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factor models

by

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# Sequential testing for structural stability in approximate factor models

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## Abstract

We develop an on-line monitoring procedure to detect a change in a large approximate factor model. Our statistics are based on a well-known property of the  $(r + 1)$ -th eigenvalue of the sample covariance matrix of the data (having defined  $r$  as the number of common factors): whilst under the null the  $(r + 1)$ -th eigenvalue is bounded, under the alternative of a change (either in the loadings, or in the number of factors itself) it becomes spiked. Given that the sample eigenvalue cannot be estimated consistently under the null, we regularise the problem by randomising the test statistic in conjunction with sample conditioning, obtaining a sequence of *i.i.d.*, asymptotically chi-square statistics which are then employed to build the monitoring scheme. Numerical evidence shows that our procedure works very well in finite samples, with a very small probability of false detections and tight detection times in presence of a genuine change-point.

*Keywords:* large factor model, change-point, sequential testing, randomised tests.

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# 1 Introduction

In this paper, we investigate the issue of testing for the stability of a large factor model:

$$X_{i,t} = a_i' f_t + u_{i,t}, \quad (1)$$

where  $\{X_{i,t}, 1 \leq i \leq N, 1 \leq t \leq T\}$  is a panel of  $N$  time series observed for  $T$  periods;  $a_i$  and  $f_t$  are latent vectors of loadings and factors, respectively, both of dimension  $r < N$  and representing the “signal” component of the data. In particular, we focus on the *sequential monitoring* of the stability of (1) - that is, we propose a test to check whether there are any breaks in (1) as new data come in. Factor models have been paid significant attention in virtually all applied sciences, as a tool to reduce dimensionality while preserving the information content of a large dataset. In particular, in the context of social sciences and economics, the use of factor models has been popularised by the seminal paper by Chamberlain and Rothschild (1983); thereafter, factor models have acquired a huge popularity in various applications, such as business cycle analysis, asset pricing and economic monitoring and forecasting - see the review by Stock and Watson (2011) for a comprehensive list of references.

Model (1) is usually characterized by the identifying assumption that, as  $N \rightarrow \infty$ , the covariance matrix of  $\{X_{i,t}\}_{i=1}^N$  has  $r$  spiked eigenvalues diverging to infinity, while the remaining ones stay bounded for any  $N$ . Numerous contributions have developed a full-fledged inferential theory for (1) under general assumptions, such as weak serial and cross-correlation of the error terms  $u_{i,t}$ . In particular, in the case of stationary data the estimation by means of principal component analysis of the “signal” part of (1) has been developed for high-dimensional, i.e. large  $N$ , data, e.g. by Bai (2003) and Fan, Liao and Mincheva (2013). The literature has also produced many results on the determination of the number of common factors  $r$  - see, *inter alia*, Bai and Ng (2002), Alessi, Barigozzi and Capasso (2010), Onatski (2010), Ahn and Horenstein (2013), and Trapani (2017). The factors in equation (1) have also proven to be very effective to forecast large datasets, overcoming the curse of dimensionality issue - see e.g. Stock and Watson (2002). Extensions to the case in which the common factors  $f_t$  are explicitly allowed to have a linear process representation, have been studied also - see e.g. Forni, Giannone, Lippi and Reichlin (2009).

In comparison with this huge body of literature, the issue of testing for the structural

stability of (1) can be still considered underdeveloped, with some notable exceptions. Indeed, Stock and Watson (2002) and Bates, Plagborg-Møller, Stock and Watson (2013) argue that, at least in the presence of “small” breaks and a constant number of factors, inference on the factor space is not hampered, thus making the change-point problem less compelling than in other contexts. Nevertheless, stylised facts show that in many applications the assumptions of a negligible break size and a stable number of factors are not, in general, correct. Most importantly, it has been argued that, in presence of a crisis, co-movements become stronger, which may suggest that the economy is driven by a different number of factors than in quieter periods - see e.g. Stock and Watson (2009), Cheng, Liao and Schorfheide (2016) and Li, Todorov, Tauchen and Lin (2017). In such cases, the impact of a change-point is bound to invalidate standard inference and subsequent applications such as forecasting. Recently, the literature has proposed a series of tests for the in-sample detection of breaks in factor structures: examples include the works by Breitung and Eickmeier (2011), Chen, Dolado and Gonzalo (2014), Han and Inoue (2015), Corradi and Swanson (2014), Yamamoto and Tanaka (2015), Baltagi, Kao and Wang (2017), Cheng et al. (2016), Massacci (2017) and Barigozzi, Cho and Fryzlewicz (2017).

Sequential detection of breaks in (1) is important for at least four reasons. First, the general motivation put forward by Chu, Stinchcombe and White (1996) holds true in the context of factor models also: it is important to verify whether a model, which has been valid thus far, is still capable of adequately approximate the behaviour of new data. Second, the aforementioned (substantial) empirical evidence that factor structures do tend to change over time, especially in presence of a crisis, illustrates the importance of a timely detection of such changes. Third, inference on factor models can be severely marred by the presence of a break (see the comments in Baltagi et al., 2017), which again shows the importance of detecting a break in real time, rather than realising this *a posteriori* after inference has been carried out and employed, e.g. for the purpose of forecasting. Finally, in the context economics and finance, data are collected and made available automatically, so that the cost of monitoring is almost negligible, especially if compared with the potential costs of employing a model which is no longer valid. Sequential detection of breaks in a univariate or small dimensional, i.e. finite  $N$ , setting has been studied e.g. in Lai (1995), Chu et al. (1996), Aue and Horváth (2004), Horváth, Hušková, Kokoszka and Steinebach (2004), Andreou and Ghysels (2006), Horváth, Kokoszka and Steinebach (2007), Brodsky (2010),

Aue, Hörmann, Horváth, Hušková and Steinebach (2012), Kirch and Tadjuidje Kamgaing (2015), and Groen, Kapetanios and Price (2013).

## 1.1 Hypotheses of interest and main results of the paper

There are several possible ways in which model (1) may undergo a change at a point in time  $\tau$ ; however, despite such a wide variety, in all cases it may be argued that a change in the factor structure of the data will result in a change in the covariance matrix of  $\{X_{i,t}\}_{i=1}^N$ . More specifically, since common factors determine the presence and number of spiked eigenvalue of the covariance of  $\{X_{i,t}\}_{i=1}^N$  (defined as eigenvalues which are not bounded, but grow with the dimension of the dataset), it is natural to investigate whether a change has occurred in the factor structure of (1) by verifying whether changes have occurred in the spectrum of the covariance matrix. Formally, in this paper we test for the null hypothesis that the factor structure does not change, viz.:

$$H_0 : X_{i,t} = \sum_{j=1}^r a_{ij} f_{j,t} + u_{i,t}, \quad 1 \leq t \leq T.$$

As far as alternatives are concerned, we focus on two different possible breaks at a point in time  $\tau$ : (1) changes in the loadings attached to one or more common factor:

$$H_{A,1} : \begin{cases} X_{i,t} = \sum_{j=1}^r a_{ij} f_{j,t} + u_{i,t} & \text{for } 1 \leq t < \tau \\ X_{i,t} = \sum_{j=1}^r \tilde{a}_{ij} f_{j,t} + u_{i,t} & \tau \leq t \leq T \end{cases}, \quad (2)$$

where  $\tilde{a}_{ij} \neq a_{ij}$  for all  $i$  and some  $j$ , and (2) the appearance of  $q \geq 1$  new factors:

$$H_{A,2} : \begin{cases} X_{i,t} = \sum_{j=1}^r a_{ij} f_{j,t} + u_{i,t} & \text{for } 1 \leq t < \tau \\ X_{i,t} = \sum_{j=1}^r a_{ij} f_{j,t} + \sum_{j=1}^q b_{ij} g_{j,t} + u_{i,t} & \tau \leq t \leq T \end{cases}. \quad (3)$$

Hypothesis  $H_{A,1}$  is the typical case considered in all the above cited literature on change-points in factor models. On the other hand hypothesis  $H_{A,2}$  has received less attention from the literature - see for example Cheng et al. (2016) and Barigozzi et al. (2017). Whilst in this paper we mainly focus on  $H_{A,1}$  and  $H_{A,2}$ , other alternatives, as disappearing factors or less pervasive changes in the loadings, can also be accommodated in our framework - see the discussion in Section 4.

We show that, under both  $H_{A,1}$  and  $H_{A,2}$ , the  $(r+1)$ -th largest eigenvalue of the

covariance matrix of  $\{X_{i,t}\}_{i=1}^N$  becomes unbounded at time  $\tau + 1$ , passing to infinity as fast as the sample size  $N$ . Conversely, it stays bounded under the null of no break. Thus, we base our test on the estimated  $(r + 1)$ -th eigenvalue of the sample covariance matrix of  $\{X_{i,t}\}_{i=1}^N$ . Although using the sample eigenvalues of the sample covariance matrix for testing is not uncommon in the context of factor models (Onatski, 2010; Trapani, 2017), in our context such an approach is fraught with difficulties. The main issue is that, under the null of no break, the  $(r + 1)$ -th sample eigenvalue does not have a known distribution, and indeed it cannot even be estimated consistently: as Wang and Fan (2017) explain, there is too much noise (due to  $N$  being large) to be able to identify the small signal coming from a bounded eigenvalue.

Given that the only thing we know is that the  $(r + 1)$ -th sample eigenvalue may be bounded or unbounded, we propose to use a randomised test in order to regularize the problem. Randomisation is a widely employed approach, dating back at least to Pearson (1950); various authors have employed different ways of introducing randomness into a statistic - see Corradi and Swanson (2006), Bandi and Corradi (2014), and Trapani (2017), among others. Our methodology is based on the same approach, but with a different scope. In essence, the approach which we propose takes, at each  $t$ , the  $(r + 1)$ -th sample eigenvalue as input, and returns, as output, an *i.i.d.* sequence, with known (asymptotic) distribution, first and second moments that can be approximated with a negligible error, and finite moments up to any order. Such sequence is then used to replace the  $(r + 1)$ -th sample eigenvalue in the construction of the monitoring process, thus allowing us to use the standard asymptotic theory already developed for partial sum processes of *i.i.d.* sequences - see Horváth et al. (2004) and Kirch and Tadjuidje Kamgaing (2015). Although our results are derived conditionally on the sample (see the comments in Section 3 on the meaning of randomisation under sample conditioning), we construct a monitoring procedure which falsely identifies a break under the null with probability smaller than a prescribed level, and which identifies a break with probability one when this is present. This is a desirable feature of sequential testing since as more data come in the probability of type I errors is anyway likely to increase - see for example the comments in Chapter 9 by Sen (1981). Indeed, numerical evidence suggests that our procedure works extremely well, with a short delay in finding breaks. In principle, our test can be applied also under more general circumstances, including the presence of weak factors or less pervasive loadings changes,

the case of heteroskedastic idiosyncratic components, and the disappearance of one or more factors. All these extensions are discussed in Sections 4 and 7.

The rest of the paper is organised as follows. In Section 2 we spell out the main assumptions, and we study the inference on the  $(r + 1)$ -th eigenvalue of the covariance matrix. Section 3 discusses the construction of the test statistic, including the double randomisation procedure and all the relevant intermediate results. Some straightforward extensions of our framework to more general circumstances are discussed in Section 4. Numerical evidence from Monte Carlo experiments and a real data application on US industrial production monthly data are given in Sections 5 and 6, respectively. Section 7 discusses further possible extensions and concludes. All proofs are in the Appendix.

NOTATION. We let  $C_0, C_1, \dots$  denote generic, finite positive constants that do not depend on the sample size, and whose value may change from line to line; “ $\rightarrow$ ” denotes the ordinary limit; orders of magnitude for an a.s. convergent sequence (say  $s_T$ ) are denoted as  $O(T^\varsigma)$  and  $o(T^\varsigma)$  when, for some  $\epsilon > 0$  and  $\tilde{T} < \infty$ ,  $P\left[|T^{-\varsigma}s_T| < \epsilon \text{ for all } T \geq \tilde{T}\right] = 1$  and  $T^{-\varsigma}s_T \rightarrow 0$  a.s., respectively;  $I_A(x)$  is the indicator function of a set  $A$ .

## 2 Assumptions and preliminary theory

Recall the factor model in (1), where we now make explicit the possibility of changes over time in the “signal” component

$$X_{i,t} = a_i'(t)f_t + u_{i,t}, \quad 1 \leq i \leq N, \quad 1 \leq t \leq T. \quad (4)$$

We use  $r(t)$  to denote the number of factors at a given time  $t$ , i.e. the vectors of loadings  $a_i(t)$  and of factors  $f_t$  have dimension  $r(t)$ . Consider also the matrix form of (4):

$$X_t = A(t)f_t + u_t, \quad 1 \leq t \leq T, \quad (5)$$

where,  $A(t) = [a_1(t)|\dots|a_N(t)]'$  is the  $N \times r(t)$  loadings matrix and  $u_t = [u_{1,t}, \dots, u_{N,t}]'$  is the “idiosyncratic” component.

We define the covariance matrix of the data at time  $t$  as  $\Sigma_X(t) = E(X_t X_t')$ , assuming

$X_t$  has zero-mean. Consider the (population) rolling covariance matrix

$$\Sigma_m(t) = \frac{1}{m} \sum_{k=t-m+1}^t \Sigma_X(k), \quad m \leq t \leq T, \quad (6)$$

and its sample counterpart

$$\widehat{\Sigma}_m(t) = \frac{1}{m} \sum_{k=t-m+1}^t X_k X_k', \quad m \leq t \leq T. \quad (7)$$

Based on (6) and (7), in what follows  $m$  will denote our sample size when estimating the model; hence, our asymptotics is for  $m \rightarrow \infty$ . We assume that for the first  $m$  periods no change-point is present and we have  $r$  factors. Moreover, for simplicity, we also assume that our monitoring procedure will last until  $T > m$ . Therefore, the total number of observations  $T$  includes both the estimation and the monitoring period. Note that, in real applications, the monitoring may be expected to go on indefinitely, so that  $T \rightarrow \infty$ .

We start with the following assumption.

**Assumption 1.** *It holds that (i)  $E(X_{i,t}) = 0$  for all  $1 \leq i \leq N$  and  $1 \leq t \leq T$ ; (ii)  $E(f_{j,t} u_{i,t}) = 0$  for all  $i, j, t$ ; (iii)  $r(t) = r$  for  $1 \leq t \leq m$ ; (iv)  $r(t) < N$  for  $1 \leq t \leq T$  and for any  $N \in \mathbb{N}$ .*

Parts (i) and (ii) of the assumption are made only for convenience and could be relaxed. Clearly from part (iii) we have that, in presence of breaks, the change-point location  $\tau$  is such that  $\tau > m$ . Finally, part (iv) is a reasonable requirement for the number of factors to be finite at any point in time. By Assumption 1 the covariance is decomposed as

$$\Sigma_X(t) = A(t) \Sigma_F(t) A(t)' + \Sigma_u(t),$$

having defined  $\Sigma_F(t) = E(f_t f_t')$  and  $\Sigma_u(t) = E(u_t u_t')$ . Henceforth, we denote the  $k$ -th largest eigenvalue of  $\Sigma_m(t)$  as  $\lambda^{(k)}(t)$ , the  $k$ -th eigenvalue of  $A(t) \Sigma_F(t) A(t)'$  as  $\gamma^{(k)}(t)$ ; and, finally, the  $k$ -th eigenvalue of  $\Sigma_u(t)$  as  $\omega^{(k)}(t)$ ; similarly, we denote the  $k$ -th largest eigenvalue of  $\widehat{\Sigma}_m(t)$  as  $\widehat{\lambda}^{(k)}(t)$ . In order to derive our results on the population and sample eigenvalues, we make the following assumptions.

**Assumption 2.** *It holds that (i)  $\gamma^{(k)}(t) = C_k(t) N$  for all  $1 \leq k \leq r(t)$ , some finite  $C_k(t) > 0$  and for  $m \leq t \leq T$ ; (ii)  $\omega^{(k)}(t) \leq C_0$  for all  $1 \leq k \leq N$  and  $m \leq t \leq T$ .*



**Assumption 3.** *It holds that (i)  $E|X_{i,t}|^{4+\epsilon} \leq C_0$  for  $1 \leq i \leq N$ ,  $1 \leq t \leq T$  and some  $\epsilon > 0$ ; (ii)  $E \left[ \max_{t_0 \leq \tilde{t} \leq t_0+m-1} \left| \sum_{t=1}^{\tilde{t}} X_{h,t} X_{j,t} - E(X_{h,t} X_{j,t}) \right|^2 \right] \leq C_1 m$  for  $1 \leq h, j \leq N$  and  $1 \leq t_0 \leq T - m$ .*

Assumption 2 is typical of high-dimensional factor analysis and is analogous to the assumptions in Chamberlain and Rothschild (1983) and Forni et al. (2009). In particular, as far as the non-zero  $\gamma^{(k)}(t)$ 's are concerned, part (i) of the assumption requires that they diverge to positive infinity, as  $N \rightarrow \infty$ , at a rate  $N$ . Equivalently, we could follow Bai and Ng (2002) and Fan et al. (2013) and require that, in addition to  $\Sigma_F(t)$  being positive definite,  $N^{-1}A(t)'A(t)$  tends to a positive definite matrix, which is tantamount to assuming that  $\gamma^{(k)}(t)$  passes to infinity at a rate  $N$ . As far as the  $\omega^{(k)}(t)$ 's are concerned, in part (ii) of the assumption, the same condition could be derived from the assumptions in Fan et al. (2013) - see also Bai and Ng (2002). Note also that we do not require the  $\omega^{(k)}(t)$ 's to be constant over  $t$ : unconditional heteroskedasticity is allowed for, in principle - see also the comments in Section 4. Assumption 2 determines the behaviour of the population eigenvalues of  $\Sigma_m(t)$ . In particular, at  $t = m$ , by Weyl's inequality we have that  $\lambda^{(k)}(m) = C_0 N$  for  $1 \leq k \leq r$ , while  $\lambda^{(k)}(m) \leq C_1$  for  $r + 1 \leq k \leq N$ . This condition allows us to identify  $r$  in the pre-break sample.

Assumption 3(ii) is a high-level condition which, in essence, poses a constraint on the amount of serial correlation that one can have in the process  $\{X_{h,t} X_{j,t}\}_{t=1}^T$  and therefore, albeit indirectly, in  $\{X_{i,t}\}_{t=1}^T$ . In general, this assumption is satisfied by any linear process with summable fourth cumulants (see e.g., Hannan, 1970, Theorem 6, page 210). Some examples under which Assumption 3 holds are reported in Trapani (2017) and include the case of stationary, causal processes - see Wu (2005). This family of processes in turn includes several popular examples such as Volterra series and ARCH/GARCH processes, thus allowing for the case of conditional heteroskedasticity.

Finally, note that Assumptions 2 and 3 allow for some degree of cross-sectional and serial dependence in the panel of idiosyncratic components,  $\{u_{i,t}, 1 \leq i \leq N, 1 \leq t \leq T\}$ ; thus, (1) defines an ‘‘approximate’’ factor model, as opposed to an ‘‘exact’’ one, which would require cross-sectionally and serially *i.i.d.* errors.

The following result characterizes the behaviour of the population eigenvalues of  $\Sigma_m(t)$ .

**Lemma 1.** *Under Assumptions 1 and 2, it holds that*

$$\lambda^{(r+1)}(t) \leq C_0, \quad m \leq t \leq T, \quad \text{under } H_0. \quad (8)$$

*Further, it holds that*

$$\lambda^{(r+1)}(t) \begin{cases} \leq C_0 & m \leq t < \tau, \\ \geq C_1 \min\{\frac{t-\tau+1}{m}, \frac{\tau+m-t-1}{m}\}N & \tau \leq t < \tau + m, \\ \leq C_0 & \tau + m \leq t \leq T, \end{cases} \quad \text{under } H_{A,1}, \quad (9)$$

$$\lambda^{(r+1)}(t) \begin{cases} \leq C_0 & m \leq t < \tau, \\ \geq C_1 \frac{t-\tau+1}{m}N & \tau \leq t < \tau + m, \\ \geq C_1 N & \tau + m \leq t \leq T, \end{cases} \quad \text{under } H_{A,2}. \quad (10)$$

The sample counterpart to Lemma 1 is the following result, derived in Trapani (2017).

**Lemma 2.** *Under Assumptions 1 and 3, it holds that*

$$\widehat{\lambda}^{(r+1)}(t) = \lambda^{(r+1)}(t) + O_{a.s.} \left( \frac{N}{m^{1/2}} l(m, N) \right), \quad m \leq t \leq T, \quad (11)$$

where

$$l(m, N) = (\ln N)^{1+\epsilon} (\ln m)^{\frac{1+\epsilon}{2}},$$

for some  $\epsilon > 0$ .

Lemma 2 provides a strong rate for the estimation error  $(\widehat{\lambda}^{(r+1)}(t) - \lambda^{(r+1)}(t))$ , which is valid for any combination of  $N$  and  $m$ , and indeed for all estimated eigenvalues,  $\widehat{\lambda}^{(k)}(t)$  for  $1 \leq k \leq N$ . The lemma does not require any assumptions on  $\lambda^{(k)}(t)$ : some of these may be non-distinct, non well-separated, or even equal to zero. Equation (11) states that the estimation error can be quite large. It is, however, comparatively small for the spiked eigenvalues, which, by Assumption 2, are of order  $N$ . Conversely, the error term in (11) can be quite large for the bounded eigenvalues; in this case, the rate is probably not the sharpest one, although it suffices for the construction of the monitoring procedure. The result of Lemma 2 can also be compared with the results from Random Matrix literature for spiked covariance models – see e.g. El Karoui (2007), Paul (2007), Johnstone and Lu (2009), Jung and Marron (2009), Bai and Yao (2012), and Onatski, Moreira and Hallin (2014).

### 3 Testing procedure and asymptotics

In this section, we propose an algorithm to “regularise” the behaviour of the eigenvalues so as to be able to construct a monitoring procedure. As a consequence of Lemmas 1 and 2, we are unable to use  $\widehat{\lambda}^{(r+1)}(t)$ , due to the lack of a known limiting distribution under the null, and of the dependence structure across  $t$ . We therefore propose a randomisation algorithm, whose output is a sequence of *i.i.d.* random variables with finite moments of arbitrarily high order and, under the null, (asymptotically) chi-square distributed. We subsequently employ (the standardised version of) such random variables to construct a partial sum process, which we use as the relevant test statistic in an analogous way as Horváth et al. (2004) and Horváth et al. (2007).

#### 3.1 The randomisation algorithm

Define  $\delta \in (0, 1)$  such that

$$\delta \begin{cases} > 0 \\ > 1 - \frac{1}{2} \frac{\ln m}{\ln N} \end{cases} \quad \text{according as} \quad \begin{cases} N \leq m^{1/2} \\ N > m^{1/2} \end{cases} ; \quad (12)$$

note that the choice of  $\delta$  is uniquely determined by  $N$  and  $m$ , with no need to estimate it. We consider the statistic

$$\phi_{N,m}(t) = g \left( \frac{N^{-\delta} \widehat{\lambda}^{(r+1)}(t)}{\frac{1}{N} \sum_{k=1}^N \widehat{\lambda}^{(k)}(t)} \right), \quad m \leq t \leq T, \quad (13)$$

where  $g(\cdot)$  is a monotonically increasing function such that  $g(0) = 0$  and  $\lim_{x \rightarrow \infty} g(x) = \infty$ ; in this paper, we use  $g(a) = a$ , but other choices are also possible. The denominator in (13) makes the argument of  $g(\cdot)$  scale invariant.

The quantity  $\delta$ , defined in (12), plays a very important role in the remainder of the paper. Based on Lemma 2, it can be expected that  $\widehat{\lambda}^{(r+1)}(t)$  may diverge to positive infinity even when  $\lambda^{(r+1)}(t)$  is bounded; in this case, the divergence rate is  $O(Nm^{-1/2})$ , modulo the logarithmic terms. On the other hand,  $\widehat{\lambda}^{(r+1)}(t)$  diverges at the faster rate  $O(N)$  under the alternative. The purpose of  $\delta$  is to annihilate the estimation error: based on (12), it can be seen that  $N^\delta$  is larger than  $Nm^{-1/2}l(m, N)$ : thus, under the null of no break, it can be expected that  $N^{-\delta} \widehat{\lambda}^{(r+1)}(t)$  will drift to zero. Under the alternative, it

still passes to infinity (since  $\delta < 1$ ), albeit at a slower rate than  $\lambda^{(r+1)}(t)$  itself. Note that this would hold also for very large values of  $N$ : indeed, no restriction is required between the relative rate of divergence of  $N$  and  $m$  as they pass to infinity, and one could also allow for  $N = \exp(m)$ ; in this case, after some algebra it can be shown that  $\delta \in (1 - \frac{1}{2} \frac{\ln m}{m}, 1)$ , which still yields that  $N^{-\delta} \widehat{\lambda}^{(r+1)}(t)$  drifts to zero or diverges to infinity according as the null or the alternative is true.

On account of the comments above, and of Lemmas 1 and 2, it holds that

$$\begin{aligned} \lim_{N,m \rightarrow \infty} \phi_{N,m}(t) &= g(0) = 0, \quad \text{w.p. 1, when } N^{-\delta} \lambda^{(r+1)}(t) \rightarrow 0, \\ \lim_{N,m \rightarrow \infty} \phi_{N,m}(t) &= g(\infty) = \infty, \quad \text{w.p. 1, when } N^{-\delta} \lambda^{(r+1)}(t) \rightarrow \infty. \end{aligned}$$

Letting  $t_{N,m}^*$  be a point in time such that

$$\lim_{N,m \rightarrow \infty} \frac{N^{1-\delta}}{m} (t_{N,m}^* - \tau + 1) = \infty,$$

we therefore have that

$$\lim_{N,m \rightarrow \infty} \phi_{N,m}(t) = 0, \quad m \leq t \leq T, \quad \text{under } H_0,$$

while

$$\lim_{N,m \rightarrow \infty} \phi_{N,m}(t) = \begin{cases} 0 & m \leq t < \tau, \\ \infty & t_{N,m}^* \leq t < 2\tau + m - 2 - t_{N,m}^*, \\ 0 & \tau + m \leq t \leq T, \end{cases} \quad \text{under } H_{A,1},$$

or

$$\lim_{N,m \rightarrow \infty} \phi_{N,m}(t) = \begin{cases} 0 & m \leq t < \tau, \\ \infty & t_{N,m}^* \leq t < \tau + m, \end{cases} \quad \text{under } H_{A,2};$$

between  $\tau$  and  $t_{N,m}^*$ ,  $\phi_{N,m}(t)$  is growing from 0 to  $\infty$ .

Given that the results above entail that we only have rates for  $\phi_{N,m}(t)$ , we propose a to use a randomised version of it, built according to the following steps.

*Step A1.* At each given  $t \geq m$ , generate an *i.i.d.* sample  $\{\xi_j(t)\}_{j=1}^R$  with common distribution  $G_\phi$  such that  $G_\phi(0) \neq 0$  or 1.

*Step A2.* For any  $u$  drawn from a distribution  $F_\phi(u)$ , define

$$\zeta_j(u; t) = I \left[ \xi_j(t) \leq u\phi_{N,m}^{-1}(t) \right].$$

*Step A3.* Compute

$$\vartheta(u; t) = \frac{1}{\sqrt{R}} \sum_{j=1}^R \frac{\zeta_j(u; t) - G_\phi(0)}{\sqrt{G_\phi(0)[1 - G_\phi(0)]}}.$$

*Step A4.* Compute

$$\Theta_t = \int_{-\infty}^{+\infty} |\vartheta(u; t)|^2 dF_\phi(u).$$

Although the details of the behaviour of  $\Theta_t$  under the null and the alternative are spelt out later on, a heuristic preview of the main argument may be helpful. In essence, under the alternative the Bernoulli random variable  $\zeta_j(u; t)$  should be equal to 1 or 0 with probability  $G_\phi(0)$  and  $1 - G_\phi(0)$  respectively, and thus have mean  $G_\phi(0)$ . In this case, when constructing  $\vartheta(u; t)$ , a Central Limit Theorem holds and therefore we expect  $\Theta_t$  to have a chi-square distribution. On the other hand, under the null  $\zeta_j(u; t)$  should be (heuristically) 0 or 1 with probability 0 or 1 (depending on the sign of  $u$ ) - thus, its mean should be different than  $G_\phi(0)$  (and equal to 0 or 1 depending on the sign of  $u$ ) and a Law of Large Numbers should hold. Note that, by construction, conditionally on the sample the sequence  $\{\Theta_t\}_{t=m}^T$  is independent across  $t$ . In order to study  $\Theta_t$ , we need the following assumptions.

**Assumption 4.** *It holds that: (i)  $G_\phi(\cdot)$  has a bounded density; (ii)  $\int_{-\infty}^{+\infty} u^2 dF_\phi(u) < \infty$ .*

**Assumption 5.** *It holds that, as  $\min(N, m, R) \rightarrow \infty$ : (i)*

$$R^{1/2} \left[ g \left( N^{1-\delta} \frac{t - \tau + 1}{m} \right) \right]^{-1} \rightarrow 0, \quad \begin{array}{l} \text{under } H_{A,1}, \text{ for } t_{N,m}^* \leq t < 2\tau + m - 2 - t_{N,m}^*, \\ \text{under } H_{A,2}, \text{ for } t_{N,m}^* \leq t \leq T; \end{array}$$

(ii)  $R^{1/2} [g(N^{1-\delta})]^{-1} \rightarrow 0$  under  $H_{A,1}$ , for  $\tau + m \leq t \leq T$ .

Considering Assumption 4,  $G_\phi$  is usually chosen as the standard normal distribution, and  $F_\phi$  as a discrete uniform distribution (see e.g. Bandi and Corradi (2014)).

Let now  $P^*$  represent the conditional probability with respect to  $\{X_{i,t}, 1 \leq i \leq N, 1 \leq t \leq T\}$ ; “ $\xrightarrow{D^*}$ ” and “ $\xrightarrow{P^*}$ ” denote, respectively, conditional convergence in distribution

and in probability according to  $P^*$ .

**Theorem 1.** *Under Assumptions 1-5, as  $\min(N, m, R) \rightarrow \infty$ , it holds that*

$$\Theta_t \xrightarrow{\mathcal{D}^*} \chi_1^2, \quad \begin{array}{ll} \text{under } H_{A,1}, & \text{for } t_{N,m}^* \leq t < 2\tau + m - 2 - t_{N,m}^*, \\ \text{under } H_{A,2}, & \text{for } t_{N,m}^* \leq t \leq T, \end{array} \quad (14)$$

for almost all realisations of  $\{X_{i,t}, 1 \leq i \leq N, 1 \leq t \leq T\}$ .

Under Assumptions 1-4, as  $\min(N, m, R) \rightarrow \infty$ , it holds that

$$\frac{1}{R} \Theta_t \xrightarrow{P^*} \frac{\int_{-\infty}^{+\infty} |I_{[0,\infty)}(u) - G_\phi(0)|^2 dF_\phi(u)}{G_\phi(0) [1 - G_\phi(0)]}, \quad \begin{array}{ll} \text{under } H_0, & \text{for } m \leq t \leq T, \\ \text{under } H_{A,1}, & \text{for } m \leq t < \tau, \\ \text{and } \tau + m \leq t \leq T, \\ \text{under } H_{A,2}, & \text{for } m \leq t < \tau, \end{array} \quad (15)$$

for almost all realisations of  $\{X_{i,t}, 1 \leq i \leq N, 1 \leq t \leq T\}$ .

Theorem 1 is an intermediate result: in order to be able to construct a test for the “classical” null of no changes in the factor structure, it is necessary to have a statistic which diverges under the null and is bounded under the alternative. Several options are possible, the most obvious one being using  $1/\Theta_t$ . However, we have tried this approach and results were not satisfactory. Based on those preliminary, unreported simulations, we propose to randomise  $\Theta_t$ , with a second randomisation based on

$$\psi_{N,m,R}(t) = h\left(\frac{\Theta_t}{\tilde{l}(N, m, R)}\right), \quad m \leq t \leq T, \quad (16)$$

where

$$\tilde{l}(N, m, R) = (\ln N)^{2+\epsilon} (\ln m)^{2+\epsilon} (\ln R)^{2+\epsilon},$$

for some  $\epsilon > 0$  - in practice, any small value of  $\epsilon$  works well.

In (16), the function  $h(\cdot)$ , similarly to  $g(\cdot)$  in (13), is a monotonically increasing function such that  $h(0) = 0$  and  $\lim_{x \rightarrow \infty} h(x) = \infty$ ; again, we use  $h(a) = a$ .

Similarly to the case of  $\phi_{N,m}(t)$ , Theorem 1 entails that

$$\lim_{N,m,R \rightarrow \infty} \psi_{N,m,R}(t) = \infty, \quad m \leq t \leq T, \quad \text{under } H_0,$$

and

$$\lim_{N,m,R \rightarrow \infty} \psi_{N,m,R}(t) = \begin{cases} \infty & m \leq t < \tau, \\ 0 & t_{N,m}^* \leq t < 2\tau + m - 2 - t_{N,m}^*, \\ \infty & \tau + m \leq t \leq T, \end{cases} \quad \text{under } H_{A,1},$$

while

$$\lim_{N,m,R \rightarrow \infty} \psi_{N,m,R}(t) = \begin{cases} \infty & m \leq t < \tau, \\ 0 & t_{N,m}^* \leq t < \tau + m, \end{cases} \quad \text{under } H_{A,2};$$

Consider now the second randomisation.

*Step B1.* At each given  $t \geq m$ , generate an *i.i.d.* sample  $\{\tilde{\xi}_j(t)\}_{j=1}^W$  with common distribution  $G_\psi$  such that  $G_\psi(0) \neq 0$  or 1.

*Step B2.* For any  $u$  drawn from a distribution  $F_\psi(u)$ , define

$$\tilde{\zeta}_j(u; t) = I \left[ \tilde{\xi}_j(t) \leq u \psi_{N,m,R}^{-1}(t) \right].$$

*Step B3.* Compute

$$\gamma(u; t) = \frac{1}{\sqrt{W}} \sum_{j=1}^W \frac{\tilde{\zeta}_j(u; t) - G_\psi(0)}{\sqrt{G_\psi(0)[1 - G_\psi(0)]}}.$$

*Step B4.* Compute

$$\Gamma_t = \int_{-\infty}^{+\infty} |\gamma(u; t)|^2 dF_\psi(u).$$

The following assumptions are needed in order to study the asymptotic behavior of  $\Gamma_t$ ; note their similarity with Assumptions 4 and 5.

**Assumption 6.** *It holds that: (i)  $G_\psi(\cdot)$  has a bounded density; (ii)  $\int_{-\infty}^{+\infty} u^4 dF_\psi(u) < \infty$ .*

**Assumption 7.** *It holds that, as  $\min(N, m, R, W) \rightarrow \infty$*

$$W^{1/2} \left[ h \left( \frac{R}{\tilde{l}(N, m, R)} \right) \right]^{-1} \rightarrow 0.$$

As above, in Assumption 6 we can choose  $G_\psi$  to be the standard normal distribution, and  $F_\psi$  to be a discrete uniform distribution. The restrictions in Assumption 7 provide a selection rule for  $W$ ; for most choices of the function  $h(\cdot)$ , the choice  $W = R$  is appropriate.

Let  $P^\dagger$  represent the conditional probability with respect to  $\{X_{i,t}, 1 \leq i \leq N, 1 \leq t \leq T\}$  and  $\{\xi_j(t), 1 \leq j \leq R, m \leq t \leq T\}$ ; we use the notation “ $\xrightarrow{D^\dagger}$ ” and “ $\xrightarrow{P^\dagger}$ ” to define, respectively, conditional convergence in distribution and in probability according to  $P^\dagger$ .

**Theorem 2.** *Under Assumptions 1-7, as  $\min(N, m, R, W) \rightarrow \infty$ , it holds that*

$$\begin{aligned} & \text{under } H_0, \quad \text{for } m \leq t \leq T, \\ \Gamma_t & \xrightarrow{D^\dagger} \chi_1^2, \quad \text{under } H_{A,1}, \quad \text{for } m \leq t < \tau \text{ and } \tau + m \leq t \leq T, \\ & \text{under } H_{A,2}, \quad \text{for } m \leq t < \tau, \end{aligned} \quad (17)$$

for almost all realisations of  $\{X_{i,t}, 1 \leq i \leq N, 1 \leq t \leq T\}$  and  $\{\xi_j(t), 1 \leq j \leq R, m \leq t \leq T\}$ .

Under Assumptions 1-5, as  $\min(N, m, R, W) \rightarrow \infty$ , it holds that

$$\frac{1}{W} \Gamma_t \xrightarrow{P^\dagger} \frac{\int_{-\infty}^{+\infty} |I_{[0,\infty)}(u) - G_\psi(0)|^2 dF_\psi(u)}{G_\psi(0) [1 - G_\psi(0)]}, \quad \begin{aligned} & \text{under } H_{A,1}, \quad \text{for } t_{N,m}^* \leq t < 2\tau + m - 2 - t_{N,m}^*, \\ & \text{under } H_{A,2}, \quad \text{for } t_{N,m}^* \leq t \leq T, \end{aligned} \quad (18)$$

for almost all realisations of  $\{X_{i,t}, 1 \leq i \leq N, 1 \leq t \leq T\}$  and  $\{\xi_j(t), 1 \leq j \leq R, m \leq t \leq T\}$ .

Theorem 2 is, again, an intermediate result. It states that  $\Gamma_t$  has (asymptotically) a chi-square distribution under the null of no breaks; further, by construction the sequence  $\{\Gamma_t\}_{t=m}^T$  is independent across  $t$  conditional on the sample. We now discuss how these two basic facts can be employed in order to propose a monitoring scheme for the on-line detection of breaks in the factor structure.

### 3.2 Sequential monitoring of factor models

We base our sequential monitoring procedure on the theory developed in Horváth et al. (2004). Recall that, after collecting  $m$  observations, we monitor our model over the period  $m + 1 \leq t \leq T$ , which has size denoted as  $T_m = T - m$ . We then consider a monitoring procedure based on the detector

$$d(k; m) = \left| \sum_{t=m+1}^{m+k} \frac{\Gamma_t - 1}{\sqrt{2}} \right|, \quad 1 \leq k \leq T_m, \quad (19)$$



which covers the entire monitoring period. In other words our detector is made of the cumulative sum of the centered and standardized version of the sequence  $\{\Gamma_t\}_{t=m}^T$ , obtained by double randomisation. Other detectors, differing from (19) only with respect to the start of the monitoring period, could be also suggested, and we refer to Kirch and Weber (2017) for a discussion.

Given the stopping rule

$$\widehat{k}_m = \begin{cases} \inf \{1 \leq k \leq T_m, \text{ such that } d(k; m) \geq \nu(k; m)\}, \\ T_m \text{ if the above does not hold in } 1 \leq k \leq T_m, \end{cases} \quad (20)$$

we define the estimated change-point location as  $\widehat{\tau}_m = \widehat{k}_m + m$ . The threshold function in (20) is defined as (see Horváth et al., 2004 and Horváth et al., 2007)

$$\nu(k; m) = c_{\alpha, m} \nu^*(k; m), \quad (21)$$

$$\nu^*(k; m) = m^{1/2} \left(1 + \frac{k}{m}\right) \left(\frac{k}{k+m}\right)^\eta, \quad \eta \in \left[0, \frac{1}{2}\right], \quad (22)$$

where  $c_{\alpha, m}$  is a critical value corresponding to a pre-specified level  $\alpha$ . Depending on the choice of  $\eta$ , the critical value is defined as

$$P\left(\sup_{0 \leq t \leq 1} \frac{|B(t)|}{t^\eta} \leq c_{\alpha, m}\right) = 1 - \alpha, \quad \text{for } \eta \in \left[0, \frac{1}{2}\right], \quad (23)$$

where  $\{B(t), 0 \leq t \leq 1\}$  denotes a standard Wiener process, or

$$c_{\alpha, m} = \frac{D_m - \ln[-\ln(1 - \alpha)]}{A_m}, \quad \text{for } \eta = \frac{1}{2}, \quad (24)$$

with  $A_m = (2 \ln \ln m)^{1/2}$  and  $D_m = 2 \ln \ln m + \frac{1}{2} \ln \ln \ln m - \frac{1}{2} \ln \pi$ . Note that in (23)  $c_{\alpha, m}$  does not depend on  $m$ , whilst it does in (24).

Note that Chu et al. (1996), albeit in a different context, choose  $\eta = 0$ . It is well known that tests based on  $\eta = 0$  have the smallest power, which on the contrary increases as  $\eta$  increases (see the discussion in Horváth et al., 2004).

In order to derive our main theorem, we also need the following assumptions.

**Assumption 8.** *It holds that (i)  $T_m = O(m^\varkappa)$  for some  $\varkappa \geq 1$ ; (ii)  $\liminf_{m \rightarrow \infty} \frac{T_m}{m} > 0$ ; (iii)  $T_m > \tau + C_0 m^{1/2+\epsilon}$  for  $\epsilon > 0$  such that  $\frac{N^{1-\delta}}{m^{1/2-\epsilon}} \rightarrow C_1$ .*

**Assumption 9.** It holds that (i)  $\int_{-\infty}^{+\infty} |u|^{4+2\delta} dF_\psi(u) < \infty$ ; (ii)

$$m^{1/2+\epsilon} \left\{ W^{-1} + W \left[ h \left( \frac{R}{\tilde{l}(N, m, R)} \right) \right]^{-2} + \left[ h \left( \frac{R}{\tilde{l}(N, m, R)} \right) \right]^{-1} \right\} \rightarrow 0,$$

for some  $\epsilon > 0$ .

Assumption 8 is the same as equation (1.12) in Horváth et al. (2007), and it essentially requires that the monitoring goes on for a sufficiently long time, longer than the initial training period  $m$ . Assumption 9 strengthens Assumption 6(ii), and it is needed to prove a moment condition for the sequence  $\{\Gamma_t\}_{t=m}^T$  which will enable a Central Limit Theory to hold. Our main result follows.

**Theorem 3.** Let Assumptions 1-9 hold. Under  $H_0$  it holds that, as  $\min(N, m, R, W) \rightarrow \infty$

$$P^\dagger \left( \max_{1 \leq k \leq T_m} \frac{d(k; m)}{\nu^*(k; m)} \leq x \right) \rightarrow P \left( \sup_{0 \leq t \leq 1} \frac{|B(t)|}{t^\eta} \leq x \right), \quad \text{for } \eta \in \left[ 0, \frac{1}{2} \right), \quad (25)$$

$$P^\dagger \left( \max_{1 \leq k \leq T_m} \frac{d(k; m)}{\nu^*(k; m)} \leq \frac{x + D_m}{A_m} \right) \rightarrow e^{-e^{-x}}, \quad \text{for } \eta = \frac{1}{2}, \quad (26)$$

for almost all realisations of  $\{X_{i,t}, 1 \leq i \leq N, 1 \leq t \leq T\}$  and  $\{\xi_j(t), 1 \leq j \leq R, m \leq t \leq T\}$  and for  $x \in \mathbb{R}$ .

Under  $H_{A,1}$  and  $H_{A,2}$ , as  $\min(N, m, R, W) \rightarrow \infty$ , and for a given significance level  $\alpha$ , it holds that

$$c_{\alpha, m}^{-1} \max_{1 \leq k \leq T_m} \frac{d(k; m)}{\nu^*(k; m)} \xrightarrow{P^\dagger} \infty, \quad \text{for all } \eta \in \left[ 0, \frac{1}{2} \right], \quad (27)$$

for almost all realisations of  $\{X_{i,t}, 1 \leq i \leq N, 1 \leq t \leq T\}$  and  $\{\xi_j(t), 1 \leq j \leq R, m \leq t \leq T\}$  and for  $x \in \mathbb{R}$  and where  $c_{\alpha, m}$  is defined in (23) when  $\eta < \frac{1}{2}$  and in (24) when  $\eta = \frac{1}{2}$ .

The main implication of Theorem 3 is summarized in the following result:

**Corollary 1.** Under the assumptions of Theorem 3 it holds that:

$$\lim_{\min(N, m, R, W) \rightarrow \infty} P^\dagger(\hat{\tau}_m < T) \leq \alpha, \quad \text{under } H_0, \quad (28)$$

$$\lim_{\min(N, m, R, W) \rightarrow \infty} P^\dagger(\hat{\tau}_m < T) = 1, \quad \text{under } H_{A,1} \text{ and } H_{A,2}, \quad (29)$$

for almost all realisations of  $\{X_{i,t}, 1 \leq i \leq N, 1 \leq t \leq T\}$  and  $\{\xi_j(t), 1 \leq j \leq R, m \leq t \leq T\}$ .

The notion of size implied by (28), in this context, is very different from the one usually considered in the literature. The purpose of the procedure is to keep the false rejection probability as little as possible, and therefore (at a minimum) below the threshold  $\alpha$ , rather than making it close to  $\alpha$ . This makes the monitoring procedure different from the standard Neyman-Pearson paradigm (and, in general, from a multiple testing exercise): given that the monitoring horizon keeps expanding, the purpose of  $c_{\alpha,m}$  is to ensure that the chance of a false break detection is as little as possible - see also similar comments in Horváth et al. (2007).

## 4 Applying the test under general circumstances

The purpose of this section is to discuss how the test could be applied under slightly different assumptions than the ones above, and up to which extent such assumptions can be relaxed. More substantive extensions, which involve modifications of the test, are briefly discussed in the concluding remarks in Section 7.

### 4.1 Detection of breaks

A consequence of our approach is that monitoring for a structural change (despite being in a high-dimensional set-up) can be treated as in a classical time series framework. In particular, in addition to the consistency of the procedure, a natural question is how much would the delay be in detecting a break. In order to formally address this issue, one can directly use the results by Aue and Horváth (2004); hereafter, we provide a heuristic discussion of the magnitude of the delay within our setup.

Consider the notation  $a_n = \Omega(b_n)$  to indicate that the magnitude of the sequence  $a_n$  is not smaller than that of  $b_n$ , viz.  $a_n > Cb_n > 0$ . Then, by construction,  $\{\Gamma_t\}_{t=m}^T$  has, under the alternative, a “large” shift in the mean after  $t_{N,m}^*$ , where  $t_{N,m}^*$  is such that

$$t_{N,m}^* - \tau = \Omega\left(\frac{m}{N^{1-\delta}}\right). \quad (30)$$

Defining  $\beta$  such that  $N = m^\beta$ , and using (12), it is possible to analyse (30) for various relative rates of divergence of  $m$  and  $N$  as they pass to infinity. When  $\beta > \frac{1}{2}$ , we have that

$\delta = 1 - \frac{1}{2\beta} + \epsilon$  for an arbitrarily small value of  $\epsilon$ . Thus, by (30)

$$t_{N,m}^* = \tau + \Omega\left(\frac{m}{m^{\beta(1-\delta)}}\right) = \tau + \Omega\left(m^{1/2+\epsilon'}\right),$$

where  $\epsilon' > 0$  is arbitrarily small. Thus, when  $N$  is not much smaller than  $m$ , or even larger, the change-point is detected with a delay,  $t_{N,m}^* - \tau$ , which is of order at least  $m^{1/2}$ . By the same token, whenever  $\beta \leq \frac{1}{2}$ , we have that  $\delta = \epsilon$  for an arbitrarily small value of  $\epsilon$ , so that

$$t_{N,m}^* = \tau + \Omega\left(m^{1-\beta(1-\delta)}\right),$$

and, by elementary arguments, it follows that  $m^{1-\beta(1-\delta)} = \Omega(m^{1/2+\epsilon'})$ : the delay, in this case, will be bigger. This is in line with the intuition that a break will cause  $\Gamma_t$  - and consequently the detector - to diverge as fast as  $N$ : the lower  $N$ , the lower the divergence rate, and the less effective the detection of breaks. Finally, it is interesting to consider the ultra-dimensional case,  $N = \exp(m)$ . By (12), it holds that  $\delta = 1 - (1 - \epsilon)\frac{\ln m}{2m}$  for an arbitrarily small value of  $\epsilon$ . Hence, (30) yields

$$t_{N,m}^* = \tau + \Omega\left(\frac{m}{\exp((1-\delta)m)}\right) = \tau + \Omega\left(m^{1/2+\epsilon'}\right),$$

again. In essence, in all cases considered there is a delay in the detection of breaks which is greater than  $C_0 m^{1/2}$ , but smaller than  $C_0 m$  - that is, rescaling the delay by the sample size, this vanishes.

## 4.2 Weak factors and local alternatives

The theory developed in this paper - starting from Assumption 2 - implicitly require that, when a new factor appears as a consequence of a break, this should be a pervasive factor. Indeed, part (i) of the assumption entails that spiked eigenvalues must diverge at a rate  $N$ , i.e. a “strong” factor model. However, the literature has also considered cases in which one or more common factor may be less pervasive, thus leading to a covariance matrix which has some eigenvalues passing to infinity at a rate  $N^\kappa$ , for  $\kappa \in (0, 1)$ . A possible example of weak factors arises when considering jointly macroeconomic data of different countries: global factors are strong since they are likely to affect all countries; however national factors, although strong within a given country, will affect only a subset of all variables considered

and can be seen as weak - see e.g. the empirical study in Moench, Ng and Potter (2013). Estimation of factor models in the presence of such “weak” or “local” factors have been paid considerable attention by the literature - see De Mol, Giannone and Reichlin (2008), Onatski (2012), in the same setting as ours and, in a slightly different context, Lam and Yao (2012). The notion of weak factors is intertwined with that of a local alternative hypothesis where the break does happen but it is “small”, for example when a break is caused by a change of only some, but not all, loadings. Consistently with the literature on weak factors, we characterise such situations by the fact that, the  $(r + 1)$ -th eigenvalue, behaves as

$$\lambda^{(r+1)}(t) = C_0 N^\kappa, \quad \begin{array}{ll} \text{under } H_{A,1}, & \text{for } \tau \leq t < \tau + m, \\ \text{under } H_{A,2}, & \text{for } \tau \leq t \leq T, \end{array} \quad (31)$$

for  $\kappa \in (0, 1)$ , while it is bounded for all other values of  $t$ .

We now discuss heuristically under which conditions such small breaks can be detected; we consider for simplicity the case  $\eta < \frac{1}{2}$ . We know that, based on Theorem 2, a break in the  $(r + 1)$ -th largest eigenvalue enters the sequence  $\{\Gamma_t\}_{t=m}^T$  as a shift in its mean: this is essentially the way in which the monitoring procedure picks up the presence of a break. In particular, from (18), for any  $t \geq t_{N,m}^*$  for which either  $H_{A,1}$  or  $H_{A,2}$  holds, we have

$$\Gamma_t \approx W \int_{-\infty}^{+\infty} \left| G_\psi \left( u \psi_{N,m,R}^{-1}(t) \right) - G_\psi(0) \right|^2 dF_\psi(u) \approx C_0 W \psi_{N,m,R}^{-2}(t), \quad (32)$$

having used a Mean Value argument. Moreover, by analysing the proof of Theorem 1, it follows that

$$\Theta_t \approx R \int_{-\infty}^{+\infty} \left| G_\phi \left( u \phi_{N,m}^{-1}(t) \right) - G_\phi(0) \right|^2 dF_\phi(u) \approx C_0 R \phi_{N,m}^{-2}(t), \quad (33)$$

for the same values of  $t \geq \tau$  for which (32) holds.

Consider the case (also considered in the simulations and in the empirical application) where  $W = R = N$ . Then, letting  $N = m^\beta$  and  $h(\cdot)$  in (16) be the identity function, we have  $\ln N = \ln R = \beta \ln m$ . Moreover, assuming that  $g(\cdot)$  in (13) is also the identity

function, by (31) we have  $\phi_{N,m}(t) \approx N^{\kappa-\delta}$ . Therefore, because of (32) and (33)

$$\Gamma_t \approx \Delta_N = \begin{cases} \frac{(\ln N)^{12+\epsilon}}{N} N^{4(\kappa-\delta)}, & \text{under } H_{A,1}, \quad \text{for } t_{N,m}^* \leq t < 2\tau + m - 2 - t_{N,m}^*, \\ \text{under } H_{A,2}, & \text{for } t_{N,m}^* \leq t \leq T, \end{cases} \quad (34)$$

where we denoted by  $\Delta_N$  the magnitude of the shift in the mean of  $\{\Gamma_t\}_{t=m}^T$ .

Based on (34) and condition (1.9) in Aue and Horváth (2004), we therefore have a detectable shrinking break when both  $\Delta_N \rightarrow 0$  and  $m^{1/2}\Delta_N \rightarrow \infty$  hold, which imply the constraints on  $\kappa$ :

$$\frac{1}{4}\left(1 - \frac{1}{2\beta}\right) + \delta < \kappa < \frac{1}{4} + \delta.$$

### 4.3 Heteroskedasticity in the idiosyncratic component

The main assumptions in the paper are spelt out with respect to  $X_{i,t}$ , avoiding to make any comments on the properties of  $u_{i,t}$  across time. We now discuss the behaviour of the test in the presence of heteroskedasticity, which is not explicitly considered (although not ruled out) by Assumption 2. For the sake of simplicity, we consider the case of an abrupt change in the covariance matrix of  $\{u_{i,t}\}_{i=1}^N$ , although more general forms of heteroskedasticity could also be considered.

To illustrate this, we consider a simple example where the covariance matrix  $E(u_t u_t')$  undergoes an abrupt change of size  $\Delta_u$  after a point in time, say  $\tau^*$ :

$$E(u_t u_t') = \begin{cases} \Sigma_u & \text{for } m \leq t < \tau^*, \\ \Sigma_u + \Delta_u & \tau^* \leq t \leq T, \end{cases}$$

where  $\Delta_u$  affects some or even all covariances. The only condition we require in order for our test to be applicable is  $\omega^{(1)}(m^{-1} \sum_{k=t-m+1}^t E(u_k u_k')) \leq C_0$  for each  $t \geq m$ , where the notation  $\omega^{(1)}(A)$  is understood to represent the largest eigenvalue of a matrix  $A$ . This holds, when  $t < \tau^*$ , as long as  $\omega^{(1)}(\Sigma_u) \leq C_0$ . When  $t \geq \tau^*$ , using Weyl's inequality it follows that

$$\omega^{(1)}\left(\frac{1}{m} \sum_{k=t-m+1}^t E(u_k u_k')\right) \leq \omega^{(1)}(\Sigma_u) + \omega^{(1)}(\Delta_u), \quad (35)$$

which is bounded as long as  $\omega^{(1)}(\Sigma_u) \leq C_0$  and  $\omega^{(1)}(\Delta_u) \leq C_1$ . In essence, as long as the perturbation matrix  $\Delta_u$  is not too big, and therefore as long as the changes in the covariance structure of the idiosyncratic are not too big, our test can still be applied.

Condition (35) has interesting implications. Consider a break such that  $\Delta_u = \text{diag}\{d_i\}$ , with  $0 \leq d_i \leq C_0$  for all  $1 \leq i \leq N$ . In such a case, where the variances of the error terms all undergo a change (potentially), but the covariance structure does not change, it would hold that  $\omega^{(1)}(\Delta_u) \leq C_0$ : even a huge (but of finite size) break in the variance of the idiosyncratic components does not alter the structure of the eigenvalues of  $E(X_t X_t')$ , by introducing a spurious spiked eigenvalue. Thus, an interesting question about the robustness of our procedure is: when is a break in the idiosyncratic component strong enough to be confused with a break in the factor structure? By the same (heuristic) token as above, the eigenvalue structure of  $E(X_t X_t')$  would change if, for argument's sake,  $\omega^{(1)}(m^{-1} \sum_{k=t-m+1}^t E(u_t u_t')) = C_0 N^\varepsilon$  with  $\varepsilon \in (0, 1]$ . By Weyl's inequality assuming for simplicity that there is no break in the factor component

$$\omega^{(1)}\left(\frac{1}{m} \sum_{k=t-m+1}^t E(u_t u_t')\right) \geq \omega^{(N)}(\Sigma_u) + \omega^{(1)}(\Delta_u) \geq \omega^{(1)}(\Delta_u). \quad (36)$$

Therefore, a sufficient condition would be  $\omega^{(1)}(\Delta_u) = C_0 N^\varepsilon$ . Moreover, given that  $\omega^{(1)}(\Delta_u) \geq N^{-1} \sum_{i=1}^N \sum_{j=1}^N \{\Delta_u\}_{i,j}$ , then (36) suggests that a break which is ‘‘sufficiently pervasive’’, so that it affects not merely the variances of the idiosyncratic components, but also their covariances (without needing to be necessarily huge), could introduce a spiked eigenvalue in  $E(X_t X_t')$ . In such cases our procedure might detect  $\tau^*$  as a change-point even if the signal component does not change.

#### 4.4 Extensions to consider further alternative hypotheses

So far, we have focused our attention onto two empirically relevant but very specific forms of alternative hypotheses: a possible change in the loadings -  $H_{A,1}$  - and a possible increase in the number of factors -  $H_{A,2}$ . However, our methodology is sufficiently general to be adapted (with minor modifications) to other cases also. A leading example is the case in which  $q \geq 1$  factors vanish, viz.

$$H_{A,3} : \begin{cases} X_{i,t} = \sum_{j=1}^r a_{ij} f_{jt} + u_{i,t} & \text{for } 1 \leq t < \tau \\ X_{i,t} = \sum_{j=1}^{r-q} \tilde{a}_{ij} f_{jt} + u_{i,t} & \tau \leq t \leq T \end{cases}. \quad (37)$$

Note that, in (37), we can entertain the possibility that the loadings of the non-vanishing factors may also be subject to changes, although this is not required. For simplicity consider

the case  $q = 1$ , then under (37), it can be noted that the  $r$ -th eigenvalue of the covariance matrix of  $X_{i,t}$  is spiked before  $\tau$ , and bounded thereafter. This suggests that testing for (37) can be based on  $\widehat{\lambda}^{(r)}(t)$ . Since under the null (in essence, on account of Lemma 2)  $N^{-\delta}\widehat{\lambda}^{(r)}(t) \rightarrow \infty$ , whereas under the alternative  $N^{-\delta}\widehat{\lambda}^{(r)}(t) \rightarrow 0$ , one round of randomisation is enough to have a sequence of test statistics which behaves like  $\{\Gamma_t\}_{t=m}^T$  under the null - that is, which (conditional on the sample) is *i.i.d.*, has moments that exist up to any order, and has an asymptotic chi-square distribution, with mean and variance that can be approximated with a polynomially vanishing error. Hence, monitoring can be again carried out as proposed in Section 3.

## 5 Monte Carlo simulations

Under  $H_0$  we simulate data according to the stable factor model (4):

$$X_{i,t} = a_i' f_t + u_{i,t}, \quad 1 \leq i \leq N, \quad 1 \leq t \leq T.$$

In particular, we fix  $N = 100$ , and we consider  $r \in \{1, 2, 3, 4\}$  factors. As far as the time dimension is concerned, we consider burn-in periods and thus sample sizes of dimension  $m \in \{50, 75, 100, 125, 150, 175, 200, 225, 250\}$ . We monitor our model for 1000 periods (that is, we set  $T = 1000$ ). We simulate each element of the loadings vector  $a_i$  as *i.i.d.*  $\mathcal{N}(0, 1)$ ; we assume some time dependence in the common factors through a causal VAR(1) process

$$f_t = H f_{t-1} + e_t, \quad 1 \leq t \leq T,$$

where  $e_t \sim i.i.d.\mathcal{N}(0, I_r)$  and the matrix  $H$  has maximum absolute value of the eigenvalues equal to 0.7. The  $N \times T$  matrix of idiosyncratic components  $u$  is generated as  $u = D\varepsilon G$ , where the  $NT \times 1$  vector of stacked columns of  $\varepsilon$  is *i.i.d.*  $\mathcal{N}(0, I_{NT})$  and  $D$  and  $G$  are two  $N \times N$  and  $T \times T$  Toeplitz matrices with entries, in the  $k$ -th diagonal place, given by  $0.3^{k-1}$  and  $0.5^{k-1}$  respectively. Finally, we have set the signal-to-noise ratio to  $\frac{\text{Var}(X_{i,t})}{\text{Var}(u_{i,t})} = 2$  for all  $1 \leq i \leq N$ .

Under the alternative, we consider breaks to occur at the change-point  $\tau = 500$  under



the two schemes:

$$X_{i,t} = a'_i f_t I[t < \tau] + \tilde{a}'_i f_t I[t \geq \tau] + u_t, \quad 1 \leq i \leq N, 1 \leq t \leq T, \quad (38)$$

$$X_{i,t} = a'_i f_t + b g_t I[t \geq \tau] + u_t, \quad 1 \leq i \leq N, 1 \leq t \leq T. \quad (39)$$

In (38), we consider the case in which all loadings undergo a change, i.e.  $H_{A,1}$ ; all the elements of  $a_i$  and  $\tilde{a}_i$  are generated as *i.i.d.* $\mathcal{N}(0, 1)$ . Scheme (39) refers to a break owing to a new common factor,  $g_t$ , appearing, i.e.  $H_{A,2}$ ; the elements in  $b$  are generated as *i.i.d.* $\mathcal{N}(0, 1)$ , and we simulate  $g_t$  as the causal AR(1)

$$g_t = \varphi g_{t-1} + v_t, \quad 1 \leq t \leq T,$$

with  $v_t \sim i.i.d.\mathcal{N}(0, 1)$  and  $\varphi = 0.7$ .

All results of the test are computed when setting  $\eta = 0.45$  and  $\eta = 0.5$ . The critical values used in the case  $\eta = 0.45$  are taken from Horváth et al. (2004); in particular, when the significance level is  $\alpha = 0.05$  the critical value is  $c_{0.05} = 2.7992$  and when  $\alpha = 0.1$  we have  $c_{0.1} = 2.5437$ . Regarding the double randomisation, we choose the functions  $g(\cdot)$  in (13) and  $h(\cdot)$  in (16) to be the identity, we set  $W = R = N$ , the distributions  $G_\phi$  and  $G_\psi$  in steps A1 and B1 are chosen to be standard normals, while  $F_\phi$  and  $F_\psi$  in steps A2 and B2 are chosen to have non-zero and equal mass at  $u = \pm\sqrt{2}$ .

In order to evaluate the performance of our procedure, we repeat simulations 500 times, and we consider a series of indicators.

- (1) In Table 1 we report the fraction of false rejections over the whole monitoring period ( $m + 1 \leq t \leq T$ ), when no break is present, i.e. under  $H_0$ , and when testing at 5% and 10% significance levels. As expected the empirical size is always below the significance level.
- (2) In Tables 2, 3, 4 and 5 we show the fraction of detections for which  $\tau \leq \hat{\tau}_m < \tau + m$ , when a break takes place under  $H_{A,1}$  or  $H_{A,2}$  and when testing at 5% and 10% significance levels, setting either  $\eta = 0.45$  or  $\eta = 0.5$ . Results show that the test does have power versus the two alternative hypotheses considered in this paper. As the construction of the test and the theory would suggest, the power declines as  $r$ , the original, pre-break number of factors, increases: in essence, the test checks whether

Table 1: Empirical size - 5% and 10% significance  
 Fraction of detections in  $[m + 1, T]$

		$m = 50$				$m = 75$				$m = 100$			
$r$	$\eta = 0.45$		$\eta = 0.5$		$\eta = 0.45$		$\eta = 0.5$		$\eta = 0.45$		$\eta = 0.5$		
	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	
1	0.03	0.05	0.03	0.06	0.04	0.05	0.03	0.05	0.04	0.06	0.03	0.06	
2	0.04	0.05	0.04	0.06	0.03	0.04	0.02	0.05	0.04	0.06	0.04	0.06	
3	0.03	0.05	0.03	0.05	0.03	0.05	0.03	0.06	0.04	0.06	0.04	0.06	
4	0.03	0.05	0.03	0.06	0.02	0.05	0.02	0.06	0.04	0.05	0.03	0.06	
		$m = 125$				$m = 150$				$m = 175$			
$r$	$\eta = 0.45$		$\eta = 0.5$		$\eta = 0.45$		$\eta = 0.5$		$\eta = 0.45$		$\eta = 0.5$		
	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	
1	0.05	0.06	0.05	0.06	0.04	0.05	0.03	0.05	0.04	0.07	0.04	0.06	
2	0.03	0.05	0.03	0.06	0.04	0.05	0.03	0.05	0.05	0.06	0.04	0.07	
3	0.03	0.05	0.03	0.05	0.03	0.05	0.03	0.05	0.04	0.08	0.04	0.08	
4	0.03	0.06	0.03	0.07	0.04	0.07	0.05	0.08	0.04	0.06	0.05	0.06	
		$m = 200$				$m = 225$				$m = 250$			
$r$	$\eta = 0.45$		$\eta = 0.5$		$\eta = 0.45$		$\eta = 0.5$		$\eta = 0.45$		$\eta = 0.5$		
	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	
1	0.05	0.07	0.05	0.07	0.05	0.07	0.05	0.07	0.04	0.07	0.04	0.08	
2	0.04	0.05	0.03	0.05	0.04	0.06	0.04	0.07	0.04	0.07	0.04	0.07	
3	0.05	0.07	0.04	0.08	0.03	0.04	0.03	0.04	0.03	0.05	0.04	0.05	
4	0.04	0.06	0.04	0.06	0.04	0.06	0.04	0.07	0.04	0.07	0.04	0.08	

Table 2: Power - loadings change - 5% significance  
 Fraction of detections in  $[\tau, \tau + m]$

$\eta = 0.45$		$m$								
$r$	50	75	100	125	150	175	200	225	250	
1	0.96	0.95	0.96	0.96	0.96	0.96	0.96	0.95	0.95	
2	0.58	0.97	0.97	0.96	0.96	0.96	0.97	0.95	0.98	
3	0.01	0.74	0.97	0.97	0.96	0.97	0.96	0.97	0.96	
4	0.00	0.03	0.80	0.94	0.96	0.94	0.96	0.96	0.96	
$\eta = 0.5$		$m$								
$r$	50	75	100	125	150	175	200	225	250	
1	0.96	0.95	0.96	0.96	0.97	0.95	0.96	0.96	0.96	
2	0.44	0.97	0.97	0.96	0.96	0.96	0.97	0.95	0.98	
3	0.00	0.62	0.97	0.97	0.96	0.96	0.97	0.96	0.97	
4	0.00	0.01	0.65	0.95	0.97	0.95	0.96	0.96	0.97	

an eigenvalue is large, and the magnitude of the  $r + 1$ -th largest eigenvalue declines with  $r$ . Still, even when  $r = 4$ , the test has high power when  $m \geq 100$  in all cases considered, and, in presence of a new factor appearing (see Tables 4 and 5), even when  $m \geq 50$ . An interesting feature of the test is the case  $\eta = 0.5$ : although

Table 3: Power - loadings change - 10% significance  
 Fraction of detections in  $[\tau, \tau + m)$

$\eta = 0.45$		$m$								
$r$	50	75	100	125	150	175	200	225	250	
1	0.94	0.93	0.95	0.93	0.94	0.93	0.94	0.94	0.93	
2	0.66	0.95	0.95	0.93	0.95	0.93	0.94	0.94	0.95	
3	0.01	0.81	0.95	0.95	0.94	0.95	0.95	0.94	0.94	
4	0.00	0.06	0.87	0.94	0.95	0.92	0.93	0.94	0.94	

$\eta = 0.5$		$m$								
$r$	50	75	100	125	150	175	200	225	250	
1	0.94	0.93	0.95	0.92	0.93	0.93	0.93	0.93	0.93	
2	0.59	0.94	0.95	0.92	0.94	0.93	0.94	0.93	0.95	
3	0.01	0.73	0.94	0.95	0.94	0.95	0.94	0.93	0.94	
4	0.00	0.03	0.80	0.93	0.94	0.91	0.93	0.94	0.93	

Table 4: Power - new factor appears - 5% significance  
 Fraction of detections in  $[\tau, \tau + m)$

$\eta = 0.45$		$m$								
$r$	50	75	100	125	150	175	200	225	250	
1	0.93	0.93	0.92	0.95	0.92	0.95	0.93	0.93	0.92	
2	0.78	0.96	0.97	0.95	0.95	0.96	0.96	0.95	0.96	
3	0.10	0.89	0.97	0.98	0.96	0.97	0.95	0.95	0.97	
4	0.00	0.27	0.91	0.96	0.96	0.96	0.95	0.95	0.96	

$\eta = 0.5$		$m$								
$r$	50	75	100	125	150	175	200	225	250	
1	0.95	0.94	0.93	0.96	0.95	0.96	0.95	0.93	0.94	
2	0.71	0.96	0.97	0.96	0.95	0.96	0.96	0.96	0.97	
3	0.06	0.85	0.97	0.98	0.96	0.98	0.96	0.96	0.98	
4	0.00	0.16	0.89	0.96	0.96	0.95	0.95	0.96	0.97	

in theory this choice yields the highest power, it is well known that convergence to the extreme value distribution is very slow, leading to larger than correct critical values, and, consequently, to lower power (see the comments in Csörgő and Horváth, 1997). However, considering the discrepancy between the power when  $\eta = 0.45$  and  $\eta = 0.5$ , this is not always the case: tests based on the choice  $\eta = 0.5$  have roughly the same power as for the case  $\eta = 0.45$  whenever there is a change in the loadings, and also when there is a new factor appearing (at least for a sample size  $m \geq 100$ ). Last, notice that when considering  $H_{A,2}$  (a new factor appearing), then we could also detect a change-point when  $\tau + m \leq t \leq T$ , but we do not report results in this case since power can only increase with respect to what shown in Tables 4 and 5.

Table 5: Power - new factor appears - 10% significance  
 Fraction of detections in  $[\tau, \tau + m)$

$\eta = 0.45$		$m$								
$r$	50	75	100	125	150	175	200	225	250	
1	0.89	0.88	0.88	0.90	0.88	0.91	0.92	0.88	0.88	
2	0.81	0.94	0.95	0.93	0.91	0.93	0.93	0.93	0.93	
3	0.14	0.88	0.94	0.95	0.94	0.95	0.92	0.93	0.95	
4	0.00	0.36	0.92	0.94	0.94	0.93	0.93	0.93	0.94	
$\eta = 0.5$		$m$								
$r$	50	75	100	125	150	175	200	225	250	
1	0.90	0.89	0.88	0.91	0.89	0.92	0.91	0.89	0.89	
2	0.76	0.94	0.95	0.92	0.92	0.93	0.93	0.93	0.93	
3	0.10	0.86	0.94	0.96	0.94	0.95	0.93	0.93	0.94	
4	0.00	0.27	0.89	0.94	0.93	0.93	0.92	0.92	0.94	

- (3) In Tables 6 and 7 we report the minimum, maximum, the 25<sup>th</sup>, 50<sup>th</sup> and 75<sup>th</sup> percentiles of the distribution of the estimated change-point locations, whenever under  $H_{A,1}$  or  $H_{A,2}$  a break is detected at  $\hat{\tau}_m$  such that  $\tau \leq \hat{\tau}_m \leq T$  and when testing at 10% significance levels. Given the results in Tables 3 and 5 we report those statistics only for  $m = 100, 175, 250$ . It is evident that the test detects a break with a delay which increases as  $r$  increases - this is in line with the comments in Section 4.2, since, as  $r$  grows, the  $r$ -th eigenvalue becomes smaller and smaller, thus being closer to a weak factor. Interestingly, there are virtually no differences between the cases of  $\eta = 0.45$  and  $\eta = 0.5$ ; similarly, different values of  $m$  also do not seem to alter results. Note that, as expected, the minimum values of the distribution of the estimated locations are, roughly speaking, of order  $m^{1/2}$  all across the table.

## 6 An application to US industrial production data

We conclude with an application to a panel of US industrial production indexes. Specifically, we consider monthly growth rates for  $N = 224$  sectorial indices, over the period from January 1972 to November 2015, for a total of  $T = 527$  observations. Estimation is based on a sample of size  $m = 60$ , i.e. 5 years. Analysis of the whole dataset using rolling samples of size  $m$  suggests between one and two factors throughout - this result consistently follows using different procedures - namely, Trapani (2017) testing procedure and the criteria by Bai and Ng (2002) and Alessi et al. (2010). Therefore, we run our sequential testing procedure

Table 6: Location distribution - loadings change  
(true change-point at  $\tau = 500$ )

$m$	$r$	$\eta = 0.45$					$\eta = 0.5$				
		min	25 <sup>th</sup>	50 <sup>th</sup>	75 <sup>th</sup>	max	min	25 <sup>th</sup>	50 <sup>th</sup>	75 <sup>th</sup>	max
100	1	504	516	520	526	551	505	516	521	526	551
	2	507	523	528	533	553	507	524	529	534	554
	3	509	529	536	543	573	509	531	538	545	586
	4	520	548	559	571	$> T$	520	553	563	578	$> T$
175	1	504	516	522	528	550	505	517	523	529	550
	2	508	523	529	535	553	509	524	530	537	555
	3	507	531	537	543	564	507	532	538	544	566
	4	514	536	544	551	580	514	538	546	553	591
250	1	502	518	523	529	551	505	519	524	529	553
	2	508	525	531	537	558	508	526	532	538	565
	3	508	531	538	547	568	508	533	540	548	570
	4	516	538	545	554	581	516	539	547	555	584

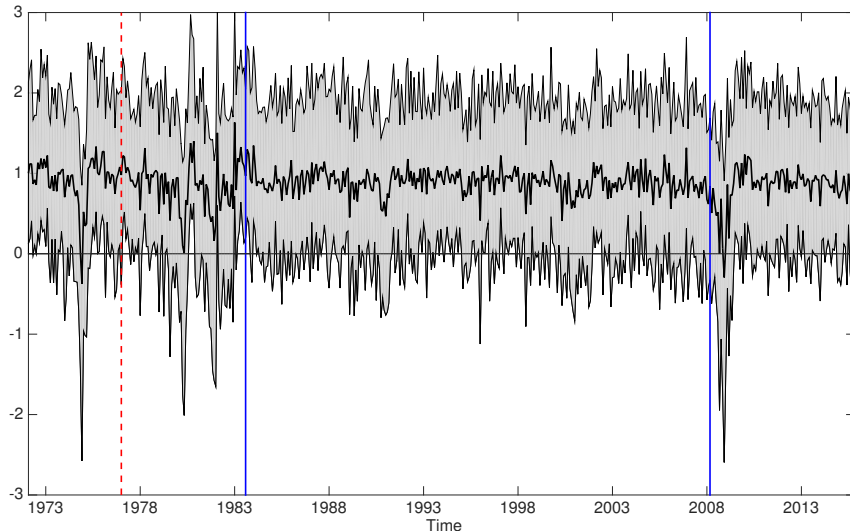
Table 7: Location distribution - new factor appears  
(true change-point at  $\tau = 500$ )

$m$	$r$	$\eta = 0.45$					$\eta = 0.5$				
		min	25 <sup>th</sup>	50 <sup>th</sup>	75 <sup>th</sup>	max	min	25 <sup>th</sup>	50 <sup>th</sup>	75 <sup>th</sup>	max
100	1	502	514	519	525	547	505	515	520	525	548
	2	505	522	530	537	574	506	523	531	538	576
	3	513	532	541	550	598	514	534	542	552	598
	4	519	546	557	569	648	520	548	560	574	669
175	1	501	515	520	525	546	502	516	521	526	552
	2	507	524	531	537	564	507	524	532	539	572
	3	508	532	541	550	578	508	534	542	552	579
	4	518	543	552	562	598	518	544	555	565	599
250	1	502	514	520	527	552	502	514	521	527	553
	2	508	524	531	540	571	508	525	532	541	571
	3	510	533	542	551	590	510	534	543	552	589
	4	515	542	552	562	600	515	544	553	564	603

monitoring the first four factors, thus accounting both for at most two new factors emerging and for a change in all loadings. The test is run at 5% significance level and setting  $\eta = 0.5$ , hence using the critical values in (24).

The monitoring is implemented as follows. We begin at  $t = m + 1$ ; once the first change-point is detected at  $\hat{\tau}_1 \geq m + 1$ , we restart the estimation at  $t = \hat{\tau}_1$  and after  $m$  periods we restart monitoring at  $t = \hat{\tau}_1 + m + 1$ . In general, given an estimated change-point  $\hat{\tau}_j$ , with  $j \geq 1$ , we restart monitoring by computing the detector defined in (19) which in this case

Figure 1: Estimated change-point locations for US industrial production indexes



Black line: cross-sectional median of the data (monthly growth rates on yearly basis); blue lines: estimated change-point locations; red line: first period used for testing.

is defined as

$$d(k; m) = \left| \sum_{t=\hat{\tau}_j+m+1}^{m+k} \frac{\Gamma_t - 1}{\sqrt{2}} \right|, \quad \hat{\tau}_j + 1 \leq k \leq T - m.$$

Therefore, the monitoring window after the  $j$ -th change-point is of size  $T - \hat{\tau}_j - m$  and the estimated change-points  $\hat{\tau}_j$  are such that  $\hat{\tau}_{j+1} - \hat{\tau}_j \geq m + 1$ . We keep restarting the procedure as long as we have a monitoring window of non-zero length, that is as long as  $T - \hat{\tau}_j > m$ ; this allows the possibility for the last change-point to be detected in the interval  $T - m \leq t \leq T$ .

We find evidence of two change-points dated:  $\hat{\tau}_1$ : August, 1983;  $\hat{\tau}_2$ : March, 2008, The estimated locations are also shown in Figure 1 together with the joint panel of data. The first estimated change-point ( $\hat{\tau}_1$ ) clearly mark the start of the Great Moderation while the second one ( $\hat{\tau}_2$ ) takes place at the start of the Great Financial Crisis.

## 7 Conclusions

In this paper we develop a a family of monitoring procedures to detect a break in the signal component of a large factor model; to the best of our knowledge, this is the first contribution in high-dimensional factor models which proposes a sequential monitoring and

testing procedure, as opposed to the extant literature where ex-post detection of breaks is usually considered. Our statistics are based on a well-known property of the  $(r + 1)$ -th eigenvalue of the sample covariance matrix of the data: whilst under the null the  $(r + 1)$ -th eigenvalue is bounded, under the alternative of a break (either in the loadings, or in the number of factors itself) it becomes spiked. Given that the sample eigenvalue does not have a known limiting distribution under the null, we regularise the problem by (doubly) randomising the test statistic in conjunction with sample conditioning, obtaining a sequence of *i.i.d.*, asymptotically chi-square statistics which are then employed to build the monitoring scheme. Numerical evidence shows that our procedure works very well in finite samples, with a very small probability of false detections and tight detection times in presence of a genuine change-point.

Building on the methodology proposed in this paper, there are at least two possible extensions which could be considered. Firstly, the results and methodology in this paper could be also used in the context of a non-stationary factor model, similar to the one considered in Bai (2004), where the factors are allowed to have unit roots. In such case, the key theoretical result would be to show that in presence of  $r$  factors the first  $r$  eigenvalues of the matrix  $m^{-2} \sum_{t=1}^m X_t X_t'$  diverge to positive infinity almost surely at some rate, whereas the remaining factors are a.s. bounded. Secondly, it is possible to extend the theory developed in this paper to the context of the generalised dynamic factor model by Forni, Hallin, Lippi and Reichlin (2000) or the factor model by Lam and Yao (2012), which are based on the asymptotic behavior of the eigenvalues of the spectral density or the long-run covariance matrices, respectively. By studying the asymptotic behavior of the estimated eigenvalues of those matrices an appropriate test statistic based on these can be built. These, and other, extensions are under current investigations by the authors.

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## A Technical appendix

### A.1 Preliminary lemmas

This section contains technical results which are useful to prove the main theorems in the paper. Throughout this and the next section,  $E^*$  denotes expectation calculated with respect to  $P^*$ ; similarly,  $E^\dagger$  and  $V^\dagger$  denote expectation and variance calculated with respect to  $P^\dagger$ . Also, whenever possible, we omit for ease of notation the dependence of  $\xi_j(t)$ , and of related quantities, on  $t$ .

**Lemma 3.** *Let*

$$\bar{\lambda}_N(t) = \frac{1}{N} \sum_{p=1}^N \lambda^{(p)}(t); \quad (40)$$

*under Assumptions 1 and 2, it holds that*

$$\begin{cases} \limsup_{N \rightarrow \infty} \bar{\lambda}_N(t) = \bar{\lambda}^{\text{sup}}(t) < \infty \\ \liminf_{N \rightarrow \infty} \bar{\lambda}_N(t) = \bar{\lambda}^{\text{inf}}(t) > 0 \end{cases}, \quad (41)$$

*for all  $m \leq t \leq T$ . Further, under Assumptions 1-3, it holds that, as  $\min\{N, m\} \rightarrow \infty$*

$$\begin{cases} \limsup_{N \rightarrow \infty} \widehat{\bar{\lambda}}_N(t) = \bar{\lambda}^{\text{sup}}(t) < \infty \\ \liminf_{N \rightarrow \infty} \widehat{\bar{\lambda}}_N(t) = \bar{\lambda}^{\text{inf}}(t) > 0 \end{cases}.$$

*Proof.* See Lemma A.1 in Trapani (2017). □

**Lemma 4.** *Under Assumptions 1-4, it holds that, under  $H_A$*

$$\lim_{N, m, R \rightarrow \infty} \sup \frac{\Theta_t}{\tilde{l}(N, m, R)} = 0 \quad a.s.; \quad (42)$$

*under  $H_0$ , it holds that*

$$\frac{\tilde{l}(N, m, R)}{R} \times \frac{\Theta_t}{\tilde{l}(N, m, R)} \rightarrow C > 0 \quad a.s., \quad (43)$$

*as  $\min(N, m, R) \rightarrow \infty$ .*

*Proof.* We begin with (42). Define

$$U_{n,s,r} = \int_{-\infty}^{+\infty} \left| r^{-1/2} \sum_{j=1}^r \{I[\xi_j \leq u\phi_{n,s}^{-1}(t)] - G_\phi(0)\} \right|^2 dF_\phi(u).$$

We begin by showing that

$$\sum_{N=1}^{\infty} \sum_{m=1}^{\infty} \sum_{R=1}^{\infty} \frac{1}{mNR} P^* \left[ \max_{1 \leq n \leq N, 1 \leq s \leq m, 1 \leq r \leq R} U_{n,s,r} > \epsilon \tilde{l}(N, m, R) \right] < \infty; \quad (44)$$

using the short-hand notation  $\max_{n,s,r}$  for  $\max_{1 \leq n \leq N, 1 \leq s \leq m, 1 \leq r \leq R}$ , Markov inequality im-

plies that this follows if

$$\sum_{N=1}^{\infty} \sum_{m=1}^{\infty} \sum_{R=1}^{\infty} \frac{1}{mNR\tilde{l}(N, m, R)} E^* \left| \max_{n,s,r} U_{n,s,r} \right| < \infty. \quad (45)$$

The maximal inequality contained in Theorem 2 in Moricz (1983) entails that

$$E^* \left| \max_{n,s,r} U_{n,s,r} \right| \leq C_0 E^* |U_{N,m,R}| (\ln N) (\ln m) (\ln R). \quad (46)$$

Further, it holds that

$$E^* |U_{N,m,R}| \leq C_0 + C_1 R \phi_{N,m}^{-2}(t) \leq C_2,$$

where the first inequality follows from the proof of Theorem 1, and the second one from Assumption 5. Then (44) follows immediately from (45).

Note now that for every triple  $(N, m, R)$ , there is a triple of positive integers  $(k_1, k_2, k_3)$  such that  $2^{k_1} \leq N < 2^{k_1+1}$ ,  $2^{k_2} \leq m < 2^{k_2+1}$ ,  $2^{k_3} \leq R < 2^{k_3+1}$ . Further, there is also a triple of real numbers defined over  $[0, 1)$ , say  $(\rho_1, \rho_2, \rho_3)$ , such that  $N = 2^{k_1+\rho_1}$ , etc... Define now the short-hand notation

$$A_{k_1, k_2, k_3} = \left\{ \omega : \max_{1 \leq k_1 \leq 2^{k_1+\rho_1}, 1 \leq k_2 \leq 2^{k_2+\rho_2}, 1 \leq k_3 \leq 2^{k_3+\rho_3}} |U_{k_1, k_2, k_3}| > \tilde{e}l(k_1, k_2, k_3) \right\}.$$

By (44), it holds that

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{2^{k_1+1} 2^{k_2+1} 2^{k_3+1}}{(2^{k_1+1} - 1)(2^{k_2+1} - 1)(2^{k_3+1} - 1)} P^*(A_{k_1, k_2, k_3}) < \infty;$$

thus

$$\begin{aligned} & \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} P^*(A_{k_1, k_2, k_3}) \leq \\ & \leq \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{2^{k_1+1} 2^{k_2+1} 2^{k_3+1}}{(2^{k_1+1} - 1)(2^{k_2+1} - 1)(2^{k_3+1} - 1)} P^*(A_{k_1, k_2, k_3}) < \infty. \end{aligned} \quad (47)$$

This result entails that  $P^*(A_{k_1, k_2, k_3} \text{ i.o.}) = 1$ , which is a *conditional* result. Let now  $X_{k_1, k_2, k_3}$  be the indicator of  $A_{k_1, k_2, k_3}$ , and note that  $A_{k_1, k_2, k_3}$  is conditional on the  $\sigma$ -field  $\mathcal{F}_{k_1, k_2, k_3} = \{X_{i,t}, 1 \leq i \leq N, 1 \leq t \leq m\} \cup \{\xi_j, 1 \leq j \leq R\}$ , which is non decreasing.

Equation (47) implies that

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} E(X_{k_1, k_2, k_3} | \mathcal{F}_{k_1, k_2, k_3}) < \infty;$$

hence, by Theorem 1 in Chen (1978), it holds that

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} X_{k_1, k_2, k_3} < \infty \quad \text{a.s.} \quad (48)$$

We note that result by Chen (1978) is for a series indexed by a single index, but his arguments can be readily generalised to the case of multi-index series. Equation (48) can be equivalently rewritten as

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} P(A_{k_1, k_2, k_3}) < \infty, \quad (49)$$

which is an *unconditional* result. From (49), it is easy to see that

$$\frac{\max_{k_1, k_2, k_3} |U_{k_1, k_2, k_3}|}{\tilde{l}(k_1, k_2, k_3)} \rightarrow 0 \quad \text{a.s.};$$

this entails that

$$\frac{|U_{N, m, R}|}{\tilde{l}(N, m, R)} \leq \frac{\max_{k_1, k_2, k_3} |U_{k_1, k_2, k_3}| \tilde{l}(k_1, k_2, k_3)}{\tilde{l}(k_1, k_2, k_3)} \leq \frac{\max_{k_1, k_2, k_3} |U_{k_1, k_2, k_3}|}{\tilde{l}(k_1, k_2, k_3)} \rightarrow 0 \quad \text{a.s.},$$

so that finally

$$\lim_{N, m, R \rightarrow \infty} \sup \frac{|U_{N, m, R}|}{\tilde{l}(N, m, R)} = 0 \quad \text{a.s.},$$

which proves (42).

Consider now (43); the proof is based on similar passages as the proof of Theorem 3.2 in Horváth and Trapani (2017). Under the null, Lemma 1 entails that

$$P \left\{ \omega : \lim_{N, m \rightarrow \infty} \phi_{N, m}(t) = 0 \right\} = 1,$$

so that we can assume henceforth that  $\lim_{N, m \rightarrow \infty} \phi_{N, m}(t) = 0$ . Also, by definition it holds

that  $E^* I \left[ \xi_j \leq u\phi_{N,m}^{-1}(t) \right] = G_\phi \left( u\phi_{N,m}^{-1}(t) \right)$ . Therefore

$$\begin{aligned}
& G_\phi(0) [1 - G_\phi(0)] \Theta_t = \\
& = \int_{-\infty}^{+\infty} \left| R^{-1/2} \sum_{j=1}^R \left\{ I \left[ \xi_j \leq u\phi_{N,m}^{-1}(t) \right] \pm G_\phi \left( u\phi_{N,m}^{-1}(t) \right) - G_\phi(0) \right\} \right|^2 dF_\phi(u) = \\
& = \int_{-\infty}^{+\infty} \left| R^{-1/2} \sum_{j=1}^R \left\{ I \left[ \xi_j \leq u\phi_{N,m}^{-1}(t) \right] - G_\phi \left( u\phi_{N,m}^{-1}(t) \right) \right\} \right|^2 dF_\phi(u) + \\
& + R \int_{-\infty}^{+\infty} \left| G_\phi \left( u\phi_{N,m}^{-1}(t) \right) - G_\phi(0) \right|^2 dF_\phi(u) + \\
& + 2 \int_{-\infty}^{+\infty} R^{-1/2} \sum_{j=1}^R \left\{ I \left[ \xi_j \leq u\phi_{N,m}^{-1}(t) \right] - G_\phi \left( u\phi_{N,m}^{-1}(t) \right) \right\} \left[ G_\phi \left( u\phi_{N,m}^{-1}(t) \right) + \right. \\
& \left. - G_\phi(0) \right] dF_\phi(u),
\end{aligned}$$

and note that  $G_\phi \left( u\phi_{N,m}^{-1}(t) \right) - G_\phi(0) = I_{[0,\infty)}(u) - G_\phi(0)$  as  $N, m \rightarrow \infty$ . We can easily show, following the proof of (42), that

$$\limsup_{N,m,R \rightarrow \infty} \frac{\int_{-\infty}^{+\infty} \left| R^{-1/2} \sum_{j=1}^R \left\{ I \left[ \xi_j \leq u\phi_{N,m}^{-1}(t) \right] - G_\phi \left( u\phi_{N,m}^{-1}(t) \right) \right\} \right|^2 dF_\phi(u)}{\tilde{l}(N, m, R)} = 0 \text{ a.s.};$$

equation (43) follows directly from this, Assumption 4(iii) and from the Cauchy-Schwartz inequality.  $\square$

**Lemma 5.** *Under Assumptions 1-6 and 9(ii), it holds that, under  $H_0$*

$$\max_{1 \leq k \leq T_m} \sqrt{\frac{m}{k(k+m)}} \left| \sum_{t=m+1}^{m+k} \left( E^\dagger(\Gamma_t) - 1 \right) \right| = O(m^{-\epsilon}), \quad (50)$$

$$\max_{1 \leq k \leq T_m} \sqrt{\frac{m}{k(k+m)}} \left| \sum_{t=m+1}^{m+k} \left( V^\dagger(\Gamma_t) - 2 \right) \right| = O(m^{-\epsilon}), \quad (51)$$

for some  $\epsilon > 0$ . Also

$$E^\dagger |\Gamma_t|^{2+\delta} < \infty, \quad (52)$$

for some  $\delta > 0$ .



*Proof.* We start with equation (50). By construction

$$\begin{aligned}
\frac{E^\dagger(\Gamma_t)}{G_\psi(0)[1-G_\psi(0)]} &= \frac{E^\dagger \int_{-\infty}^{+\infty} |\gamma(u;t)|^2 dF_\psi(u)}{G_\psi(0)[1-G_\psi(0)]} = \\
&= E^\dagger \int_{-\infty}^{+\infty} \left| W^{-1/2} \sum_{j=1}^W [\tilde{\zeta}_j(u;t) - G_\psi(0)] \right|^2 dF_\psi(u) = \\
&= W^{-1} \int_{-\infty}^{+\infty} E^\dagger \left| \tilde{\zeta}_j(u;t) - G_\psi(0) \right|^2 dF_\psi(u);
\end{aligned}$$

by similar passages as in the proof of Theorem 1, it can be shown that

$$\frac{E^\dagger(\Gamma_t)}{G_\psi(0)[1-G_\psi(0)]} - 1 \leq C_0 \left[ Wh^{-2} \left( \frac{R}{\ln R} \right) + h^{-1} \left( \frac{R}{\ln R} \right) \right].$$

Thus

$$\begin{aligned}
&\max_{1 \leq k \leq T_m} \sqrt{\frac{m}{k(k+m)}} \left| \sum_{t=m+1}^{m+k} (E^\dagger(\Gamma_t) - 1) \right| \leq \\
&\leq C_0 W \left( \frac{\ln R}{R} \right)^2 \max_{1 \leq k \leq T_m} \sqrt{\frac{m}{k(k+m)}} k \leq \\
&\leq C_0 m^{1/2} \left[ Wh^{-2} \left( \frac{R}{\ln R} \right) + h^{-1} \left( \frac{R}{\ln R} \right) \right],
\end{aligned}$$

which is  $O(m^{-\epsilon})$  on account of Assumption 9(ii).

We now turn to (51). Let  $\gamma(0;t) = W^{-1/2} \sum_{j=1}^W [\tilde{\zeta}_j(0;t) - G_\psi(0)]$ ; we have

$$\begin{aligned}
&\left( \int_{-\infty}^{+\infty} |\gamma(u;t)|^2 dF_\psi(u) \right)^2 - \left( \int_{-\infty}^{+\infty} |\gamma(0;t)|^2 dF_\psi(u) \right)^2 = \tag{53} \\
&= \int_{-\infty}^{+\infty} \left( \left| W^{-1/2} \sum_{j=1}^W [\tilde{\zeta}_j(u;t) - \tilde{\zeta}_j(0;t)] \right|^2 + \right. \\
&\quad \left. + 2W^{-1} \sum_{j=1}^W [\tilde{\zeta}_j(u;t) - \tilde{\zeta}_j(0;t)] \sum_{j=1}^W [\tilde{\zeta}_j(0;t) - G_\psi(0)]^2 \right) dF_\psi(u) + \\
&\quad + 2 \int_{-\infty}^{+\infty} W^{-1/2} \sum_{j=1}^W [\tilde{\zeta}_j(0;t) - G_\psi(0)] \times \left( \left| W^{-1/2} \sum_{j=1}^W [\tilde{\zeta}_j(u;t) - \tilde{\zeta}_j(0;t)] \right|^2 + \right. \\
&\quad \left. + 2W^{-1} \sum_{j=1}^W [\tilde{\zeta}_j(u;t) - \tilde{\zeta}_j(0;t)] \sum_{j=1}^W [\tilde{\zeta}_j(0;t) - G_\psi(0)] \right) dF_\psi(u).
\end{aligned}$$

By Rosenthal's inequality

$$\begin{aligned}
W^{-2}E^\dagger \left| \sum_{j=1}^W [\tilde{\zeta}_j(u;t) - \tilde{\zeta}_j(0;t)] \right|^4 &\leq \tag{54} \\
&\leq C_0 W^{-2} \left[ \left| \sum_{j=1}^W E^\dagger (\tilde{\zeta}_j(u;t) - \tilde{\zeta}_j(0;t)) \right|^4 + \sum_{j=1}^W E^\dagger |\tilde{\zeta}_j(u;t) - \tilde{\zeta}_j(0;t)|^4 \right] \leq \\
&\leq C_0 W^{-2} \left[ W^4 \left| E^\dagger (\tilde{\zeta}_j(u;t) - \tilde{\zeta}_j(0;t)) \right|^4 + C_1 W \right] \leq \\
&\leq C_0 W^{-2} \left[ W^4 \left| G_\psi(\psi_{N,m,R}^{-1}(t)) - G_\psi(0) \right|^4 + C_1 W \right] \leq \\
&\leq C_0 W^{-1} + C_1 W^2 u^4 \psi_{N,m,R}^{-4}(t);
\end{aligned}$$

the same logic also yields

$$\begin{aligned}
W^{-1}E^\dagger \left| \sum_{j=1}^W [\tilde{\zeta}_j(u;t) - \tilde{\zeta}_j(0;t)] \right|^2 &\leq C_0 u \psi_{N,m,R}^{-1}(t) + C_1 W u^2 \psi_{N,m,R}^{-2}(t), \\
E^\dagger \left| W^{-1/2} \sum_{j=1}^W [\tilde{\zeta}_j(0;t) - G_\psi(0)] \right|^2 &\leq C_0.
\end{aligned}$$

Repeated applications of the  $C_r$ -inequality and of the Cauchy-Schwartz inequality to (53) yield

$$E^\dagger \left( \int_{-\infty}^{+\infty} |\gamma(u;t)|^2 dF_\psi(u) \right)^2 - E^\dagger \left( \int_{-\infty}^{+\infty} |\gamma(0;t)|^2 dF_\psi(u) \right)^2 \leq C_0 W \psi_{N,m,R}^{-2}(t).$$

Finally, tedious but elementary calculations yield

$$E^\dagger \left| W^{-1/2} \sum_{j=1}^W [\tilde{\zeta}_j(0;t) - G_\psi(0)] \right|^4 = 3(G_\psi(0)[1 - G_\psi(0)])^2 + O(W^{-1}).$$

Putting all together and using (50), it follows that

$$V^\dagger(\Gamma_t) - 2 \leq C_0 \left( W \psi_{N,m,R}^{-2}(t) + W^{-1} \right),$$

whence (51) follows.

Consider now (52). By convexity

$$\left| \int_{-\infty}^{+\infty} |\gamma(u; t)|^2 dF_\psi(u) \right|^{2+\delta} \leq \int_{-\infty}^{+\infty} |\gamma(u; t)|^{4+2\delta} dF_\psi(u);$$

also, applying the  $C_r$ -inequality

$$\begin{aligned} & \int_{-\infty}^{+\infty} |\gamma(u; t)|^{4+2\delta} dF_\psi(u) \leq \\ & \leq C_0 \int_{-\infty}^{+\infty} \left| W^{-1/2} \sum_{j=1}^W [\tilde{\zeta}_j(u; t) - G_\psi(u\psi_{N,m,R}^{-1}(t))] \right|^{4+2\delta} dF_\psi(u) + \\ & + C_0 \int_{-\infty}^{+\infty} \left| W^{-1/2} \sum_{j=1}^W [G_\psi(u\psi_{N,m,R}^{-1}(t)) - G_\psi(0)] \right|^{4+2\delta} dF_\psi(u). \end{aligned}$$

Note that, by applying Burkholder's inequality and convexity

$$\begin{aligned} & \int_{-\infty}^{+\infty} E^\dagger \left| W^{-1/2} \sum_{j=1}^W [\tilde{\zeta}_j(u; t) - G_\psi(u\psi_{N,m,R}^{-1}(t))] \right|^{4+2\delta} dF_\psi(u) \leq \\ & \leq \int_{-\infty}^{+\infty} E^\dagger \left| W^{-1} \sum_{j=1}^W [\tilde{\zeta}_j(u; t) - G_\psi(u\psi_{N,m,R}^{-1}(t))] \right|^{2+2\delta} dF_\psi(u) \leq \\ & \leq W^{-1} \sum_{j=1}^W \int_{-\infty}^{+\infty} E^\dagger \left| [\tilde{\zeta}_j(u; t) - G_\psi(u\psi_{N,m,R}^{-1}(t))] \right|^{2+2\delta} dF_\psi(u) \leq C_0, \end{aligned}$$

since  $\tilde{\zeta}_j(u; t)$  has finite moments of any order. Also

$$\begin{aligned} & \int_{-\infty}^{+\infty} \left| W^{-1/2} \sum_{j=1}^W [G_\psi(u\psi_{N,m,R}^{-1}(t)) - G_\psi(0)] \right|^{4+2\delta} dF_\psi(u) \leq \\ & \leq W^{2+\delta} \int_{-\infty}^{+\infty} |G_\psi(u\psi_{N,m,R}^{-1}(t)) - G_\psi(0)|^{4+2\delta} dF_\psi(u) \leq \\ & \leq W^{2+\delta} \left( \frac{m_{G,\psi}}{\psi_{N,m,R}(t)} \right)^{4+2\delta} \int_{-\infty}^{+\infty} |u|^{4+2\delta} dF_\psi(u) \leq C_0, \end{aligned}$$

where  $m_{G,\psi}$  is the upper bound of the density of  $G_\psi(\cdot)$ ; the final estimate follows from Assumptions 7 and 9(i). This proves (52).  $\square$

## A.2 Proofs of main results

*Proof of Lemma 1.* The proof is similar to that of Lemma 1 in Trapani (2017), and therefore only the main arguments are reported. Recall the definition

$$\Sigma_m(t) = \frac{1}{m} \sum_{k=t-m+1}^t E(X_k X_k');$$

if  $t < \tau$ , there is no change in the  $(r+1)$ -th eigenvalue of  $E(X_k X_k')$ , and therefore the proof that  $\lambda^{(r+1)}(t)$  is finite is exactly the same as in Trapani (2017). We begin with the case of  $H_{A,2}$ . When  $t \geq \tau$ , consider

$$\begin{aligned} \Sigma_m(t) &= \frac{1}{m} \sum_{k=t-m+1}^{\tau-1} E(X_k X_k') + \frac{1}{m} \sum_{k=\tau}^t E(X_k X_k') \\ &= \frac{\tau - t + m - 1}{m} \Sigma_m^{(1)}(t) + \frac{t - \tau + 1}{m} \Sigma_m^{(2)}(t). \end{aligned}$$

Let  $\lambda_1^{(r+1)}(t)$  and  $\lambda_2^{(r+1)}(t)$  be the  $(r+1)$ -th eigenvalue of  $\Sigma_m^{(1)}(t)$  and  $\Sigma_m^{(2)}(t)$  respectively. In the pre-break period, we have  $\lambda_1^{(r+1)}(t) < \infty$ ; as far as  $\lambda_2^{(r+1)}(t)$  is concerned, this is equal to  $C_0 N$ . The desired result follows immediately from Weyl's dual inequality (see Tao (2012)). The proof for the case of  $H_{A,1}$  is based on exactly the same arguments and is therefore omitted.  $\square$

*Proof of Lemma 2.* The proof follows from combining Lemma 1 and the proof of Lemma 2 in Trapani (2017).  $\square$

*Proof of Theorem 1.* Consider (14); its proof follows similar passages as the proof of Theorem 3.1 in Horváth and Trapani (2017), although it is actually a refinement of that result. In the presence of a break, it follows from Lemmas 1 and 2 that

$$P \left\{ \omega : \lim_{N, m \rightarrow \infty} \phi_{N, m}(t) = \infty \right\} = 1,$$

for each  $t \geq \tau$ , as long as

$$\begin{aligned} N^{1-\delta} \min \left\{ \frac{t - \tau + 1}{m}, \frac{\tau + m - t - 1}{m} \right\} &\rightarrow \infty \text{ for } \tau \leq t \leq \tau + m - 1, \\ N^{1-\delta} &\rightarrow \infty \text{ for } s \geq \tau + m, \end{aligned}$$

hold. Thus, we can assume from now on that  $\lim_{N,m \rightarrow \infty} \phi_{N,m}(t) = \infty$  holds. Note that

$$\begin{aligned} R^{-1/2} \sum_{i=1}^R [\zeta_i(u; t) - G_\phi(0)] &= R^{-1/2} \sum_{i=1}^R [I\{\xi_i \leq 0\} - G_\phi(0)] + \\ &+ R^{-1/2} \sum_{i=1}^R \left[ I\{\xi_i \leq u\phi_{N,m}^{-1}(t)\} - I\{\xi_i \leq 0\} - (G_\phi(u\phi_{N,m}^{-1}(t)) - G_\phi(0)) \right] + \\ &+ R^{-1/2} \sum_{i=1}^R \left[ G_\phi(u\phi_{N,m}^{-1}(t)) - G_\phi(0) \right]. \end{aligned}$$

By construction,

$$\begin{aligned} E^* \zeta_j(u; t) &= G_\phi(u\phi_{N,m}^{-1}(t)), \\ E^* (\zeta_j(u; t) - E^* \zeta_j(u; t))^2 &= G_\phi(u\phi_{N,m}^{-1}(t)) \left[ 1 - G_\phi(u\phi_{N,m}^{-1}(t)) \right]. \end{aligned}$$

Consider now the following passages:

$$\begin{aligned} E^* \int_{-\infty}^{\infty} \left( R^{-1/2} \sum_{i=1}^R \left[ I\{\xi_i \leq u\phi_{N,m}^{-1}(t)\} - I\{\xi_i \leq 0\} - (G_\phi(u\phi_{N,m}^{-1}(t)) - G_\phi(0)) \right] \right)^2 dF_\phi(u) &= \\ = \int_{-\infty}^{\infty} E^* \left[ I\{\xi_1 \leq u\phi_{N,m}^{-1}(t)\} - I\{\xi_1 \leq 0\} - (G_\phi(u\phi_{N,m}^{-1}(t)) - G_\phi(0)) \right]^2 dF_\phi(u), \end{aligned}$$

on account of the independence of the  $\xi_i$ s. Also, note that the random variable  $I\{\xi_1 \leq u\phi_{N,m}^{-1}(t)\} - I\{\xi_1 \leq 0\}$  has expected value given by  $G_\phi(u\phi_{N,m}^{-1}(t)) - G_\phi(0)$ , and variance equal to

$$\begin{aligned} E^* \left[ I\{\xi_1 \leq u\phi_{N,m}^{-1}(t)\} - I\{\xi_1 \leq 0\} - G_\phi(u\phi_{N,m}^{-1}(t)) + G_\phi(0) \right]^2 &= \\ = \left( G_\phi(u\phi_{N,m}^{-1}(t)) - G_\phi(0) \right) \left[ 1 - G_\phi(u\phi_{N,m}^{-1}(t)) + G_\phi(0) \right] &\leq \\ \leq G_\phi(u\phi_{N,m}^{-1}(t)) - G_\phi(0). \end{aligned}$$

Hence, we have

$$\begin{aligned} \int_{-\infty}^{\infty} E^* \left[ I\{\xi_1 \leq u\phi_{N,m}^{-1}(t)\} - I\{\xi_1 \leq 0\} - (G_\phi(u\phi_{N,m}^{-1}(t)) - G_\phi(0)) \right]^2 dF_\phi(u) &\leq \\ \leq \int_{-\infty}^{\infty} \left[ G_\phi(u\phi_{N,m}^{-1}(t)) - G_\phi(0) \right] dF_\phi(u) &\leq \frac{m_G}{\phi_{N,m}(t)} \int_{-\infty}^{\infty} |u| dF_\phi(u), \end{aligned}$$

where the last passage follows from Assumption 6(i), with  $m_G$  an upper bound for the

density of  $G$ . Also

$$\int_{-\infty}^{\infty} \left( R^{1/2} \left[ G_{\phi}(u\phi_{N,m}^{-1}(t)) - G_{\phi}(0) \right] \right)^2 dF_{\phi}(u) \leq \frac{R}{\phi_{N,m}^2(t)} m_G \int_{-\infty}^{\infty} u^2 dF_{\phi}(u).$$

Hence, using Assumptions 4 and 5, we conclude via Markov's inequality that

$$\begin{aligned} \Theta_t &= \int_{-\infty}^{\infty} \left\{ \frac{1}{\sqrt{G_{\phi}(0)} [1 - G_{\phi}(0)] R^{1/2}} \sum_{i=1}^R [I\{\xi_i \leq 0\} - G_{\phi}(0)] \right\}^2 dF_{\phi}(u) + o_{P^*}(1) = \\ &= \left\{ \frac{1}{\sqrt{G_{\phi}(0)} [1 - G_{\phi}(0)] R^{1/2}} \sum_{i=1}^R [I\{\xi_i \leq 0\} - G_{\phi}(0)] \right\}^2 + o_{P^*}(1), \end{aligned}$$

and therefore the result follows from the Central Limit Theorem for Bernoulli random variables. The proof of (15) follows from exactly the same passages as the proof of (43).  $\square$

*Proof of Theorem 2.* The proof of the theorem is exactly the same as that of Theorem 1, based on Lemma 4.  $\square$

*Proof of Theorem 3.* By (50) and (51), we have

$$\max_{1 \leq k \leq T_m} \sqrt{\frac{m}{k(k+m)}} \left| \sum_{t=m+1}^{m+k} \frac{\Gamma_t - 1}{\sqrt{2}} \right| = \max_{1 \leq k \leq T_m} \sqrt{\frac{m}{k(k+m)}} \left| \sum_{t=m+1}^{m+k} Z_t \right| + O(m^{-\epsilon}), \quad (55)$$

where

$$Z_t = \frac{\Gamma_t - E^{\dagger}(\Gamma_t)}{\sqrt{V^{\dagger}(\Gamma_t)}}$$

is an *i.i.d.* sequence with mean zero, unit variance and finite moments of order  $2 + \delta$ . Consider (25); on account of (55), this holds immediately, following the same passages as in the proof of Theorem 2.1 in Horváth et al. (2004). As far as (26) is concerned, due to the polynomial rate of approximation in (55), it suffices to prove that

$$P^{\dagger} \left( A_m \max_{1 \leq k \leq T_m} \sqrt{\frac{m}{k(k+m)}} \left| \sum_{t=m+1}^{m+k} Z_t \right| \geq x + D_m \right) = e^{-e^{-x}},$$

as  $\min(N, m, R, W) \rightarrow \infty$ . This is a relatively standard exercise, and it is very similar to the proof of Theorem 1.1 in Horváth et al. (2007); we therefore report only the main

passages. Let  $a(m) = (\ln m)^2$ ; by virtue of (52), it holds that

$$\sup_{a(m) \leq k < \infty} \sqrt{\frac{m}{k(k+m)}} \left| \sum_{t=m+1}^{m+k} Z_t - B(k) \right| = O_{P^\dagger} \left( a(m)^{-\frac{\epsilon}{2(2+\epsilon)}} \right), \quad (56)$$

where  $\{B(t), 0 \leq t < \infty\}$  is a standard Wiener process - see Komlós, Major and Tusnády (1975, 1976). We now show that

$$\max_{a(m) \leq k \leq \frac{cm}{\ln m}} \frac{B\left(\frac{k}{k+m}\right)}{\sqrt{\frac{k}{m}}} = \max_{a(m) \leq k \leq \frac{cm}{\ln m}} \frac{B\left(\frac{k}{k+m}\right)}{\sqrt{\frac{k}{k+m}}} + O_{P^\dagger} \left( \frac{(\ln \ln m)^{1/2}}{\ln m} \right); \quad (57)$$

given that

$$\left| \frac{k}{m} - \frac{k}{k+m} \right| \leq \left( \frac{k}{m} \right)^2,$$

using the modulus of continuity of the Wiener process we obtain

$$\max_{a(m) \leq k \leq \frac{cm}{\ln m}} \left| \frac{B\left(\frac{k}{k+m}\right)}{\sqrt{\frac{k}{m}}} - \frac{B\left(\frac{k}{k+m}\right)}{\sqrt{\frac{k}{k+m}}} \right| = O_{P^\dagger}(1) \max_{a(m) \leq k \leq \frac{cm}{\ln m}} \frac{k}{m} \left( \ln \frac{m}{k} \right)^{1/2},$$

whence (57) follows. Consequently, the following results hold:

$$\frac{1}{\sqrt{2 \ln \ln m}} \max_{1 \leq k \leq T_m} \frac{B\left(\frac{k}{k+m}\right)}{\sqrt{\frac{k}{m}}} \xrightarrow{P^\dagger} 1, \quad (58)$$

$$A_m \max_{1 \leq k \leq a(m)} \frac{B\left(\frac{k}{k+m}\right)}{\sqrt{\frac{k}{m}}} - D_m \xrightarrow{P^\dagger} -\infty, \quad (59)$$

$$A_m \max_{\frac{cm}{\ln m} \leq k \leq T_m} \frac{B\left(\frac{k}{k+m}\right)}{\sqrt{\frac{k}{m}}} - D_m \xrightarrow{P^\dagger} -\infty; \quad (60)$$

the results above are shown, for  $\sqrt{\frac{k+m}{m}} B\left(\frac{k}{k+m}\right)$ , in Lemmas 3.4, 3.5 and (in the proof of) Lemma 3.6 in Horváth et al. (2007); in (60) we have used the fact that, by Assumption 8, there exists a  $c > 0$  such that  $T_m > cm$ . Combining (56), (58), (59), (60) and (57) together,

we obtain

$$\begin{aligned} & P^\dagger \left( A_m \max_{1 \leq k \leq T_m} \frac{1}{\sqrt{k(k+m)}} \left| \sum_{t=m+1}^{m+k} Z_t \right| \geq x + D_m \right) = \\ & = P^\dagger \left( A_m \max_{a(m) \leq k \leq \frac{cm}{\ln m}} \frac{\left| B \left( \frac{k}{k+m} \right) \right|}{\sqrt{\frac{k}{k+m}}} \geq x + D_m \right) + o(1); \end{aligned}$$

then the desired result follows from Lemma 3.6 in Horváth et al. (2007).

Consider now (27); it is convenient to prove the result under  $H_{A,2}$  first. On account of (9), Lemmas 1 and 2, Assumption 4(i) and (18), it holds that

$$\Gamma_t = C_0 W + o_{P^\dagger}(W), \text{ for } t \geq \tau + C_1 m^{1/2+\epsilon},$$

where  $\epsilon > 0$  is such that  $\frac{N^{1-\delta}}{m^{1/2-\epsilon}} \rightarrow C_2 \in (0, +\infty)$  and

$$C_0 = \int_{-\infty}^{+\infty} \frac{|G_\psi(u) - G_\psi(0)|^2}{G_\psi(0)[1 - G_\psi(0)]} dF_\psi(u).$$

Thus, standard algebra yields that under  $H_{A,2}$

$$\sum_{m+1}^{m+k} \frac{\Gamma_t - 1}{\sqrt{2}} = O_{P^\dagger}(1) \left[ m + k - \left( \tau + C_1 m^{1/2+\epsilon} \right) \right] W + o_{P^\dagger}(W),$$

whenever  $k \geq \tau + C_1 m^{1/2+\epsilon}$ . Therefore,

$$\Lambda_m = O_{P^\dagger}(1) m^{1/2} W \max_{1 \leq k \leq T_m} \frac{m + k - \left( \tau + C_1 m^{1/2+\epsilon} \right)}{k^{1/2} (m + k)^{1/2}} + o_{P^\dagger}(W);$$

elementary algebra yields

$$\max_{1 \leq k \leq T_m} \frac{m + k - \left( \tau + C_1 m^{1/2+\epsilon} \right)}{k^{1/2} (m + k)^{1/2}} \geq C_2 > 0;$$

thus, Assumption 8 implies (27). Under  $H_{A,1}$  the logic is similar, and therefore only the main passages are reported. Under  $H_{A,1}$  we have

$$\Gamma_t = C_0 W + o_{P^\dagger}(W), \text{ for } \tau + C_1 m^{1/2+\epsilon} \leq t \leq \tau + m - C_1 m^{1/2+\epsilon},$$



with the same notation as above; it is then easy to see that

$$\max_{1 \leq k \leq T_m} \frac{m + k - (\tau + C_1 m^{1/2+\epsilon})}{k^{1/2} (m + k)^{1/2}} \geq C_2 > 0;$$

the proof is now the same as before.  $\square$

*Proof of Corollary 1.* The corollary is an immediate consequence of Theorem 3 and its proof. Considering (28), note that  $P^\dagger(\widehat{\tau}_m < T)$  is monotonically nondecreasing in  $T$ ; by definition

$$\begin{aligned} P^\dagger(\widehat{\tau}_m < T) &= P\left(\max_{1 \leq k \leq T} \frac{d(k; m)}{\nu^*(k; m)} > c_{\alpha, m}\right) \leq P\left(\max_{1 \leq k < \infty} \frac{d(k; m)}{\nu^*(k; m)} > c_{\alpha, m}\right) = \\ &= P\left(\sup_{0 \leq t \leq 1} \frac{|B(t)|}{t^\eta} > c_{\alpha, m}\right) + o(1) = \alpha, \end{aligned}$$

which proves (25); (26) follows from the same passages. Similarly, as far as (29) is concerned, note that

$$P^\dagger(\widehat{\tau}_m \leq T) = P\left(c_{\alpha, m}^{-1} \max_{1 \leq k \leq T_m} \frac{d(k; m)}{\nu^*(k; m)} > 1\right) = 1,$$

by (27).  $\square$