloor\tau T\rfloor}} \xrightarrow{p} \frac{\lambda_\varepsilon^2\int_0^\tau\omega^2/\tau}{\lambda_\varepsilon^2\int_\tau^1\omega^2/(1-\tau)} = \frac{1-\tau}{\tau}\left(\frac{\eta(\tau)}{1-\eta(\tau)}\right) =: \kappa_\omega(\tau) \ .$$

The stated result then follows directly from Theorem 1 and an application of the CMT.

Proof of Theorem 3. The joint distribution of the sequences of statistics used in forming the statistics $\mathcal{K}_i$, $\mathcal{K}_i^*$, $\mathcal{K}_j'$, and $\mathcal{K}_j'^*$, $i=1,...,6$, $j=1,2,3$, follows from the fixed τ representations given in Theorems 1 and 2, using arguments proved in Zivot and Andrews (1992). The stated results then follow directly from applications of the CMT, noting the continuity of the functions defining the various test statistics.

Proof of Theorem 4. Define $S_T^1(\cdot) := T^{-1/2}\sum_{t=1}^{\lfloor\cdot T\rfloor}z_{t,1}$ on $[0,\tau^*]$ and $S_T^2(\cdot) := T^{-1/2}\sum_{t=\lfloor\tau^*T\rfloor}^{\lfloor\cdot T\rfloor}z_{t,2}$ on $[\tau^*,1]$. As in Theorem 1, $S_T^1 \xrightarrow{w} \bar{\omega}\lambda_u B_\omega$ on $[0,\tau^*]$. Moreover, under the stated conditions, using the same arguments as in the proof of Theorem 1 and since $z_{\lfloor\tau^*T\rfloor,2}$ is of $O_p(1)$, the weak convergence result $T^{-1/2}z_{\lfloor\cdot T\rfloor,2} \xrightarrow{w} \bar{\omega}\lambda_\varepsilon(B_\omega(\cdot) - B_\omega(\tau^*))$ holds on $\mathcal{D}[0,1]$.

Consider first the case $0<\tau^*<\tau<1$. Using the above results and since $\mathbf{x}_t$ contains a constant, we have, after some tedious algebra

$$\frac{1}{T^{1/2}}\breve{\varepsilon}_{\lfloor\cdot T\rfloor,\tau} \xrightarrow{w} \bar{\omega}\lambda_\varepsilon\left(B_\omega(\cdot) - \int_\tau^1 B_\omega\mathbf{x}'\left(\int_\tau^1\mathbf{x}\mathbf{x}'\right)^{-1}\mathbf{x}(\cdot)\right) = \bar{\omega}\lambda_\varepsilon\mathcal{P}_{\mathbf{x}}^{\perp}B_\omega(\cdot;\tau,1)$$

and, by using the equality $\int_a^\cdot\mathcal{P}_X^{\perp}Y(s;a,b)\,ds = \mathcal{Q}_X^{\perp}\mathbb{Y}(\cdot;a,b) - \mathbb{Y}(a)$, where $\mathbb{Y}(\cdot) := \int_0^\cdot Y$,

$$\frac{1}{T^{3/2}}\breve{S}_{\lfloor\cdot T\rfloor}(\tau) = \frac{1}{T^{3/2}}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor\cdot T\rfloor}\breve{\varepsilon}_{t,\tau} \xrightarrow{w} \bar{\omega}\lambda_\varepsilon\int_\tau^\cdot\mathcal{P}_{\mathbf{x}}^{\perp}B_\omega(s;\tau,1)\,ds$$

$$= \bar{\omega}\lambda_\varepsilon\left(\mathcal{Q}_{\mathbf{x}}^{\perp}\mathbb{B}_\omega(\cdot;\tau,1) - \mathbb{B}_\omega(\tau)\right)$$

from which the following weak convergence result for the numerator of the ratio statistic

$$T^{-2}(T-\lfloor \tau T\rfloor)^{-2}\sum_{t=\lfloor \tau T\rfloor+1}^{T}(\breve{S}_t(\tau))^2 \xrightarrow{w} \frac{\bar{\omega}^2\lambda_\varepsilon^2}{(1-\tau)^2}\int_\tau^1 \left(\mathcal{Q}_{\mathbf{x}}^{\perp}\mathbb{B}_\omega(s;\tau,1)-\mathbb{B}_\omega(\tau)\right)^2 ds$$

follows immediately. Similarly, and since $T^{-1/2}z_{t,1}$ is of $o_p(1)$,

$$\frac{1}{T^{1/2}}\hat{\varepsilon}_{\lfloor \cdot T\rfloor,\tau} \xrightarrow{w} \bar{\omega}\lambda_\varepsilon \mathcal{P}_{\mathbf{x}}^{\perp} B_\omega^*(\cdot;0,\tau) \tag{A.1}$$

and, hence, $T^{-2}(\lfloor \tau T\rfloor)^{-2}\sum_{t=1}^{\lfloor \tau T\rfloor}(\hat{S}_t(\tau))^2 \xrightarrow{w} \frac{\bar{\omega}^2\lambda_\varepsilon^2}{\tau^2}\int_0^\tau \left(\mathcal{Q}_{\mathbf{x}}^{\perp}\mathbb{B}_\omega^*(s;0,\tau)\right)^2 ds$. This yields the result that

$$\mathcal{K}(\tau) \xrightarrow{w} \frac{\tau^2}{(1-\tau)^2}\frac{\int_\tau^1 \left(\mathcal{Q}_{\mathbf{x}}^{\perp}\mathbb{B}_\omega(s;\tau,1)-\mathbb{B}_\omega(\tau)\right)^2 ds}{\int_0^\tau \mathcal{Q}_{\mathbf{x}}^{\perp}\mathbb{B}_\omega^*(s;0,\tau)^2 ds} = O_p(1).$$

Now, consider the case $\tau^* > \tau$. Similarly to the previous part of the proof we have

$$\frac{1}{T^{1/2}}\breve{\varepsilon}_{\lfloor \cdot T\rfloor,\tau} \xrightarrow{w} \mathcal{P}_{\mathbf{x}}^{\perp} B_\omega^*(\cdot;\tau,1) \tag{A.2}$$

and

$$T^{-2}(T-\lfloor \tau T\rfloor)^{-2}\sum_{t=\lfloor \tau T\rfloor+1}^{T}(\breve{S}_t(\tau))^2 \xrightarrow{w} \frac{\bar{\omega}^2\lambda_\varepsilon^2}{(1-\tau)^2}\int_\tau^1 \left(\mathcal{Q}_{\mathbf{x}}^{\perp}\mathbb{B}_\omega^*(s;\tau,1)-\mathbb{B}_\omega^*(\tau)\right)^2 ds\ .$$

However, since for $t \le \lfloor \tau T\rfloor < \lfloor \tau^* T\rfloor$ the series has no $I(1)$ (stochastic) trend, we may refer back to the invariance principle introduced in the proof of Theorem 1, thereby obtaining that

$$(\lfloor \tau T\rfloor)^{-2}\sum_{t=1}^{\lfloor \tau T\rfloor}(\hat{S}_t(\tau))^2 \xrightarrow{w} \frac{\bar{\omega}^2\lambda_u^2}{\tau^2}\left(\int_0^\tau \hat{B}_\omega(s,\tau)^2 ds\right).$$

Taken together these results imply that

$$\frac{1}{T^2}\mathcal{K}(\tau) \xrightarrow{w} \frac{\tau^2}{(1-\tau)^2}\frac{\lambda_\varepsilon^2}{\lambda_u^2}\frac{\int_\tau^1 \left(\mathcal{Q}_{\mathbf{x}}^{\perp}\mathbb{B}_\omega^*(s;\tau,1)-\mathbb{B}_\omega^*(\tau)\right)^2 ds}{\left(\int_0^\tau \hat{B}_\omega(s,\tau)^2 ds\right)}\ .$$

Proof of Theorem 5. As in the Proof of Theorem 2, we need to prove the additional results for the estimated long run variance ratio $\hat{\lambda}^2_{m_T,\lfloor \tau T\rfloor}/\breve{\lambda}^2_{m_T,\lfloor \tau T\rfloor}$. Consider the case $0 < \tau^* < \tau < 1$ under Assumption $\mathcal{K}$ first. From the previous proof, we have that $T^{-1/2}\breve{\varepsilon}_{\lfloor \cdot T\rfloor,\tau} \xrightarrow{w} \bar{\omega}\lambda_\varepsilon \mathcal{P}_{\mathbf{x}}^{\perp} B_\omega(\cdot;\tau,1)$, and, hence, as in Phillips (1991), the long-run variance estimator from the second sub-sample satisfies

$$\frac{1}{m_T T}\breve{\lambda}^2_{m_T,\lfloor \tau T\rfloor} \xrightarrow{w} \frac{\bar{\omega}^2\lambda_\varepsilon^2}{1-\tau}\int_{-\infty}^{+\infty} k \int_\tau^1 \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_\omega(s;\tau,1)\right)^2 ds. \tag{A.3}$$

Turning to the first sub-sample, $T^{-1/2}\hat{\varepsilon}_{\lfloor \cdot T \rfloor,\tau} \xrightarrow{w} \mathcal{P}_{\mathbf{x}}^{\perp} B_{\omega}^{*}(\cdot; 0, \tau)$ and, hence,

$$\frac{1}{m_T T}\hat{\lambda}^2_{m_T,\lfloor \tau T \rfloor} \xrightarrow{w} \frac{\bar{\omega}^2 \lambda_{\varepsilon}^2}{\tau} \int_{-\infty}^{+\infty} k \int_0^{\tau} \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_{\omega}^{*}(s; 0, \tau)\right)^2 ds. \tag{A.4}$$

Consequently, from (A.3), (A.4) and the CMT, we obtain that

$$\frac{\hat{\lambda}^2_{m_T,\lfloor \tau T \rfloor}}{\breve{\lambda}^2_{m_T,\lfloor \tau T \rfloor}} \xrightarrow{w} \frac{1-\tau}{\tau} \frac{\int_0^{\tau} \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_{\omega}^{*}(s; 0, \tau)\right)^2 ds}{\int_{\tau}^{1} \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_{\omega}(s; \tau, 1)\right)^2 ds}. \tag{A.5}$$

The result in the first part of the theorem then follows upon substitution.

In the case where $0 < \tau \leq \tau^* < 1$ and Assumption $\mathcal{K}$ holds, using the same reasoning as above we obtain for the second sub-sample that

$$\frac{1}{m_T T}\breve{\lambda}^2_{m_T,\lfloor \tau T \rfloor} \xrightarrow{w} \frac{\bar{\omega}^2 \lambda_{\varepsilon}^2}{1-\tau} \int_{-\infty}^{+\infty} k \int_{\tau}^{1} \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_{\omega}^{*}(s; \tau, 1)\right)^2 ds \tag{A.6}$$

while, for the first sub-sample, Theorem 4 in Cavaliere (2004b) delivers the result that

$$\hat{\lambda}^2_{m_T,\lfloor \tau T \rfloor} \xrightarrow{p} \lambda_u^2 \frac{\int_0^{\tau} \omega^2}{\tau} \tag{A.7}$$

which, together with (A.6), an application of the CMT and the equality $\eta(\tau) = \bar{\omega}^{-2}\int_0^{\tau} \omega^2$, yields the result that

$$m_T T \frac{\hat{\lambda}^2_{m_T,\lfloor \tau T \rfloor}}{\breve{\lambda}^2_{m_T,\lfloor \tau T \rfloor}} \xrightarrow{w} \frac{(1-\tau)\lambda_u^2 \eta(\tau)}{\tau \lambda_{\varepsilon}^2} \frac{1}{\int_{-\infty}^{+\infty} k} \frac{1}{\int_{\tau}^{1} \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_{\omega}^{*}(s; \tau, 1)\right)^2 ds}.$$

Again, the result in the second part of the theorem then follows upon substitution.

Proof of Theorem 6. Consider first the proof of the results in part (i). To that end, define $\hat{S}_T^b(\cdot) := T^{-1/2}\sum_{t=1}^{\lfloor \cdot T \rfloor} \hat{\varepsilon}_{t,\tau}^b$ on $[0,\tau]$, and $\breve{S}_T^b(\cdot) := T^{-1/2}\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \cdot T \rfloor} \breve{\varepsilon}_{t,\tau}^b$ on $[\tau, 1]$, and consider $\hat{S}_T^b(\cdot)$ first. By standard least squares algebra, $\hat{S}_T^b(\cdot) = S_T^*(\cdot) - M_T^*(\tau)'$ $\left(T^{-1}\sum_{t=1}^{\lfloor \tau T \rfloor} \delta_T x_t x_t' \delta_T\right)^{-1} T^{-1}\sum_{t=1}^{\lfloor \cdot T \rfloor} \delta_T x_t$, where $\hat{S}_T^*(\cdot) := T^{-1/2}\sum_{i=1}^{\lfloor \cdot T \rfloor} y_t^b$ and $\hat{M}_T^*(\cdot) := T^{-1/2}\sum_{t=1}^{\lfloor \cdot T \rfloor} y_t^b \mathbf{x}_t \delta_T$, $y_t^b := \tilde{\varepsilon}_t w_t$. Conditionally on the sample (i.e., on $\tilde{\varepsilon}_1, ..., \tilde{\varepsilon}_T$), $\hat{M}_T^*$ is Gaussian with covariance kernel $\hat{\Lambda}_T^M(r,s) = \hat{\Lambda}_T^M(\min\{r,s\}) := T^{-1}\sum_{t=1}^{\lfloor \min\{r,s\}T \rfloor} \tilde{\varepsilon}_t^2 \delta_T \mathbf{x}_t \mathbf{x}_t' \delta_T$, while $\hat{S}_T^*$ is Gaussian with covariance kernel $\hat{\Lambda}_T(r,s) = \hat{\Lambda}_T(\min\{r,s\}) := T^{-1}\sum_{t=1}^{\lfloor \min\{r,s\}T \rfloor} \tilde{\varepsilon}_t^2$. Since $\hat{S}_T^*(\cdot) = (1, 0')\hat{M}_T^*(\cdot)$, the asymptotic distribution of $\hat{S}_T^*$ follows from that of $\hat{M}_T^*$. As in Cavaliere and Taylor (2005), using the fact that for $t \leq \lfloor \tau T \rfloor$, $\tilde{\varepsilon}_t^2 = \sigma_t^2 \varepsilon_t^2 + o_p(1) = \sigma_t^2 + \sigma_t^2(\varepsilon_t^2 - 1) + o_p(1)$, $(\varepsilon_t^2 - 1)$ being a zero mean mixing process, we have $\hat{\Lambda}_T^M(\cdot) \xrightarrow{w} \int_0^{\cdot} \omega^2 \mathbf{x}\mathbf{x}'$, which implies (see, e.g., Hansen, 2000) the weak convergence in probability result $\hat{M}_T^*(\cdot) \xrightarrow{w}_p \bar{\omega}\int_0^{\cdot} \mathbf{x} dB_\omega$ and, by the CMT, that $\hat{S}_T^* \xrightarrow{w}_p \bar{\omega}B_\omega$. Again, by the CMT and on $[0,\tau]$

$$T^{-1/2}\hat{S}_T^b(\cdot) \xrightarrow{w}_p \bar{\omega}B_\omega(\cdot) - \bar{\omega}\int_0^\tau dB_\omega \mathbf{x}' \left(\int_0^\tau \mathbf{x}\mathbf{x}'\right)^{-1} \int_0^{\cdot} \mathbf{x}$$
$$= \bar{\omega}\left(\mathcal{Q}_X^\perp B_\omega(\cdot;0,\tau)\right) = \bar{\omega}\hat{B}_\omega(\cdot,\tau)$$

Similarly, on $[\tau,1]$, $\breve{S}_T^b \xrightarrow{w}_p \bar{\omega}\breve{B}_\omega(\cdot,\tau)$ which delivers, after some standard algebra, the required convergence $\mathcal{K}^b(\tau) \xrightarrow{w}_p L_\omega(\tau)$. The latter result also guarantees that $G_T^b(\cdot;\tau) \to G(\cdot;\tau)$ uniformly in probability, where $G(\cdot;\tau)$ denotes the cdf of $L_\omega(\tau)$. The remainder of the proof is identical to the proof of Theorem 5 in Hansen (2000) and is therefore omitted in the interests of brevity. As in the previous proofs, the results in part (ii) of the theorem follow from the pointwise results established above using Zivot and Andrews (1992) and the CMT.

Proof of Theorem 7. In order to prove the second part of theorem we need only show (in addition to the results given in Theorem 6) that the estimated long run variance ratio $\left(\hat{\lambda}_{m_T^b,\lfloor\tau T\rfloor}^b / \hat{\lambda}_{m_T^b,\lfloor\tau T\rfloor}^b\right)^2$ converges in probability to $\kappa_\omega(\tau)$. Consider $\hat{\lambda}_{m_T^b,\lfloor\tau T\rfloor}^{b\,2}$ first: by letting $\hat{\gamma}_{j,T}(\tau) = (1/\lfloor\tau T\rfloor)\sum_{t=j+1}^{\lfloor\tau T\rfloor} \tilde{\varepsilon}_t\tilde{\varepsilon}_{t-j}w_t w_{t-j}$ the following equalities obtain

$$\begin{aligned}
\hat{\lambda}_{m_T^b,\lfloor\tau T\rfloor}^{b\,2} &:= \frac{1}{\lfloor\tau T\rfloor}\sum_{t=1}^{\lfloor\tau T\rfloor}(\hat{\varepsilon}_{t,\tau}^b)^2 + 2\sum_{j=1}^{\lfloor\tau T\rfloor-1} k\left(j/m_T^b\right)\frac{1}{\lfloor\tau T\rfloor}\sum_{t=j+1}^{\lfloor\tau T\rfloor}\hat{\varepsilon}_{t,\tau}^b\hat{\varepsilon}_{t-j,\tau}^b \\
&= \frac{1}{\lfloor\tau T\rfloor}\sum_{t=1}^{\lfloor\tau T\rfloor}\hat{\varepsilon}_{t,\tau}^2 w_t^2 + 2\sum_{j=1}^{\lfloor\tau T\rfloor-1} k\left(j/m_T^b\right)\hat{\gamma}_{j,T}(\tau) + o_p(1) \\
&= \frac{1}{\lfloor\tau T\rfloor}\sum_{t=1}^{\lfloor\tau T\rfloor}\tilde{\varepsilon}_t^2 + \frac{1}{\lfloor\tau T\rfloor}\sum_{t=1}^{\lfloor\tau T\rfloor}\left(w_t^2-1\right)\tilde{\varepsilon}_t^2 + 2\sum_{j=1}^{\lfloor\tau T\rfloor-1} k\left(j/m_T^b\right)\hat{\gamma}_{j,T}(\tau) + o_p(1) \\
&=: f_{1,T}(\tau) + f_{2,T}(\tau) + f_{3,T}(\tau) + o_p(1)
\end{aligned}$$

where $f_{i,T}(\tau)$, $i = 1,2,3$ are defined implicitly. Now, as in Cavaliere (2004b),

$$f_{1,T} = \frac{1}{\lfloor\tau T\rfloor}\sum_{t=1}^{\lfloor\tau T\rfloor}\varepsilon_t^2 + o_p(1) \xrightarrow{p} \frac{1}{\tau}\int_0^\tau \omega^2 = \bar{\omega}^2\frac{\eta(\tau)}{\tau}. \tag{A.8}$$

Regarding $f_{2,T}(\tau)$, since, conditionally on the sample we have that

$$\begin{aligned}
E(f_{2,T}(\tau)^2) &= \frac{1}{\lfloor\tau T\rfloor^2}\sum_{t=1}^{\lfloor\tau T\rfloor}\sum_{t'=1}^{\lfloor\tau T\rfloor}\tilde{\varepsilon}_t^2\tilde{\varepsilon}_{t'}^2 E\left(\left(w_t^2-1\right)\left(w_{t'}^2-1\right)\right) \\
&= \frac{1}{\lfloor\tau T\rfloor^2}\sum_{t=1}^{\lfloor\tau T\rfloor}\tilde{\varepsilon}_t^4 E\left(\left(w_t^2-1\right)^2\right) = \frac{2}{\lfloor\tau T\rfloor}\left(\frac{1}{\lfloor\tau T\rfloor}\sum_{t=1}^{\lfloor\tau T\rfloor}\tilde{\varepsilon}_t^4\right) = o_p\left(T^{-1}\right)
\end{aligned}$$

as $\tilde{\varepsilon}_t = \sigma_t \varepsilon_t + o_p(1)$, $E(\sigma_t^4 \varepsilon_t^4)$ is bounded and w_t is iid. This establishes $f_{2,T}(\tau) \xrightarrow{p} 0$. Regarding $f_{3,T}(\tau)$, similarly to what was observed for $f_{2,T}(\tau)$, for any $j > 0$ and conditionally on the sample

$$E(\hat{\gamma}_{j,T}(\tau)^2) = \frac{1}{\lfloor \tau T \rfloor^2} \sum_{t=j+1}^{\lfloor \tau T \rfloor} \tilde{\varepsilon}_t^2 \tilde{\varepsilon}_{t-j}^2 E\left(w_t^2 w_{t-j}^2\right) = \frac{1}{\lfloor \tau T \rfloor^2} \sum_{t=j+1}^{\lfloor \tau T \rfloor} \tilde{\varepsilon}_t^2 \tilde{\varepsilon}_{t-j}^2$$

which is of $O_p(T^{-1})$ for all j. With $||X|| := E(X^2)^{1/2}$, as $\left(1/m_T^b\right) \sum_{j=1}^{T-1} \left|k\left(j/m_T^b\right)\right| \rightarrow \int_0^\infty |k| < \infty$ we have that

$$\|f_{3,T}(\tau)\| \leq 2 \sum_{j=1}^{T-1} \left|k\left(j/m_T^b\right)\right| \|\hat{\gamma}_{j,T}(\tau)\| = O_p\left(m_T^b / T^{1/2}\right)$$

which shows that $f_{3,T}(\tau) \xrightarrow{p} 0$ as $m_T^b = o\left(T^{1/2}\right)$. Therefore, we have that

$$\hat{\lambda}_{m_T^b, \lfloor \tau T \rfloor}^{b\,2} = f_{1,T}(\tau) + o_p(1) \xrightarrow{p} \bar{\omega}^2 \frac{\eta(\tau)}{\tau}. \tag{A.9}$$

Using similar reasoning, it is straightforward to show that

$$\breve{\lambda}_{m_T^b, \lfloor \tau T \rfloor}^{b\,2} \xrightarrow{w} \frac{1}{1-\tau} \int_\tau^1 \omega^2 = \bar{\omega}^2 \frac{\eta(1) - \eta(\tau)}{1-\tau},$$

which, together with (A.9), yields the result that

$$\left(\frac{\hat{\lambda}_{m_T^b, \lfloor \tau T \rfloor}^b}{\breve{\lambda}_{m_T^b, \lfloor \tau T \rfloor}^b} \right)^2 \xrightarrow{p} \frac{\eta(\tau)/\tau}{(\eta(1) - \eta(\tau))/(1-\tau)} = \kappa_\omega(\tau)$$

as required. □

Proof of Theorem 8. Initially, notice that, under the alternative hypothesis, the full sample residuals are of $O_p\left(T^{1/2}\right)$. Specifically, similarly to what obtained in section 4.2.1, the following weak convergence result holds:

$$\frac{1}{T^{1/2}} \tilde{\varepsilon}_{\lfloor T \cdot \rfloor} \xrightarrow{w} \bar{\omega} \mathcal{P}_{\mathbf{x}}^{\perp} B_\omega^*(\cdot; 0, 1) \tag{A.10}$$

where, as in section 4.2.1, $B_\omega^*(\cdot) = B_\omega(\cdot)\mathbb{I}(\cdot \geq \tau^*)$. Since, conditionally on the sample

$$\frac{1}{T} \sum_{t=1}^{\lfloor T \cdot \rfloor} y_t^b \mathbf{x}_t \delta_T = \frac{1}{T} \sum_{t=1}^{\lfloor T \cdot \rfloor} \tilde{\varepsilon}_t w_t \mathbf{x}_t \delta_T \sim N\left(0, \frac{1}{T^2} \sum_{t=1}^{\lfloor T \cdot \rfloor} \tilde{\varepsilon}_t^2 \delta_T \mathbf{x}_t \mathbf{x}_t' \delta_T\right)$$

and (eq. (A.10), Assumption $\mathcal{X}$ and the CMT)

$$\frac{1}{T^2} \sum_{t=1}^{\lfloor T \cdot \rfloor} \tilde{\varepsilon}_t^2 \delta_T \mathbf{x}_t \mathbf{x}_t' \delta_T \xrightarrow{w} \bar{\omega}^2 \int_0^{\cdot} \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_\omega^*(s; 0, 1)\right)^2 \mathbf{x}(s) \mathbf{x}(s)' ds$$

it follows, as in the proof of Theorem 6, that the weak convergence in probability

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\cdot\rfloor} y_t^b \mathbf{x}_t \delta_T \xrightarrow{w}_p \bar{\omega}\int_0^{\cdot} \mathcal{P}_{\mathbf{x}}^{\perp} B_\omega(s;0,1)\,\mathbf{x}(s)\,dV(s) \tag{A.11}$$

holds, $V(\cdot)$ being a Brownian motion, independent of $B_\omega^*(\cdot)$. Notice that this result does not depend on the selected breakdate, τ, since the bootstrap sample is constructed using the full sample residuals.

Consider $\hat{M}_T^*(\cdot) := T^{-1/2}\sum_{t=1}^{\lfloor \cdot T\rfloor} y_t^b \mathbf{x}_t \delta_T$ and $\hat{S}_T^*(\cdot) = (1, 0')\,\hat{M}_T^*(\cdot)$. Using (A.11), we have that

$$\frac{1}{T^{1/2}}\hat{M}_T^*(\cdot) \xrightarrow{w}_p \bar{\omega}\int_0^{\cdot}\left(\mathcal{P}_X^{\perp} B_\omega^*(s;0,1)\right)\mathbf{x}(s)\,dV(s)$$

$$\frac{1}{T^{1/2}}\hat{S}_T^*(\cdot) \xrightarrow{w}_p \bar{\omega}\int_0^{\cdot}\left(\mathcal{P}_X^{\perp} B_\omega^*(s;0,1)\right)dV(s)$$

which imply, after some algebra based on $\hat{M}_T^*(\cdot)$ and $\hat{S}_T^*(\cdot)$ (see the proof of 6), that

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{\lfloor\cdot T\rfloor}\hat{\varepsilon}_{t,\tau}^b \xrightarrow{w}_p\ & \bar{\omega}\int_0^{\cdot}\left(\mathcal{P}_X^{\perp} B_\omega^*(s;0,1)\right)dV(s)\\ & -\bar{\omega}\int_0^{\tau} dV(s)\left(\mathcal{P}_X^{\perp} B_\omega^*(s;0,1)\right)\mathbf{x}(s)'\left(\int_0^{\tau}\mathbf{x}\mathbf{x}'\right)^{-1}\int_0^{\tau}\mathbf{x} \\ & =: \bar{\omega}\mathcal{Q}_{\mathbf{X}}^{\perp}\mathbb{V}_\omega^*(\cdot;0,\tau)\end{aligned} \tag{A.12}$$

with $\mathbb{V}_\omega^*(\cdot) := \int_0^{\cdot}\left(\mathcal{P}_X^{\perp} B_\omega^*(s;0,1)\right)dV(s)$. Similarly,

$$\frac{1}{T}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor\cdot T\rfloor}\breve{\varepsilon}_{t,\tau}^b \xrightarrow{w}_p = \bar{\omega}\left(\mathcal{Q}_{\mathbf{X}}^{\perp}\mathbb{V}_\omega^*(\cdot;\tau,1) - \mathbb{V}_\omega^*(\tau)\right). \tag{A.13}$$

Results (A.12)–(A.13) allow us to establish that the numerator and denominator of the boostrap statistic $\mathcal{K}^b(\tau)$ are of the same order (precisely they are both of $O_p(T)$) and, hence, that $\mathcal{K}^b(\tau)$ satisfies

$$\mathcal{K}^b(\tau) \xrightarrow{w}_p \frac{\tau}{1-\tau}\frac{\int_\tau^1\left(\mathcal{Q}_{\mathbf{X}}^{\perp}\mathbb{V}_\omega(s;\tau,1) - \mathbb{V}_\omega(\tau)\right)^2 ds}{\int_\tau^1\left(\mathcal{Q}_{\mathbf{X}}^{\perp}\mathbb{V}_\omega^*(s;0,\tau)\right)^2 ds} \tag{A.14}$$

which is of $O_p(1)$ for all τ; as a consequence, see above, $\mathcal{K}_1^b$ is also of $O_p(1)$. Hence, as in Theorem 2 $p_T^b(\tau) \to 0$ provided that $\mathcal{K}(\tau)$ diverges, i.e. that $\tau \le \tau^*$; this also imples that $p_{1,T}^b(\tau) \xrightarrow{p} 0$ given that $[0,\tau^*]\cap[\tau_l,\tau_u] \neq \varnothing$. □

Proof of Theorem 9. The proof requires additional results for the bootstrap quantity $(\hat{\lambda}_{m_T^b,\lfloor\tau T\rfloor}^b/\breve{\lambda}_{m_T^b,\lfloor\tau T\rfloor}^b)^2$. Let us consider $\hat{\lambda}_{m_T,\lfloor\tau T\rfloor}^{b\,2}$, and recall that

$$\hat{\lambda}_{m_T^b,\lfloor\tau T\rfloor}^{b\,2} := \frac{1}{\lfloor\tau T\rfloor}\sum_{t=1}^{\lfloor\tau T\rfloor}(\hat{\varepsilon}_{t,\tau}^b)^2 + 2\sum_{j=1}^{\lfloor\tau T\rfloor-1}\omega\left(j/m_T^b\right)\frac{1}{\lfloor\tau T\rfloor}\sum_{t=j+1}^{\lfloor\tau T\rfloor}\hat{\varepsilon}_{t,\tau}^b\hat{\varepsilon}_{t-j,\tau}^b \tag{A.15}$$

We first have that

$$\sum_{t=1}^{\lfloor \tau T \rfloor} (\hat{\varepsilon}_{t,\tau}^{b})^2 = \sum_{t=1}^{\lfloor \tau T \rfloor} \tilde{\varepsilon}_t^2 + \sum_{t=1}^{\lfloor \tau T \rfloor} \tilde{\varepsilon}_t^2 \left(w_t^2 - 1\right) - \sum_{t=1}^{\lfloor \tau T \rfloor} \tilde{\varepsilon}_t w_t \mathbf{x}_t' \delta_T \left(\sum_{t=1}^{\lfloor \tau T \rfloor} \delta_T \mathbf{x}_t \mathbf{x}_t' \delta_T \right)^{-1} \sum_{t=1}^{\lfloor \tau T \rfloor} \delta_T \mathbf{x}_t w_t \tilde{\varepsilon}_t$$

which implies (using (A.10), $\sum_t \tilde{\varepsilon}_t^2 \left(w_t^2 - 1\right) = O_p\left(T\right)$ and $T^{-1} \sum_{t=1}^{\lfloor \tau T \rfloor} w_t \tilde{\varepsilon}_t \mathbf{x}_t' \delta_T = O_p\left(1\right)$, see the proof of Theorem 8)

$$\frac{1}{T^2} \sum_{t=1}^{\lfloor \tau T \rfloor} (\hat{\varepsilon}_{t,\tau}^{b})^2 = \frac{1}{T^2} \sum_{t=1}^{\lfloor \tau T \rfloor} \tilde{\varepsilon}_t^2 + o_p\left(1\right) \xrightarrow{w}_p \bar{\omega}^2 \int_0^{\tau} \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_{\omega}^{*}\left(s;0,1\right)\right)^2 ds \ . \tag{A.16}$$

Moreover, it can be shown that the remaining terms in (A.15) are of $o_p\left(T\right)$, and, hence, $T^{-2} \lfloor \tau T \rfloor \hat{\lambda}^{b\,2}_{m_T^b, \lfloor \tau T \rfloor}$ weakly converges in probability to the right member of (A.16).

Using the same steps it can be shown that $T^{-2}\left(T - \lfloor \tau T \rfloor\right) \breve{\lambda}^{b\,2}_{m_T^b, \lfloor \tau T \rfloor}$ weakly converges in probability to $\bar{\omega}^2 \int_{\tau}^1 \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_{\omega}^{*}\left(s;0,1\right)\right)^2 ds$. This finally proves that, for any τ,

$$\left(\frac{\hat{\lambda}^{b}_{m_T^b, \lfloor \tau T \rfloor}}{\breve{\lambda}^{b}_{m_T^b, \lfloor \tau T \rfloor}} \right)^2 \xrightarrow{w}_p \frac{\tau}{1-\tau} \frac{\int_0^{\tau} \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_{\omega}^{*}\left(s;0,1\right)\right)^2 ds}{\int_{\tau}^{1} \left(\mathcal{P}_{\mathbf{x}}^{\perp} B_{\omega}^{*}\left(s;0,1\right)\right)^2 ds}. \tag{A.17}$$

and hence, since by Theorem 8 $\mathcal{K}^{b}\left(\tau\right) = O_p\left(1\right)$, that $\mathcal{K}^{b*}\left(\tau\right)$ is also of $O_p\left(1\right)$. □

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Table 1: Empirical Size of Standard Persistence Change Tests: De-meaned Data. Tests Based on Asymptotic 5% Critical Values.

				Model 1			Model 2		Model 3	
T	ϕ	θ		$\delta=1$	$\delta=1/3$	$\delta=3$	$\delta=1/3$	$\delta=3$	$c=0$	$c=10$
100	0.0	0.0	$\mathcal{K}_1$	3.5	61.7	0.2	35.2	0.1	39.1	16.0
			$\mathcal{K}_1'$	3.3	0.3	60.2	0.1	31.6	39.7	15.0
			$\mathcal{K}_4$	3.5	52.4	48.5	26.3	21.5	69.5	20.4
			$\mathcal{K}_1^*$	2.9	3.8	8.0	3.7	3.2	7.9	5.5
			$\mathcal{K}_1'^*$	2.7	6.0	3.4	3.0	2.2	9.3	4.9
			$\mathcal{K}_4^*$	2.2	5.2	4.8	2.9	2.8	10.6	4.6
	0.5	0.0	$\mathcal{K}_1$	9.2	67.3	2.1	44.0	1.0	40.9	22.6
			$\mathcal{K}_1'$	8.1	1.5	65.0	1.3	40.3	41.5	18.6
			$\mathcal{K}_4$	11.8	60.7	57.0	34.9	32.3	71.9	26.7
			$\mathcal{K}_1^*$	2.9	4.2	7.1	3.5	3.8	7.7	4.4
			$\mathcal{K}_1'^*$	2.6	6.1	2.7	3.4	2.5	8.3	3.9
			$\mathcal{K}_4^*$	3.3	4.9	4.5	3.1	3.0	8.3	4.4
	0.0	0.5	$\mathcal{K}_1$	0.8	45.8	0.0	20.2	0.0	35.0	9.6
			$\mathcal{K}_1'$	0.5	0.0	43.5	0.0	16.1	35.8	7.7
			$\mathcal{K}_4$	0.4	32.8	31.1	11.9	9.6	63.1	10.8
			$\mathcal{K}_1^*$	1.7	2.7	4.2	2.0	1.9	6.9	4.2
			$\mathcal{K}_1'^*$	0.9	4.1	1.6	1.9	0.8	7.6	3.8
			$\mathcal{K}_4^*$	0.7	3.1	2.2	1.1	0.8	7.4	4.2
200	0.0	0.0	$\mathcal{K}_1$	4.9	60.8	0.5	33.8	0.5	37.4	15.6
			$\mathcal{K}_1'$	3.3	0.2	59.3	0.1	33.6	37.8	14.1
			$\mathcal{K}_4$	4.9	49.9	48.6	24.3	24.3	65.3	19.8
			$\mathcal{K}_1^*$	3.9	5.2	8.7	3.6	4.8	12.1	6.0
			$\mathcal{K}_1'^*$	2.9	7.1	5.2	4.1	3.0	11.9	6.6
			$\mathcal{K}_4^*$	3.6	6.7	8.3	4.2	4.5	14.4	6.8
	0.5	0.0	$\mathcal{K}_1$	7.8	63.9	1.2	38.5	1.1	39.4	17.6
			$\mathcal{K}_1'$	5.2	0.8	60.8	0.5	37.8	39.8	16.7
			$\mathcal{K}_4$	8.4	53.5	51.9	28.4	28.6	67.6	23.9
			$\mathcal{K}_1^*$	4.1	4.6	8.1	4.1	5.0	9.3	5.3
			$\mathcal{K}_1'^*$	2.8	6.5	3.9	3.0	2.7	9.3	4.7
			$\mathcal{K}_4^*$	3.4	6.2	6.4	3.8	4.3	10.5	5.8
	0.0	0.5	$\mathcal{K}_1$	2.8	52.1	0.0	24.1	0.0	34.0	12.2
			$\mathcal{K}_1'$	1.1	0.0	50.4	0.0	22.7	36.2	8.5
			$\mathcal{K}_4$	1.6	38.5	39.5	14.9	14.0	62.2	13.4
			$\mathcal{K}_1^*$	2.7	3.4	7.1	2.3	2.8	8.7	5.2
			$\mathcal{K}_1'^*$	2.1	5.1	4.1	3.1	2.3	11.8	4.3
			$\mathcal{K}_4^*$	1.8	3.9	4.7	2.0	3.0	12.4	4.9

Table 2: Empirical Size of Bootstrap Persistence Change Tests: De-meaned Data.

				Model 1			Model 2		Model 3	
T	ϕ	θ		$\delta=1$	$\delta=1/3$	$\delta=3$	$\delta=1/3$	$\delta=3$	$c=0$	$c=10$
100	0.0	0.0	$\mathcal{K}_1$	2.1	3.2	2.7	2.9	1.5	6.4	2.2
			$\mathcal{K}_1'$	1.9	2.4	3.2	1.9	2.4	6.0	2.7
			$\mathcal{K}_4$	1.5	3.2	3.2	2.9	2.4	9.8	2.1
			$\mathcal{K}_1^*$	5.8	6.9	8.2	6.6	6.1	8.7	7.0
			$\mathcal{K}_1'^*$	5.4	7.1	7.0	6.1	5.6	9.8	6.9
			$\mathcal{K}_4^*$	5.7	7.0	8.3	6.7	5.5	9.3	8.1
	0.5	0.0	$\mathcal{K}_1$	3.5	6.9	4.4	5.6	2.8	11.7	3.5
			$\mathcal{K}_1'$	3.5	4.3	5.7	3.5	4.9	10.9	3.7
			$\mathcal{K}_4$	4.0	6.9	5.8	5.5	4.9	18.4	3.6
			$\mathcal{K}_1^*$	6.0	6.8	6.9	6.2	5.5	6.5	5.5
			$\mathcal{K}_1'^*$	5.4	6.2	5.3	5.6	4.8	8.0	5.5
			$\mathcal{K}_4^*$	5.6	6.3	6.3	6.0	5.5	7.1	5.8
	0.0	0.5	$\mathcal{K}_1$	0.5	0.7	0.2	0.4	0.0	2.9	1.0
			$\mathcal{K}_1'$	0.2	0.2	0.2	0.0	0.2	2.8	1.0
			$\mathcal{K}_4$	0.0	0.7	0.2	0.4	0.1	4.7	0.8
			$\mathcal{K}_1^*$	3.9	4.4	3.6	4.1	3.6	5.4	4.7
			$\mathcal{K}_1'^*$	2.4	4.9	2.8	2.8	2.6	6.3	4.7
			$\mathcal{K}_4^*$	2.7	4.6	3.3	2.8	2.3	6.2	4.7
200	0.0	0.0	$\mathcal{K}_1$	3.4	3.2	3.9	3.5	3.1	6.4	3.3
			$\mathcal{K}_1'$	2.6	3.4	3.6	2.9	2.6	6.5	3.4
			$\mathcal{K}_4$	3.6	3.2	3.6	3.5	2.6	8.7	3.3
			$\mathcal{K}_1^*$	5.7	6.0	6.3	5.4	5.1	7.1	5.4
			$\mathcal{K}_1'^*$	4.5	6.0	6.3	5.0	5.4	7.1	6.0
			$\mathcal{K}_4^*$	5.5	5.7	7.9	5.4	5.3	7.6	6.6
	0.5	0.0	$\mathcal{K}_1$	4.3	4.5	5.3	5.1	4.7	9.8	4.0
			$\mathcal{K}_1'$	3.1	3.8	4.9	3.2	3.4	7.5	4.1
			$\mathcal{K}_4$	4.8	4.5	5.0	5.2	3.5	12.4	4.4
			$\mathcal{K}_1^*$	5.4	4.4	5.4	5.1	5.1	5.3	4.9
			$\mathcal{K}_1'^*$	4.0	5.1	5.0	3.8	4.3	5.4	4.4
			$\mathcal{K}_4^*$	4.5	4.9	5.7	4.5	4.7	5.4	4.9
	0.0	0.5	$\mathcal{K}_1$	1.8	1.0	0.7	1.5	1.4	2.3	1.6
			$\mathcal{K}_1'$	1.0	0.6	1.4	0.8	1.2	1.9	1.2
			$\mathcal{K}_4$	0.5	1.0	1.4	1.5	1.2	2.9	1.1
			$\mathcal{K}_1^*$	3.4	3.3	4.9	3.1	3.5	4.8	4.5
			$\mathcal{K}_1'^*$	3.3	3.4	4.6	3.9	3.4	5.9	3.8
			$\mathcal{K}_4^*$	3.6	3.4	3.8	3.1	3.5	5.5	4.3

Table 3: Empirical Power of Standard Persistence Change Tests: De-meaned Data. Tests Based on Asymptotic 5% Critical Values.

			Model 1			Model 2		Model 3	
T	τ^*		$\delta = 1$	$\delta = 1/3$	$\delta = 3$	$\delta = 1/3$	$\delta = 3$	$c = 0$	$c = 10$
100	0.25	$\mathcal{K}_1$	78.8	96.8	50.0	93.7	57.8	71.0	78.8
		$\mathcal{K}_1'$	48.1	30.1	82.2	31.9	74.4	52.8	51.1
		$\mathcal{K}_4$	83.7	96.3	83.9	93.9	82.4	90.9	85.0
		$\mathcal{K}_1^*$	27.8	30.4	25.3	29.9	26.9	29.9	29.5
		$\mathcal{K}_1'^*$	9.8	9.7	10.4	9.7	11.4	12.8	13.1
		$\mathcal{K}_4^*$	26.8	27.0	25.1	28.3	26.2	28.1	27.2
	0.50	$\mathcal{K}_1$	76.7	97.3	34.0	92.5	53.0	68.1	76.8
		$\mathcal{K}_1'$	30.4	25.8	52.1	19.1	50.0	43.5	34.5
		$\mathcal{K}_4$	75.5	96.7	62.6	91.5	67.0	86.5	79.3
		$\mathcal{K}_1^*$	29.5	33.2	23.4	33.3	26.8	32.3	30.8
		$\mathcal{K}_1'^*$	5.6	7.9	4.1	5.8	5.7	7.6	6.6
		$\mathcal{K}_4^*$	24.1	28.2	19.2	27.6	22.4	27.2	25.8
	0.75	$\mathcal{K}_1$	59.3	91.3	18.8	85.9	30.7	60.4	60.6
		$\mathcal{K}_1'$	8.4	3.8	37.5	2.7	25.6	30.5	14.5
		$\mathcal{K}_4$	57.6	89.4	43.1	81.7	42.4	78.9	63.9
		$\mathcal{K}_1^*$	19.9	24.0	14.3	24.4	15.4	22.6	19.2
		$\mathcal{K}_1'^*$	1.9	2.8	1.2	2.1	1.8	1.9	1.1
		$\mathcal{K}_4^*$	13.8	18.0	9.0	17.3	11.0	17.4	12.8
200	0.25	$\mathcal{K}_1$	89.1	97.9	71.0	96.6	76.8	77.4	88.8
		$\mathcal{K}_1'$	49.0	31.1	80.0	35.3	71.1	54.2	53.7
		$\mathcal{K}_4$	91.2	98.0	87.2	96.8	88.0	92.8	91.6
		$\mathcal{K}_1^*$	44.8	45.6	41.8	46.7	43.9	44.6	46.7
		$\mathcal{K}_1'^*$	14.8	12.5	14.9	13.6	15.8	15.3	14.6
		$\mathcal{K}_4^*$	40.9	41.9	39.0	43.6	41.3	42.3	42.2
	0.50	$\mathcal{K}_1$	89.0	100.0	52.1	96.8	70.8	73.3	88.2
		$\mathcal{K}_1'$	29.8	26.3	42.4	20.0	45.7	43.5	32.5
		$\mathcal{K}_4$	88.4	99.5	65.4	96.3	78.1	88.5	88.7
		$\mathcal{K}_1^*$	47.9	52.4	42.5	53.6	46.7	46.0	51.0
		$\mathcal{K}_1'^*$	8.0	9.6	4.2	7.5	7.8	7.2	8.2
		$\mathcal{K}_4^*$	40.8	44.5	34.7	45.5	39.0	40.3	45.3
	0.75	$\mathcal{K}_1$	73.6	95.9	26.1	92.4	41.9	61.3	72.0
		$\mathcal{K}_1'$	2.2	0.6	17.3	0.3	11.0	23.8	5.3
		$\mathcal{K}_4$	70.6	95.2	34.3	90.5	42.8	77.8	69.2
		$\mathcal{K}_1^*$	34.9	43.9	27.6	41.7	29.5	36.8	36.4
		$\mathcal{K}_1'^*$	0.3	0.9	0.3	0.3	0.5	0.9	0.5
		$\mathcal{K}_4^*$	26.1	33.1	19.9	32.8	20.7	29.6	26.8

Table 4: Size-Adjusted Power of Standard Persistence Change Tests.
De-meaned Data.

			Model 1			Model 2		Model 3	
T	τ^*		$\delta = 1$	$\delta = 1/3$	$\delta = 3$	$\delta = 1/3$	$\delta = 3$	$c = 0$	$c = 10$
100	0.25	$\mathcal{K}_1$	81.4	83.9	76.2	84.6	82.7	22.0	69.1
		$\mathcal{K}_1'$	49.6	51.7	50.3	51.7	53.9	15.3	41.5
		$\mathcal{K}_4$	84.4	84.4	55.1	86.8	69.2	24.3	73.8
		$\mathcal{K}_1^*$	39.7	39.3	24.3	44.5	35.3	19.5	30.0
		$\mathcal{K}_1'^*$	15.6	8.9	15.2	13.0	18.3	12.3	13.8
		$\mathcal{K}_4^*$	37.5	28.2	23.0	39.0	31.3	20.4	29.8
	0.50	$\mathcal{K}_1$	79.8	85.8	64.1	84.7	78.7	20.9	66.4
		$\mathcal{K}_1'$	32.0	46.7	19.8	38.0	32.5	9.9	26.8
		$\mathcal{K}_4$	78.9	86.0	25.5	85.4	53.8	21.0	66.6
		$\mathcal{K}_1^*$	41.8	43.6	24.7	48.7	36.5	22.0	32.3
		$\mathcal{K}_1'^*$	8.1	6.1	5.6	6.8	8.6	6.3	8.6
		$\mathcal{K}_4^*$	34.6	28.2	17.7	38.5	28.6	19.1	28.3
	0.75	$\mathcal{K}_1$	63.3	69.2	47.7	71.1	58.0	15.0	48.9
		$\mathcal{K}_1'$	8.4	10.9	7.6	9.0	11.3	6.6	8.3
		$\mathcal{K}_4$	59.6	69.3	9.3	71.4	24.1	14.7	45.0
		$\mathcal{K}_1^*$	30.2	33.2	17.6	36.2	23.2	15.5	22.2
		$\mathcal{K}_1'^*$	1.9	0.9	1.3	1.3	1.9	2.3	2.0
		$\mathcal{K}_4^*$	21.3	18.5	11.5	24.5	16.0	11.9	17.9
200	0.25	$\mathcal{K}_1$	90.2	89.6	87.0	92.0	90.0	27.2	79.9
		$\mathcal{K}_1'$	50.3	50.2	52.2	52.8	53.5	14.1	41.6
		$\mathcal{K}_4$	92.9	89.9	61.5	92.9	76.2	24.2	82.4
		$\mathcal{K}_1^*$	48.0	46.7	32.1	50.4	40.0	26.0	39.9
		$\mathcal{K}_1'^*$	15.9	7.8	13.9	12.8	16.2	8.1	10.6
		$\mathcal{K}_4^*$	44.4	36.8	30.2	44.0	42.8	20.8	35.0
	0.50	$\mathcal{K}_1$	89.4	92.6	73.0	90.1	88.3	25.9	77.8
		$\mathcal{K}_1'$	29.7	45.0	14.3	35.0	27.9	7.4	22.9
		$\mathcal{K}_4$	89.3	92.7	25.2	90.5	60.4	20.2	76.6
		$\mathcal{K}_1^*$	53.5	52.4	34.8	57.1	46.8	31.5	42.8
		$\mathcal{K}_1'^*$	9.0	6.0	3.7	7.9	7.6	4.6	5.9
		$\mathcal{K}_4^*$	48.2	41.7	28.4	48.2	42.5	23.7	35.9
	0.75	$\mathcal{K}_1$	73.1	76.8	53.5	76.6	66.0	18.4	60.3
		$\mathcal{K}_1'$	2.7	4.1	1.9	2.9	3.4	3.2	3.1
		$\mathcal{K}_4$	71.4	76.9	6.0	76.6	23.8	13.7	56.8
		$\mathcal{K}_1^*$	41.5	42.5	21.0	48.2	30.1	21.4	29.8
		$\mathcal{K}_1'^*$	0.5	0.4	0.4	0.5	0.6	0.7	0.7
		$\mathcal{K}_4^*$	32.7	27.2	14.6	35.1	24.5	15.0	21.3

Table 5: Empirical Power of Bootstrap Tests: De-meaned Data.

			Model 1			Model 2		Model 3	
T	τ^*		$\delta = 1$	$\delta = 1/3$	$\delta = 3$	$\delta = 1/3$	$\delta = 3$	$c = 0$	$c = 10$
100	0.25	$\mathcal{K}_1$	67.5	82.1	56.6	79.9	59.1	61.5	66.0
		$\mathcal{K}_1'$	43.7	37.9	60.4	35.3	58.3	46.7	47.0
		$\mathcal{K}_4$	67.5	82.7	63.9	79.4	66.2	74.0	67.7
		$\mathcal{K}_1^*$	33.9	36.8	27.5	37.4	33.1	30.8	34.4
		$\mathcal{K}_1'^*$	14.5	12.9	16.8	13.9	15.9	15.4	15.3
		$\mathcal{K}_4^*$	30.1	31.2	27.0	32.1	30.6	29.9	30.9
	0.50	$\mathcal{K}_1$	57.4	81.5	33.2	72.2	45.8	54.2	58.8
		$\mathcal{K}_1'$	34.9	37.4	34.9	28.2	45.0	40.0	37.5
		$\mathcal{K}_4$	59.0	81.6	40.3	72.0	54.2	68.1	61.4
		$\mathcal{K}_1^*$	37.9	40.5	28.3	41.6	35.5	34.1	37.3
		$\mathcal{K}_1'^*$	8.7	9.2	6.6	8.7	8.4	8.6	8.4
		$\mathcal{K}_4^*$	28.3	31.2	23.0	31.1	27.7	27.8	28.6
	0.75	$\mathcal{K}_1$	31.0	58.2	18.4	49.4	22.7	37.2	33.0
		$\mathcal{K}_1'$	7.5	8.2	10.8	5.7	12.7	20.4	10.2
		$\mathcal{K}_4$	31.4	58.3	16.7	49.7	23.8	47.5	34.5
		$\mathcal{K}_1^*$	24.9	29.0	17.8	27.8	22.3	21.5	23.8
		$\mathcal{K}_1'^*$	2.0	1.7	1.5	1.7	1.7	2.8	1.8
		$\mathcal{K}_4^*$	17.1	19.4	14.6	18.8	16.8	16.0	17.2
200	0.25	$\mathcal{K}_1$	79.2	87.8	73.2	85.0	74.2	70.3	78.6
		$\mathcal{K}_1'$	42.2	37.4	60.5	34.5	56.3	48.6	47.3
		$\mathcal{K}_4$	77.2	87.7	69.7	84.9	71.5	77.7	77.2
		$\mathcal{K}_1^*$	40.8	45.8	36.0	43.9	38.8	38.1	42.1
		$\mathcal{K}_1'^*$	14.6	11.4	17.0	13.2	15.9	14.8	13.8
		$\mathcal{K}_4^*$	36.6	37.7	35.1	37.7	37.5	36.0	38.2
	0.50	$\mathcal{K}_1$	71.4	88.7	46.6	80.6	61.6	61.4	69.6
		$\mathcal{K}_1'$	33.1	35.9	29.3	27.2	43.3	34.3	35.4
		$\mathcal{K}_4$	73.3	88.7	45.4	80.7	64.3	66.8	70.3
		$\mathcal{K}_1^*$	48.2	51.4	40.0	52.1	46.6	43.8	47.0
		$\mathcal{K}_1'^*$	7.9	9.3	4.6	7.7	7.5	6.9	7.0
		$\mathcal{K}_4^*$	40.2	42.0	33.6	42.9	38.8	37.2	39.6
	0.75	$\mathcal{K}_1$	42.9	67.8	20.7	60.6	28.4	39.1	43.4
		$\mathcal{K}_1'$	3.4	4.2	4.6	2.7	6.5	11.4	6.2
		$\mathcal{K}_4$	42.7	67.8	16.3	60.6	26.4	43.9	43.2
		$\mathcal{K}_1^*$	35.8	39.9	24.0	41.0	31.1	30.6	33.5
		$\mathcal{K}_1'^*$	0.5	0.5	0.4	0.5	1.0	1.3	0.8
		$\mathcal{K}_4^*$	25.4	28.9	17.6	29.3	22.7	23.4	23.8

Table 6: Persistence Change Tests for Twenty US Inflation Series.

	$\mathcal{K}_1$	$\mathcal{K}_1'$	$\mathcal{K}_4$	$\mathcal{K}_2$	$\mathcal{K}_2'$	$\mathcal{K}_5$	$\mathcal{K}_3$	$\mathcal{K}_3'$	$\mathcal{K}_6$
PWFSA	2.848	50.945	50.945	0.393	18.787	18.787	0.251	21.727	21.727
p_{hom}	0.742	0.000	0.003	0.917	0.000	0.000	0.887	0.000	0.003
p_{het}	0.727	0.013	0.018	0.922	0.005	0.005	0.902	0.010	0.013
PWFCSA	1.930	50.639	50.639	0.345	14.389	14.389	0.210	20.359	20.359
p_{hom}	0.860	0.000	0.003	0.947	0.000	0.000	0.917	0.000	0.003
p_{het}	0.882	0.010	0.015	0.947	0.008	0.010	0.930	0.010	0.015
PWIMSA	1.635	48.612	48.612	0.232	17.323	17.323	0.131	19.581	19.581
p_{hom}	0.895	0.000	0.003	0.982	0.000	0.000	0.977	0.000	0.003
p_{het}	0.887	0.040	0.048	0.947	0.013	0.015	0.950	0.038	0.048
PWCMSA	3.218	6.019	6.019	0.807	1.613	1.613	0.431	0.948	0.948
p_{hom}	0.704	0.426	0.707	0.739	0.436	0.782	0.767	0.511	0.830
p_{het}	0.902	0.180	0.897	0.920	0.123	0.910	0.920	0.160	0.925
PSCCOM	2.614	27.197	27.197	0.473	3.451	3.451	0.258	8.490	8.490
p_{hom}	0.774	0.008	0.018	0.885	0.143	0.243	0.882	0.023	0.040
p_{het}	0.677	0.080	0.088	0.820	0.246	0.306	0.827	0.100	0.113
PSM99Q	1.537	21.089	21.089	0.475	2.892	2.892	0.249	5.351	5.351
p_{hom}	0.902	0.035	0.065	0.885	0.193	0.333	0.887	0.058	0.100
p_{het}	0.885	0.100	0.118	0.815	0.328	0.404	0.830	0.138	0.158
PUNEW	12.247	107.797	107.797	1.406	17.368	17.368	1.880	49.232	49.232
p_{hom}	0.125	0.000	0.000	0.454	0.000	0.000	0.206	0.000	0.000
p_{het}	0.150	0.003	0.003	0.406	0.003	0.003	0.241	0.003	0.003
PU83	81.614	5.039	81.614	26.529	0.546	26.529	36.023	0.507	36.023
p_{hom}	0.000	0.531	0.000	0.000	0.900	0.000	0.000	0.749	0.000
p_{het}	0.005	0.296	0.005	0.003	0.684	0.003	0.005	0.481	0.005
PU84	25.259	17.286	25.259	1.479	5.154	5.154	7.070	4.998	7.070
p_{hom}	0.015	0.065	0.035	0.429	0.053	0.095	0.028	0.065	0.058
p_{het}	0.140	0.038	0.145	0.792	0.018	0.241	0.178	0.038	0.190
PU85	43.220	13.054	43.220	2.893	2.292	2.893	16.676	3.438	16.676
p_{hom}	0.003	0.120	0.003	0.140	0.276	0.333	0.003	0.118	0.003
p_{het}	0.003	0.303	0.033	0.050	0.486	0.429	0.003	0.278	0.033
PUC	6.006	24.600	24.600	1.129	4.875	4.875	0.949	8.144	8.144
p_{hom}	0.393	0.020	0.038	0.561	0.060	0.108	0.439	0.023	0.040
p_{het}	0.584	0.030	0.100	0.754	0.038	0.183	0.654	0.028	0.103
PUCD	338.640	6.571	338.640	6.045	0.959	6.045	163.729	0.795	163.729
p_{hom}	0.000	0.383	0.000	0.033	0.714	0.060	0.000	0.584	0.000
p_{het}	0.000	0.429	0.000	0.053	0.669	0.115	0.000	0.541	0.000
PUS	69.395	205.224	205.224	3.774	39.122	39.122	29.138	99.154	99.154
p_{hom}	0.000	0.000	0.000	0.085	0.000	0.000	0.000	0.000	0.000
p_{het}	0.008	0.005	0.005	0.033	0.005	0.005	0.008	0.005	0.005
PUXF	51.934	67.556	67.556	2.302	12.255	12.255	20.376	29.846	29.846
p_{hom}	0.003	0.000	0.000	0.206	0.003	0.003	0.003	0.000	0.000
p_{het}	0.015	0.008	0.015	0.306	0.008	0.015	0.015	0.005	0.013
PUXHS	8.573	54.840	54.840	1.155	8.696	8.696	1.050	23.129	23.129
p_{hom}	0.233	0.000	0.003	0.541	0.010	0.018	0.393	0.000	0.000
p_{het}	0.338	0.003	0.008	0.627	0.015	0.033	0.496	0.003	0.008
PUXM	10.263	99.367	99.367	1.289	17.269	17.269	1.487	45.014	45.014
p_{hom}	0.168	0.000	0.000	0.489	0.000	0.000	0.268	0.000	0.000
p_{het}	0.193	0.003	0.003	0.436	0.003	0.003	0.286	0.003	0.003
GMDC	10.350	244.512	244.512	1.377	31.916	31.916	1.429	117.667	117.667
p_{hom}	0.165	0.000	0.000	0.464	0.000	0.000	0.283	0.000	0.000
p_{het}	0.281	0.000	0.000	0.546	0.000	0.000	0.411	0.000	0.000
GMDCD	300.915	38.113	300.915	5.942	2.478	5.942	144.866	14.619	144.866
p_{hom}	0.000	0.003	0.000	0.035	0.243	0.068	0.000	0.003	0.000
p_{het}	0.000	0.013	0.000	0.063	0.253	0.098	0.000	0.013	0.000
GMDCN	3.365	71.087	71.087	0.706	16.077	16.077	0.507	31.563	31.563
p_{hom}	0.684	0.000	0.000	0.784	0.000	0.000	0.714	0.000	0.000
p_{het}	0.787	0.003	0.005	0.872	0.003	0.010	0.820	0.003	0.005
GMDCS	33.466	163.679	163.679	2.305	26.786	26.786	11.181	76.971	76.971
p_{hom}	0.008	0.000	0.000	0.206	0.000	0.000	0.008	0.000	0.000
p_{het}	0.085	0.000	0.000	0.451	0.000	0.000	0.088	0.000	0.000

Table 7: Standardized Persistence Change Tests for Twenty US Inflation Series.

	$\mathcal{K}_1^*$	$\mathcal{K}_1'^*$	$\mathcal{K}_4^*$	$\mathcal{K}_2^*$	$\mathcal{K}_2'^*$	$\mathcal{K}_5^*$	$\mathcal{K}_3^*$	$\mathcal{K}_3'^*$	$\mathcal{K}_6^*$
PWFSA	3.626	34.567	34.567	0.618	12.614	12.614	0.446	14.407	14.407
p_{hom}	0.634	0.000	0.003	0.815	0.005	0.010	0.759	0.000	0.003
p_{het}	0.734	0.010	0.030	0.907	0.003	0.013	0.845	0.005	0.013
PWFCSA	2.613	34.480	34.480	0.496	11.515	11.515	0.334	12.796	12.796
p_{hom}	0.757	0.000	0.003	0.880	0.008	0.013	0.822	0.000	0.003
p_{het}	0.825	0.005	0.015	0.937	0.000	0.020	0.897	0.005	0.018
PWIMSA	2.217	19.422	19.422	0.522	8.951	8.951	0.345	7.383	7.383
p_{hom}	0.825	0.043	0.080	0.862	0.008	0.013	0.812	0.023	0.050
p_{het}	0.897	0.038	0.168	0.932	0.008	0.075	0.892	0.033	0.135
PWCMSA	0.992	8.854	8.854	0.369	3.801	3.801	0.191	2.463	2.463
p_{hom}	0.967	0.231	0.404	0.935	0.123	0.213	0.932	0.168	0.303
p_{het}	0.925	0.193	0.419	0.927	0.093	0.283	0.922	0.150	0.353
PSCCOM	2.755	18.252	18.252	0.760	1.954	1.954	0.404	4.186	4.186
p_{hom}	0.742	0.053	0.098	0.757	0.341	0.602	0.784	0.083	0.160
p_{het}	0.815	0.050	0.133	0.825	0.263	0.659	0.840	0.068	0.190
PSM99Q	2.687	17.332	17.332	0.828	1.758	1.758	0.453	3.610	3.610
p_{hom}	0.749	0.060	0.108	0.712	0.393	0.704	0.754	0.105	0.203
p_{het}	0.860	0.048	0.108	0.797	0.293	0.754	0.820	0.080	0.203
PUNEW	9.653	37.805	37.805	1.621	5.449	5.449	1.485	14.948	14.948
p_{hom}	0.175	0.000	0.003	0.371	0.040	0.075	0.268	0.000	0.003
p_{het}	0.323	0.013	0.033	0.534	0.015	0.090	0.444	0.010	0.025
PU83	30.145	6.057	30.145	9.944	0.704	9.944	11.719	0.696	11.719
p_{hom}	0.008	0.414	0.010	0.005	0.827	0.013	0.003	0.624	0.005
p_{het}	0.010	0.516	0.018	0.008	0.835	0.010	0.003	0.669	0.008
PU84	11.335	25.779	25.779	0.698	7.210	7.210	1.104	8.664	8.664
p_{hom}	0.133	0.010	0.028	0.784	0.015	0.033	0.366	0.013	0.030
p_{het}	0.216	0.018	0.065	0.870	0.015	0.080	0.501	0.020	0.070
PU85	73.461	1.908	73.461	6.641	0.428	6.641	31.176	0.244	31.176
p_{hom}	0.000	0.905	0.000	0.025	0.937	0.043	0.000	0.935	0.000
p_{het}	0.000	0.825	0.000	0.033	0.862	0.038	0.000	0.855	0.000
PUC	4.791	26.145	26.145	1.037	4.314	4.314	0.756	8.968	8.968
p_{hom}	0.504	0.008	0.025	0.594	0.078	0.153	0.531	0.010	0.025
p_{het}	0.614	0.018	0.065	0.774	0.040	0.213	0.694	0.018	0.068
PUCD	164.909	2.394	164.909	5.744	0.418	5.744	76.864	0.235	76.864
p_{hom}	0.000	0.847	0.000	0.033	0.937	0.065	0.000	0.935	0.000
p_{het}	0.000	0.789	0.000	0.090	0.855	0.115	0.000	0.842	0.000
PUS	36.510	21.810	36.510	3.347	4.290	4.290	13.294	8.316	13.294
p_{hom}	0.003	0.028	0.003	0.110	0.080	0.158	0.003	0.015	0.003
p_{het}	0.020	0.073	0.048	0.185	0.045	0.158	0.020	0.053	0.045
PUXF	24.118	33.041	33.041	1.665	5.186	5.186	6.532	12.806	12.806
p_{hom}	0.023	0.003	0.005	0.348	0.043	0.085	0.033	0.000	0.003
p_{het}	0.068	0.025	0.053	0.581	0.020	0.118	0.088	0.025	0.048
PUXHS	7.843	40.648	40.648	1.389	5.401	5.401	1.128	16.215	16.215
p_{hom}	0.246	0.000	0.003	0.444	0.040	0.075	0.361	0.000	0.003
p_{het}	0.406	0.000	0.018	0.627	0.015	0.143	0.531	0.000	0.015
PUXM	7.942	37.545	37.545	1.432	5.625	5.625	1.223	14.598	14.598
p_{hom}	0.246	0.000	0.003	0.439	0.038	0.070	0.341	0.000	0.003
p_{het}	0.421	0.008	0.033	0.619	0.015	0.095	0.496	0.005	0.030
GMDC	7.567	104.189	104.189	1.552	12.786	12.786	1.178	47.732	47.732
p_{hom}	0.268	0.000	0.000	0.388	0.005	0.010	0.356	0.000	0.000
p_{het}	0.436	0.000	0.000	0.591	0.000	0.008	0.544	0.000	0.000
GMDCD	163.851	15.011	163.851	6.344	1.070	6.344	76.334	3.529	76.334
p_{hom}	0.000	0.080	0.000	0.030	0.664	0.050	0.000	0.105	0.000
p_{het}	0.000	0.063	0.000	0.053	0.516	0.068	0.000	0.073	0.000
GMDCN	2.855	62.822	62.822	0.669	13.800	13.800	0.449	28.379	28.379
p_{hom}	0.727	0.000	0.000	0.799	0.003	0.003	0.757	0.000	0.000
p_{het}	0.835	0.003	0.008	0.905	0.000	0.010	0.857	0.000	0.005
GMDCS	9.840	158.774	158.774	1.158	24.612	24.612	1.257	74.206	74.206
p_{hom}	0.170	0.000	0.000	0.529	0.000	0.000	0.326	0.000	0.000
p_{het}	0.258	0.000	0.000	0.632	0.000	0.000	0.398	0.000	0.000

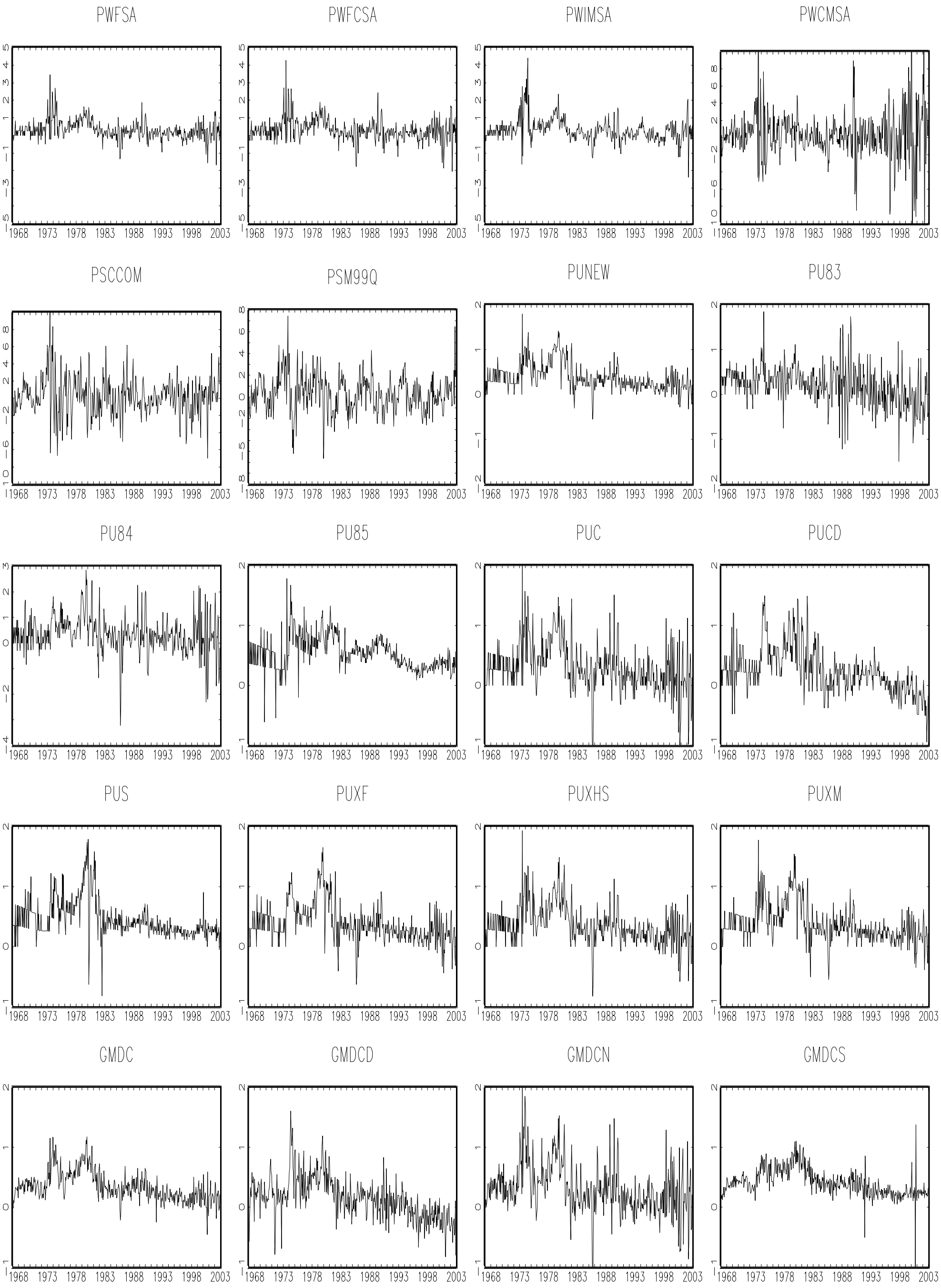

Figure 1: Twenty U.S. inflation rates, 1967–2003.

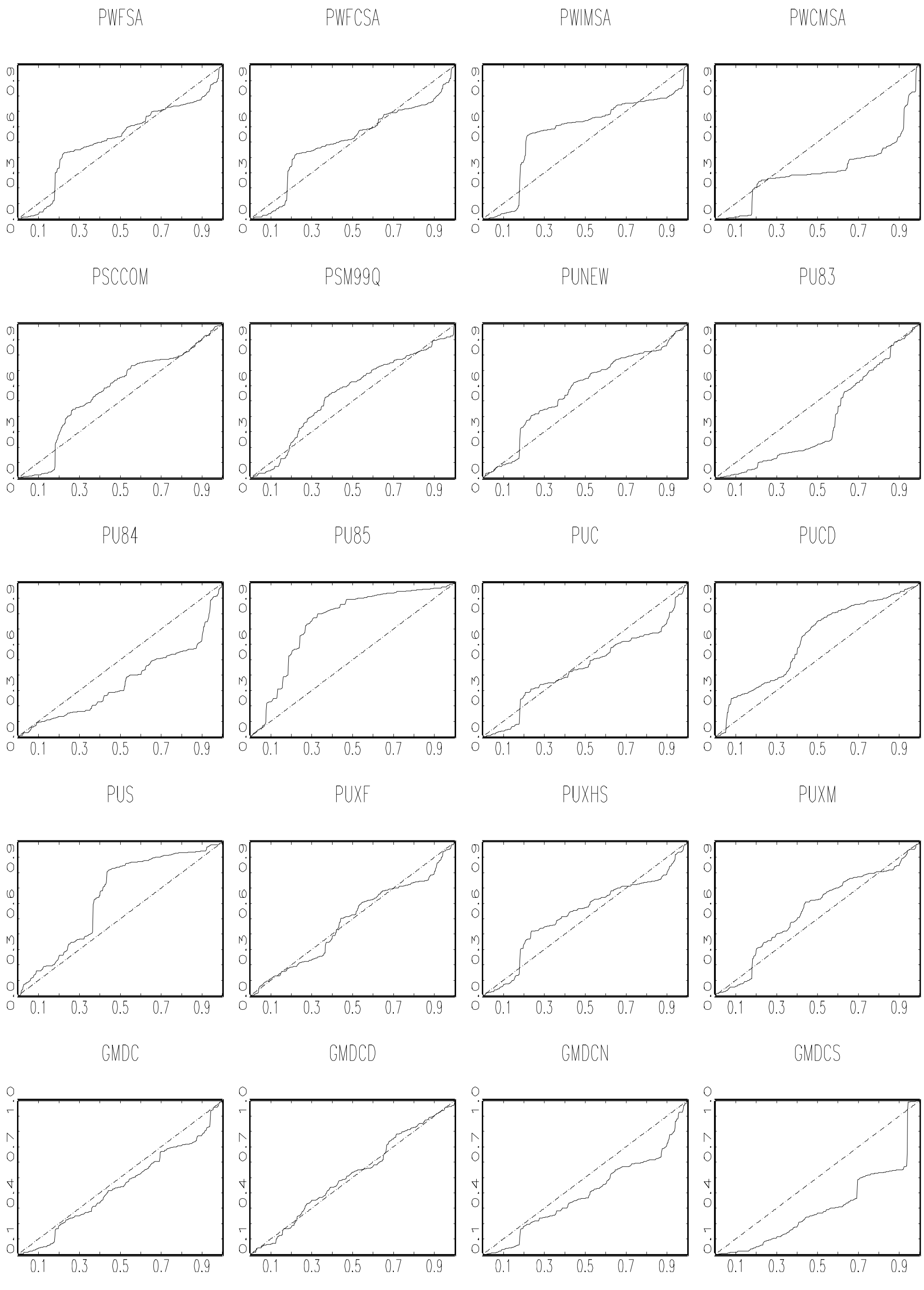

Figure 2: Twenty U.S. inflation rates, 1967–2003: estimated variance profiles.