Tests of time-invariance

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April 20, 2007

Abstract

Quantiles provide a comprehensive description of the properties of a variable and tracking changes in quantiles over time using signal extraction methods can be informative. It is shown here how stationarity tests can be generalized to test the null hypothesis that a particular quantile is constant over time by using weighted indicators. Corresponding tests based on expectiles are also proposed; these might be expected to be more powerful for distributions that are not heavy-tailed. Tests for changing dispersion and asymmetry may be based on contrasts between particular quantiles or expectiles. We report Monte Carlo experiments investigating the effectiveness of the proposed tests and then move on to consider how to test for relative time invariance, based on residuals from fitting a time-varying level or trend. Empirical examples, using stock returns and U.S. inflation, provide an indication of the practical importance of the tests.

KEYWORDS: Dispersion; expectiles; quantiles; skewness; stationarity tests; stochastic volatility, value at risk.

JEL classification: C12, C22

1 Introduction

Tests for serial correlation and stationarity are mostly applied to the level of a time series. For financial time series, movements in the variance are often of interest so serial correlation and stationarity tests are applied to squares or absolute values (possibly after first removing the mean). However, the distribution of a variable may display changes over time that are not captured by the level or variance. For example, there may be movements in skewness or the tails. Quantiles provide a more comprehensive description of the properties of a variable and tracking changes in quantiles over time can be informative. Indeed, lower tail quantiles are used to define Value at Risk (VaR), a measure that is of considerable importance in financial regulation; see, for example, Engle and Manganelli (2004). In a recent paper, De Rossi and Harvey (2006a) explain how time-varying quantiles may be estimated using state space signal extraction techniques. Figure 1, taken from that paper, shows quantiles, modeled as random walks, fitted to General Motors daily stock returns.[1] The present paper generalizes serial correlation and stationarity tests so as to test the null hypothesis that a particular quantile is constant over time. These tests are based on statistics constructed from quantile residuals coded as weighted indicators. The main concern is with stationarity tests, where the test statistics can be shown to have the usual (asymptotic) Cramér von Mises distribution under the null hypothesis. The proof is a generalization of that given by De Jong, Amsler and Schmidt (2006) for the indicator test based on the median.

De Jong, Amsler and Schmidt (2006) show that for most distributions with finite variance the indicator-based stationarity test is less powerful than the standard test based on residuals from the mean. We therefore seek to generalize the standard stationarity test to provide alternatives to the indicator tests for the constancy of quantiles. We do this by first noting that expectiles have been proposed as an alternative - or complement - to quantiles; see, for example, Newey and Powell (1987), Efron (1991) and, in a time series context, De Rossi and Harvey (2006b). We then define residuals based on expectiles and show that they can be used to construct stationarity tests with the usual asymptotic distribution. By matching up an expectile with a quantile, an expectile based test can be used to test the constancy of a specific quantile. We investigate the asymptotic properties of such tests and carry out Monte Carlo experiments to compare their performance with that

[1] The stock returns data used as illustrations are taken from Engle and Manganelli (2004). Their sample runs from April 7th, 1986, to April 7th, 1999, but the figure uese only the first 2000. The large (absolute) values near the beginning of figure 1 are associated with the great crash of 1987.

The histogram of the series from observation 501 to 2000 (avoiding the 1987 crash) shows heavy tails but no clear evidence of skewness. The excess kurtosis is 1.547 and the associated test statistic, distributed as χ^2_1 under normality, is 149.5. On the other hand, skewness is 0.039 with an associated test statistic of only 0.37.

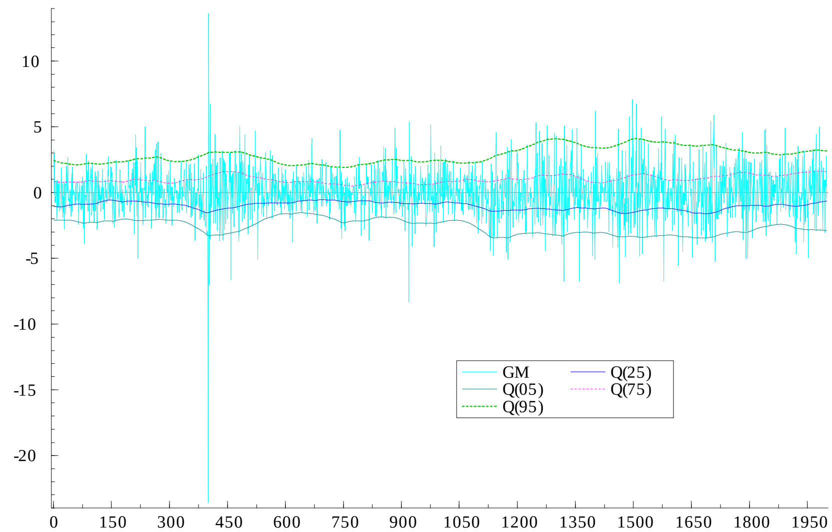

Figure 1: Quantiles fitted to GM returns

of quantile indicator tests. We also propose ways in which they may be made robust to heavy tails, a property that is a feature of quantile indicator tests.

If the level, as represented by the median, is constant, we can focus on other quantiles and contrasts between them. This is typically the case with stock returns, as in figure 1. Stationarity tests built around contrasts allow inferences to be made about possible time variation in dispersion and asymmetry. Multivariate tests can capture movements in different parts of the distribution. We report Monte Carlo experiments investigating the effectiveness of various new tests and comparing them with standard procedures and with the Box-Ljung tests based on quantile indicators that are proposed in Linton and Whang (2007). We then move on to consider how to test for relative time invariance, in other words whether the distribution around a possibly nonstationary level is constant over time. The tests are based on residuals from fitting a time-varying mean or median. We discuss the implications of working with such residuals and investigate test performance by simulations. Monthly figures on the US rate of inflation are used as an illustration.

2 Quantiles and expectiles

We first review some standard results on quantiles and then give corresponding definitions and properties for expectiles. It is then shown how it is possible to switch from one to the other. The last three sub-sections introduce the weighted indicators and residuals that are used in the tests and investigates the extent to which the serial correlation in a Gaussian series carries over to them.

2.1 Quantiles

Let $\xi(\tau)$ - or, when there is no risk of confusion, ξ - denote the $\tau - th$ quantile. The probability that an observation is less than $\xi(\tau)$ is τ, where $0 < \tau < 1$. Given a set of T observations, $y_t, t = 1, .., T$, the sample quantile, $\widetilde{\xi}(\tau)$, can be obtained by sorting the observations in ascending order. However, it is also given as the solution to minimizing

$$S_\tau = \sum_{y_t < \xi} (\tau - 1)(y_t - \xi) + \sum_{y_t \geq \xi} \tau(y_t - \xi) = \sum_t (\tau - I(y_t - \xi < 0))(y_t - \xi)$$

with respect to ξ, where $I(.)$ is the indicator function. Differentiating (minus) S_τ at all points where this is possible gives

$$\sum_{t=1}^{T} IQ(y_t - \xi(\tau)),$$

where

$$IQ(y_t - \xi(\tau)) = \begin{cases} \tau - 1, & if \quad y_t < \xi(\tau) \\ \tau, & if \quad y_t > \xi(\tau) \end{cases} \tag{1}$$

defines the *quantile indicator function*. Since the quantile indicator function is not continuous at 0, $IQ(0)$ is not determined. However, we will constrain it to lie in the range $[\tau - 1, \tau]$.

The sample quantile, $\widetilde{\xi}(\tau)$, is such that, if $T\tau$ is an integer, there are $T\tau$ observations below the quantile and $T(1 - \tau)$ above[2]. In this case any value

[2] In quantile regression, the quantile, $\xi_t(\tau)$, corresponding to the $t - th$ observation is a linear function of explanatory variables, $\mathbf{x}_t$, that is $\xi_t = \mathbf{x}_t'\boldsymbol{\beta}$. Estimates of the parameter vector, $\boldsymbol{\beta}$, may be computed by linear programming as described in Koenker (2005).

of $\widetilde{\xi}$ between the $T\tau - th$ smallest observation and the one immediately above will make $\sum IQ(y_t - \widetilde{\xi}) = 0$. If $T\tau$ is not an integer, $\widetilde{\xi}$ will coincide with one observation. This observation is the one for which $\sum IQ(y_t - \widetilde{\xi})$ changes sign. These statements need to be modified slightly if several observations take the same value and coincide with $\widetilde{\xi}$. Taking this point on board, a general definition of a sample $\tau-$quantile is a point such that the number of observations smaller, that is $y_t < \widetilde{\xi}$, is no more than $[T\tau]$ while the number greater is no more than $[T(1-\tau)]$.

2.2 Expectiles

Expectiles offer an alternative class of location measures to quantiles. The $\omega-$expectile, $\mu(\omega)$, is obtained by minimizing

$$\begin{aligned} S_\omega &= \sum_{y_t < \mu(\omega)} (1-\omega)(y_t - \mu(\omega))^2 + \sum_{y_t \geq \mu(\omega)} \omega (y_t - \mu(\omega))^2 \\ &= \sum_t |\omega - I(y_t - \mu(\omega) < 0)| \; (y_t - \mu(\omega))^2, \qquad 0 < \omega < 1 \end{aligned}$$

with respect to $\mu(\omega)$. (In a regression context this is known as asymmetric least squares.) Differentiating S_ω and dividing by minus two gives

$$\sum_{t=1}^{T} IE(y_t - \mu(\omega)) \tag{2}$$

where

$$IE(y_t - \mu(\omega)) = |\omega - I(y_t - \mu(\omega) < 0)| \; (y_t - \mu(\omega)), \qquad t = 1, ..., T. \tag{3}$$

There is no problem with defining $IE(0):$ since (3) is continuous, $IE(0) = 0$. The sample expectile, $\widetilde{\mu}(\omega)$, is the value of $\mu(\omega)$ that makes (2) equal to zero. Setting $\omega = 0.5$ gives the mean, that is $\widetilde{\mu}(0.5) = \overline{y}$. For other ω it is necessary to iterate to find $\widetilde{\mu}(\omega)$.

The population expression corresponding to (2) is

$$(1-\omega) \int_{-\infty}^{\mu(\omega)} (y - \mu(\omega))\, dF(y) + \omega \int_{\mu(\omega)}^{\infty} (y - \mu(\omega))\, dF(y), \tag{4}$$

where $F(y)$ is the cdf of y. Newey and Powell (1987) show, in their theorem 1, that when (4) is set to zero, a unique solution exists if $E(y) = \mu(0.5)$ exists.

2.3 Comparing quantiles and expectiles

As noted by Efron (1991), counting the number of observations smaller than an estimated expectile gives the value of τ needed for it to be interpreted as a quantile. Conversely, to find the value of ω for an expectile that makes it equal to the $\tau-$quantile, we note that a simple re-arrangement of (2) when it is set to zero and $\widetilde{\mu}(\omega)$ is set to $\widetilde{\xi}(\tau)$ gives

$$\widetilde{\omega} = \frac{\sum_{y_t<\xi(\tau)}(y_t - \widetilde{\xi}(\tau))}{\sum_{y_t<\widetilde{\xi}(\tau)}(y_t - \widetilde{\xi}(\tau)) - \sum_{y_t\geq\widetilde{\xi}(\tau)}(y_t - \widetilde{\xi}(\tau))} \tag{5}$$

The population analogue of (5) can be written

$$\omega = \frac{\int_{-\infty}^{\xi(\tau)}(y-\xi(\tau))dF(y)}{\int_{-\infty}^{\xi(\tau)}(y-\xi(\tau))dF(y) - \int_{\xi(\tau)}^{\infty}(y-\xi(\tau))dF(y)} = \frac{\left[\int_{-\infty}^{\xi(\tau)} ydF(y)\right] - \tau\xi(\tau)}{2\left[\int_{-\infty}^{\xi(\tau)} ydF(y)\right] - \mu + (1-2\tau)\xi(\tau)}. \tag{6}$$

When $\tau = 0.5$ and the distribution is symmetric, $\omega = 0.5$. More generally the conditions under which $\tau = \omega$ are very restrictive; see Efron (1991). For a normal distribution

$$\omega = \frac{(2\pi)^{-1/2}\exp(-\xi(\tau)^2/2) + \tau\xi(\tau)}{(2/\pi)^{1/2}\exp(-\xi(\tau)^2/2) + (2\tau-1)\xi(\tau)} \tag{7}$$

where $\xi(\tau)$ is the $\tau-$quantile from $N(0,1)$. Using this formula we find that for $\tau = 0.01, 0.05,\ 0.10, 0.15,\ 0.25,\ 0.330\ (0.331)$ the corresponding values of ω are 0.00146, 0.0124, 0.0344, 0.0652, 0.153, 0.248 (0.250). Formulae relating ω and τ can be obtained for other distributions such as uniform and Laplace. In all these cases setting $\mu(\omega) = \xi(\tau)$ implies $\omega < \tau$ for $\tau < 0.5$ (and $\omega > \tau$ for $\tau > 0.5$).

2.4 Quantics and expectics

The $\tau-$*quantile indicator* at time t is $IQ(y_t - \xi(\tau)), t = 1, ..., T$, where $IQ(.)$ is as defined in (1). The sample $\tau-$quantile indicator, or $\tau-$*quantic*, at time t

is $IQ(y_t - \widetilde{\xi}(\tau)), t = 1, ..., T$. The quantics sum to zero if, when an observation is equal to the sample quantile, the corresponding quantic is assigned a value between $\tau - 1$ and τ to ensure that this is the case. If $T\tau$ is an integer, the sample quantile will normally lie at some point between observations, in which case the mean of the quantics is automatically zero, while the sample variance is equal to $\tau(1-\tau)$.

The $\omega-$expectile residual at time t is $IE(y_t - \widetilde{\mu}(\omega)), t = 1, ..., T$, where $IE(.)$ was defined in (3). We call these $\omega-$*expectics*, even though they are only partly dependent on an indicator. An expectic may also be based on the $\tau - th$ quantile, in which case we will call it a $\tau-$*expectic* and write it as

$$IE(y_t - \widetilde{\xi}(\tau)) = \left|\widetilde{\omega} - I(y_t - \widetilde{\xi}(\tau))\right| (y_t - \widetilde{\xi}(\tau)), \qquad t = 1, ..., T, \tag{8}$$

where $\widetilde{\omega}$ is defined by (5). Sample expectics always sum to zero by construction.

2.5 Properties of population quantics and expectics

The properties of the $\tau-$quantile indicator, or *population quantic*, and the *population expectic*, (3), are needed for some of the results that follow later in the paper.

Assumption 1. $\xi(\tau)$ is the unique population $\tau-$quantile and y has a continuous positive density in the neighbourhood of $\xi(\tau)$.

Proposition 1 *If assumption 1 holds, then the population $\tau-$quantic has a mean of zero and a variance of $\tau(1-\tau)$.*

Assumption 2. The variance of y is positive and finite.

Proposition 2 *If assumption 2 holds, the population $\tau-$ and $\omega-$expectics have a mean of zero and a finite variance.*

The variance of the population $\tau-$expectic is

$$\begin{aligned} Var(IE(y_t - \xi(\tau))) &= (\omega-1)^2 E[(y-\xi(\tau))^2 \mid y < \xi(\tau)] \Pr(y < \xi(\tau)) \\ +\omega^2 E[(y-\xi(\tau))^2 &\mid y \geq \xi(\tau)] \Pr(y \geq \xi(\tau)) \end{aligned}$$

$$= \tau(\omega-1)^2 E[(y-\xi(\tau))^2 \mid y < \xi(\tau)] + \omega^2(1-\tau)E[(y-\xi(\tau))^2 \mid y \geq \xi(\tau)].$$

2.6 Serial correlation

We now investigate the properties of the population quantics and expectics for stationary processes.

Proposition 3 *If y_t is strictly stationary, then the population quantics are both strictly stationary and covariance stationary since their moments always exist. If y_t is covariance stationary (all autocovariances are finite), the population quantics and expectics are covariance stationary.*

Transforming to indicators will tend to weaken the correlation structure of a covariance stationary process. Any autocorrelation for the τ-th quantile indicator is related to the corresponding autocorrelation of y_t, denoted ρ_y, by the formula

$$\rho_\tau(\rho_y) = \frac{\tau L\left(\xi(\tau);\rho_y\right) + (1-\tau)L\left(-\xi(\tau);\rho_y\right)}{\tau(1-\tau)} - 1 \tag{9}$$

where $L(\xi(\tau);\rho) = \Pr\left(y_t > \xi(\tau),\ y_{t-k} > \xi(\tau);\rho\right)$ with k any integer. The proof is in appendix A. Note that $\rho_\tau(0) = 0$, as it should be. For a Gaussian process, the formula for the median simplifies to

$$\rho_{0.5} = 2\pi^{-1}\arcsin\left(\rho_y\right)$$

since, as shown in Abramowitz and Stegun (1964, p 936-9), $L(0;\rho) = 0.25 + (1/2\pi)\arcsin(\rho)$. In general, however, $L(h;\rho)$ must be obtained numerically.

Table 1 shows how the correlations of the quantile indicators and population expectics vary with ρ and τ for a Gaussian process. Although some progress can be made in finding analytic expressions, we found it simpler to obtain figures by simulation[3]. The values of ω correspond to τ were obtained from (7). For both quantics and expectics, the correlation weakens as we move towards the tails, but the effect is less pronounced for expectics. There is an interesting asymmetry in that negative correlations weaken much more than positive ones; as can be seen, $\left|\rho_\tau(-\rho_y)\right| < \rho_\tau(\rho_y)$ for $\rho_y > 0$ and $\tau \neq 0.5$.

[3] To be precise, data were generated from AR(1) processes that gave the required correlations using the Ox random number generator and 10,000 replications; see Doornik (1999).

3 Models of Time-Varying Quantiles and Expectiles

The signal extraction procedure for time-varying quantiles set out in De Rossi and Harvey (2006a) is based on the model

$$y_t = \xi_t(\tau) + \varepsilon_t(\tau), \qquad t = 1, ..., T, \tag{10}$$

where $\Pr(y_t < \xi_t(\tau)) = \Pr(\varepsilon_t(\tau)) < 0 = \tau$ with $0 < \tau < 1$. It is assumed that the $\varepsilon_t(\tau)'s$ are independently drawn from an asymmetric double exponential distribution, but this is simply a convenient device that leads to the appropriate criterion function for what is essentially a nonparametric estimator.

A suitable model for a stationary time-varying quantile is the first-order autoregressive process

$$\xi_t(\tau) = (1 - \phi_\tau)\xi_\tau^\dagger + \phi_\tau \xi_{t-1}(\tau) + \eta_t(\tau), \qquad |\phi_\tau| < 1, \qquad t = 1, ..., T, \tag{11}$$

where $\eta_t(\tau)$ is normally and independently distributed with mean zero and variance $\sigma^2_{\eta(\tau)}$, that is $\eta_t(\tau) \backsim NID(0, \sigma^2_{\eta(\tau)})$, and is independent of $\varepsilon_t(\tau)$, ϕ_τ is the autoregressive parameter and $\xi_\tau^\dagger$ is the unconditional mean of $\xi_t(\tau)$. The random walk quantile is obtained by setting $\phi_\tau = 1$ so that

$$\xi_t(\tau) = \xi_{t-1}(\tau) + \eta_t(\tau), \qquad t = 2, ..., T. \tag{12}$$

The initial value, $\xi_1(\tau)$, is assumed to be drawn from a $N(0, \kappa)$ distribution. Letting $\kappa \to \infty$ gives a diffuse prior; see Durbin and Koopman (2001). A nonstationary quantile can also be modelled by a local linear trend with a specification that results in the smoothed estimates being a cubic spline. The specification is close to an integrated random walk.

The signal extraction procedure developed by De Rossi and Harvey (2006b) for time-varying expectiles is also based on a signal plus noise model, but with the noise treated as coming from an asymmetric normal distribution. The algorithm is a simple iterative procedure based on the Kalman filter and smoother. It is more straightforward than for quantiles because there are no solutions where the estimated signal coincides with an observation. As with quantiles, the parameters can be estimated by cross-validation.

4 Generalized stationarity tests

4.1 Tests based on quantics: IQ tests

A test of the null hypothesis that a quantile is constant may be based on the quantics, $IQ(y_t - \widetilde{\xi}(\tau))$, $t =, ..., T$. If the alternative hypothesis is that $\xi_t(\tau)$ follows a random walk, a modified version of the basic stationarity test of Nyblom and Mäkeläinen (1983) is appropriate. The test statistic of Nyblom and Mäkeläinen uses residuals from a sample mean and its asymptotic distribution is a Cramér-von Mises (CvM) distribution; the 1%, 5% and 10% critical values are 0.743, 0.461 and 0.347 respectively. Nyblom and Harvey (2001) show that the test has high power against an integrated random walk while Harvey and Streibel (1998) note that it also has a locally best invariant interpretation as a test of constancy against a highly persistent stationary AR(1) process. This makes a modified version of the test entirely appropriate for the kind of situation we have in mind for time-varying quantiles; see our figure 1 and also figure 9 in Linton and Whang (2007).

Proposition 4 *Under the null hypothesis that the observations are IID, and assumption 1 is satisfied, the asymptotic distribution of the stationarity test statistic*

$$\eta_\tau(Q) = \frac{1}{T^2\tau(1-\tau)} \sum_{t=1}^{T} \left(\sum_{i=1}^{t} IQ(y_i - \widetilde{\xi}(\tau)) \right)^2 \tag{13}$$

is the CvM distribution.

The result comes about because the population quantics are IID, but estimating the quantile introduces a constraint. De Jong, Amsler and Schmidt (2006) - hereafter DAS - give a rigorous proof for $\tau = 0.5$. Carrying over their assumptions to all quantiles we find that, for IID observations, all that is required is for assumption 1 to hold. A formal proof can be found in appendix C. We show later that the test statistic is $O_p(T)$ when the series contains a random walk component.

4.2 Tests based on expectics: IE tests

DAS show that for most distributions with finite variance the indicator-based stationarity test is less powerful than the standard test based on residuals from the mean. This leads us to investigate stationarity tests based on expectics.

Proposition 5 *Under the null hypothesis that the observations are IID, and assumption 2 is satisfied, the asymptotic distribution of the statistic*

$$\eta_\omega(E) = \frac{\sum_{t=1}^{T}\left(\sum_{j=1}^{t} IE(y_j - \widetilde{\mu}\,(\omega))\right)^2}{T^2\widehat{\sigma}^2}, \tag{14}$$

where $\widehat{\sigma}^2 = T^{-1}\sum_{t=1}^{T} IE(y_t - \widetilde{\mu}\,(\omega))^2$, *is the* CvM *distribution.*

The proof in appendix D is based on the assumptions for theorem 3 in Newey and Powell (1987, p 826-8) which establishes the asymptotic distribution of the asymmetric least squares estimator. These assumptions require slightly higher than fourth moments. However, we conjecture that, for our purposes, the existence of second moments will suffice.

It will often be the case that we want to use expectics to test the constancy of a particular quantile, in which case we use the τ-expectics, defined in (8), to form a statistic analogous to (14). Let this statistic be denoted as $\eta_\tau(E)$.

Proposition 6 *Under the null hypothesis that the observations are IID, and assumptions 1 and 2 are satisfied, the asymptotic distribution of the statistic* $\eta_\tau(E)$ *is* CvM.

4.3 Robust expectics

Expectile tests could be made robust by Winsorizing the observations, that is observations greater in absolute value than a tail quantile are set equal to that quantile. A simple procedure is as follows:

1) estimate the $\tau-$quantile;
2) Winsorize the observations using a robust measure of range;
3) find ω and carry out the expectic test.

The precise way in which Winsorizing is carried out may vary with τ.

4.4 Serial correlation

Assuming strict stationarity (and a mixing condition), De Jong, Amsler and Schmidt (2006) modify the indicator test for $\tau = 0.5$ along the lines of Kwiatkowski *et al* (1992) so that it has the CvM distribution under the null. They call this the IKPSS test. More generally, for any τ, we can replace the term $\tau(1-\tau)$ in (13) by the nonparametric estimator of the long run variance

$$\begin{aligned} \hat{\sigma}_L^2\,(Q;m) &= \hat{\gamma}(0) + 2\sum_{j=1}^{m} k\,(j/m)\,\hat{\gamma}_\tau(j) \\ &= T^{-1}\sum_{j=-m}^{m} k\,(j/m)\,\sum_{t=1}^{T-j} IQ\left(y_t - \widetilde{\xi}(\tau)\right) IQ\left(y_{t+j} - \widetilde{\xi}(\tau)\right) \end{aligned} \tag{15}$$

where $k\,(j/m)$ is a weighting function, such as the Bartlett window, $1 - |j|/(m+1)$, and $\widehat{\gamma}_\tau(j)$ is the sample autocovariance of the quantics at lag j. Alternative options for the kernel $k(.,.)$ and guidelines for choosing lag truncation parameter, m, may be found in Andrews (1991). When modified in this way we will write the test statistic as $\eta_\tau(Q;m)$. Note that if $\widetilde{\xi}(\tau)$ does not coincide with an observation or observations, setting $m = 0$ gives exactly the same result as (13) since the sample variance, $\hat{\gamma}(0)$, is $\tau(1-\tau)$.

Under the null hypothesis that the observations are strictly stationary, that assumption 1 holds and that assumption 2 in DAS holds, the asymptotic distribution of the statistic $\eta_\tau(Q;m)$ is CvM. DAS prove that $\eta_{0.5}(Q;m)$ has the CvM distribution asymptotically given (in addition to assumption 1) strong mixing conditions, the existence of the long-run variance of the quantile indicators and regularity conditions on the kernel and the behaviour of m as T goes to infinity. It is not difficult to extend the result to all quantiles.

The expectile stationarity test statistic with the nonparametric correction for serial correlation, denoted $\eta_\omega(E;m)$, is obtained after replacing $\widehat{\sigma}^2$ by a consistent estimate of the long run variance of $IE\,(y_t - \mu(\tau))$, paralleling (15). When $\tau = 0.5$ it corresponds to the standard KPSS test statistic. The $\eta_\tau(E;m)$ test is similarly constructed. Both the $\eta_\omega(E;m)$ and $\eta_\tau(E;m)$ statistics are asymptotically distributed as CvM.

4.5 Financial time series

Test statistics for the daily GM returns of figure 1 are shown in table 2 for a range of quantiles and various values of m in the nonparametric estimator of the long-run variance.

All the quantic stationarity tests, including the median, reject the null hypothesis of time invariance at the 1% level of significance when $m = 0$.

However, the test statistic for $\tau = 0.01$ (not reported in the table) is only 0.289 which is not significant at the 10% level. This is consistent with the small estimate obtained for the signal-noise ratio. Similarly the statistic is 0.354 for $\tau = 0.99$ which is only just significant at the 10% level. The values of the test statistics fall as m increases, but they all still reject at the 5% level of significance. In a situation where independence is a reasonable assumption under the null hypothesis, this fall is to be expected given what is known about the estimator of the long-run variance in other contexts; see, in particular, the Monte Carlo results for the median indicator test in DAS.

The expectic tests, $\eta_\tau(E)$, do not reject for $\tau \leq 0.5$, except in one case where the test just rejects at the 10% level. There are more rejections for $\tau > 0.5$, but rarely at the 1% level. The outlying values around the time of the stock market crash of 1987 probably account for the low power. Robustifying the tests, by setting the observations less than the 0.01 quantile or greater than the 0.99 quantile equal to the corresponding quantile (Winsorizing), leads to far more rejections, but in most cases the test statistics are still less than the corresponding quantile statistics. This is perhaps a little surprising.

The Box-Ljung statistics are based on quantiles and here m is the number of lags in the autocorrelations, except that the column headed 0 refers to one lag. There are clear rejections in the tails, particularly $\tau = 0.05$ and 0.1, but not at the quartiles. The 5% critical values for 1,8,17 and 25 lags are 3.84, 15.5, 27.6 and 37.7 respectively.

The results indicating the non-constancy of the median are unexpected, particularly as there is no evidence for a non-stationary mean (and the Box-Ljung quantic test does not reject). De Jong et al (2006) also reject constancy of the median for some weekly exchange rates. A plot of the estimated median for GM shows that it is close to zero most of the time apart from a spell near the beginning of the series and a shorter one near the end.

The other two series in Engle and Manganelli (2004) are IBM and S & P 500. The results are somewhat similar, though in S & P the quantic statistics for $\tau = 0.5$ are much smaller (but still reject at the 5% level of significance).

4.6 Joint test for the time invariance of a distribution

A joint test to see if a group of N quantiles show evidence of changing over time can be based on a generalisation of (13), namely

$$\eta_\tau(Q;N) = T^{-2}\sum_{t=1}^{T}\left[\sum_{i=1}^{t}\mathbf{IQ}_i\right]'\mathbf{\Omega}^{-1}\left[\sum_{i=1}^{t}\mathbf{IQ}_i\right], \tag{16}$$

where the $j-th$ element of the $N\times 1$ vector $\mathbf{IQ}_i$ is $IQ(y_i-\widetilde{\xi}(\tau_j))$ and the $jk-th$ element of the $N\times N$ covariance matrix $\mathbf{\Omega}$ is $\tau_j(1-\tau_k)$ for $\tau_k>\tau_j$; see appendix B. Under the null hypothesis of IID observations, the limiting distribution of $\eta_\tau(Q;N)$ is Cramér-von Mises with N degrees of freedom, denoted $CvM(N)$. (The proof is straightforward given the proof for tests based on a single quantile - assumption 1 must hold for all the quantiles). A nonparametric modification may be made to deal with serial correlation, as in Nyblom and Harvey (2000), to give a statistic denoted $\eta_\tau(Q;N,m)$.

If the median is assumed to be constant, as might be the case with stock returns, a test on the 5,25,75 and 95 % quantiles might be regarded as a general test for a change in the distribution over time. Thus for GM the $\eta_\tau(Q;4)$ test statistic for 0.05, 0.25, 0.75 and 0.95 is 5.451. The 1% CV is 1.623 (5% is 1.237). This is a convincing rejection.

Expectics can be used to construct joint test statistics in the same way. The $\eta_\tau(E;4)$ test statistic for 0.05, 0.25, 0.75 and 0.95 is 4.420, rising to 4.470 if robustified.

5 Performance of tests

We now compare the performance of quantic and expectic tests for the model

$$y_t=\mu_t+\varepsilon_t, \qquad \mu_t=\mu_{t-1}+\eta_t, \qquad t=1,...,T, \tag{17}$$

where ε_t is strictly stationary with zero median, while $\eta_t \backsim NID(0,\sigma_\eta^2)$. Since the distribution of ε_t is time invariant, the (time-varying) quantiles and expectiles[4] of y_t are a constant distance apart. The null hypothesis for all tests corresponds to $\sigma_\eta^2=0$.

For the quantic tests, the model can be regarded as the same as (10) with μ_t being the median, $\xi(0.5)$, and $\xi_t(\tau)$ being the median plus a constant,

[4]Defined as the corresponding quantile or expectile of ε_t plus μ_t.

while $\varepsilon_t(\tau)$ is equal to ε_t, minus the same constant. Thus in terms of (12), $\sigma^2_{\eta(\tau)} = \sigma^2_\eta$, and, for a given τ, we could think of the stationarity test as testing $H_0 : \sigma^2_{\eta(\tau)} = 0$ against $H_1 : \sigma^2_{\eta(\tau)} > 0$. The probability of rejection (possibly size corrected) can then be formally interpreted as power.

For a Gaussian series the quantic test for $\tau = 0.5$ is less powerful than the standard stationarity test. This is hardly surprising since the latter is LBI and it is shown in DAS that the asymptotic distributions under the alternative are different. The extent of the difference can be seen in the simulations in DAS. On the other hand heavy-tails can favour the indicator test and the interesting question is then whether some kind of robustification can swing the balance back to a residual based test. We seek to investigate this matter for a range of τ.

Linton and Whang (2007) study the quantilogram, the plot of the autocorrelations calculated from the quantics, and investigate the performance of Box-Ljung tests for a range of τ. Assuming symmetry, they show that the statistic constructed from $\tau-$quantics, which we denote as $BL_\tau(P)$, has the usual asymptotic χ^2_P distribution, where P is the number of lags in the autocorrelations. They find a fall in power towards the tails, that is as τ approaches zero and one. One could extend their ideas by constructing the 'expectilogram' and Box-Ljung statistics from the expectics. Our main interest, though, is in seeing how the expectic stationarity tests behave as we move towards the tails. Note that, for the model under consideration, the probability of rejection with the $BL_\tau(P)$ test is likely to be somewhat below that of the corresponding stationarity test. From Harvey and Streibel (1998), the stationarity test statistic can be written $\eta = -T^{-1}\sum_k kr_k$,where r_k is the lag k sample autocorrelation, showing the importance of high order autocorrelations.

5.1 Asymptotic distribution under the alternative hypothesis

The limiting representation in DAS can be generalized to τ and ω other than 0.5. In appendix E we show that, if the observations are IID plus a random walk component, then

$$T^{-1}\eta_\tau(Q) \xrightarrow{d} \frac{1}{\tau(1-\tau)}\int_0^1 \left(\int_0^\alpha \{\tau - 0.5 + 0.5sgn(W(x) - \overline{\xi}(\tau))\}dx\right)^2 d\alpha, \tag{18}$$

where $\overline{\xi}(\tau)$ denotes a random variable with a distribution equal to the limiting distribution of $\widetilde{\xi}(\tau)$, after appropriate scaling. This expression shows that the test statistic is $O_p(T)$ under the random walk alternative and hence the test is consistent.

For the expectic tests,

$$T^{-1}\eta_\omega(E) \xrightarrow{d} \frac{\int_0^1 \left(\int_0^\alpha \{0.5 + (\omega - 0.5)sgn(W(x) - \overline{\mu}(\omega))\}(W(x) - \overline{\mu}(\omega))dx\right)^2 d\alpha}{\int_0^1 \{0.5 + (\omega - 0.5)sgn(W(x) - \overline{\mu}(\omega))\}^2(W(x) - \overline{\mu}(\omega))^2 dx}, \tag{19}$$

where $\overline{\mu}(\omega)$ is defined in a similar way to $\overline{\xi}(\tau)$. As can be seen, the limiting distribution of $T^{-1}\eta_\omega(E)$ is quite different from that of $T^{-1}\eta_\tau(Q)$, but is still of $O_p(1)$. For $\omega = 0.5$ we have $\overline{\mu}(0.5) \stackrel{d}{=} \int_0^1 W(u)du$, thereby yielding the limiting distribution of the Nyblom-Mäkeläinen statistic under the alternative hypothesis, that is

$$T^{-1}\eta_{0.5}(E) \xrightarrow{d} \frac{\int_0^1 \left(\int_0^\alpha (W(x) - \int_0^1 W(u)du)dx\right)^2 d\alpha}{\int_0^1 \left(W(x) - \int_0^1 W(u)du\right)^2 dx},$$

as in Nyblom and Harvey (2000, theorem 3).

5.2 Monte Carlo experiments: IID observations plus a random walk under the alternative

Here we present Monte-Carlo evidence on the size and power properties of the quantic and expectic tests, based on the statistics $\eta_\tau(Q)$ and $\eta_\tau(E)$, when the underlying data generating process is a random walk plus noise. To shed light on the behaviour of the tests with respect to fat tails, three cases are considered for the noise process[5], ε_t: (i) $N(0,1)$, (ii) t-distribution with three degrees of freedom, (iii) Cauchy distribution. In all cases the random walk component is driven by a Gaussian disturbance with variance equal to $\sigma_\eta^2 = c^2/T^2$, for $c = 0, 2.5, 5, 10, 25, 50$. The quantic and expectic tests are compared with the quantic Box-Ljung statistic, introduced by Linton and Whang (2007). Tables 3,4 and 5 contain the rejection frequencies of the tests for the case $T = 200$ (and 50,000 replications); additional results for

[5] DAS put the same distribution on the disturbance driving the random walk.

other sample sizes are available upon request. Because of symmetry, we only present results for $\tau \leq 0.5$.

The main conclusions for the Gaussian case, reported in Table 3, are as follows.

(a) Except when $\tau = 0.01$, the empirical sizes of the quantic and expectic tests are very close to the nominal 5%; on the other hand a test based only on the first autocorrelation coefficient, denoted $BL(1)$, does not appear to control size very well in the tails.

(b) As expected, power[6] increases with c for all tests.

(c) Power falls off as we move towards the tails.

(d) The IE test is more powerful than IQ, except[7] when $\tau = 0.01$ (when the IE test is very undersized).

(e) The IQ stationarity test is generally far more powerful than the Box-Ljung test of Linton-Whang with 5 autocorrelations, $BL(5)$, except when $c = 50$ when the rejection probabilities are close to one (using 10 autocorrelations would give a little more power, with a gain of the order of 1% to 5%). $BL(1)$ is clearly dominated.

The IKPSS test (with $m = 0$) of DAS corresponds to our IQ test for $\tau = 0.5$ and the simulation results in their table 3 for $T = 200$ are consistent with the rejection frequencies of our table 1 (note that their $\lambda = 0.01$ would imply $c = 20$).

The results for $t(3)$ errors are reported in Table 4. The IQ and IE tests still appear to control size well, except in the tails when they are somewhat undersized, particularly IE. The main difference[8] with table 1 is that IQ is now more powerful than IE. (This remains true even with size correction). Finally, when errors are taken from a Cauchy distribution the IQ test still appears to control size well and display good power properties at least in the central quantiles; see table 5. The expectic test is not valid in this case, and is undersized. (The extent of the undersizing for $\tau = 0.5$ is the same as reported by DAS). The rejection frequencies under the alternative are very low.

[6] Power here means the proportion of rejections; no attempt has been made to correct for size.

[7] It can be shown - see De Rossi and Harvey (2006b) - that the estimator of the expectile is less efficient than the quantile for $\tau = 0.01$. For $\tau = 0.05$ the efficiency is roughly the same.

[8] Power is slightly lower than in Gaussian case, but the variance of the t_3 distribution is three and so the signal-noise ratios are smaller.

5.3 Serial correlation

DAS derive the asymptotic distribution of the IKPSS statistic under the alternative hypothesis and show that $(m/T)\eta(Q;m) = O_p(1)$. We conjecture that a similar result holds for tests based on all quantics. However, the asymptotics appear to give little insight into the relative power of the various quantic and expectic tests and so we focus on simulation experiments.

Correcting the IQ and IE tests to allow for serially correlated errors implies the usual trade-off between size and power, as shown in Kwiatkowski *et al.* (1992) for the standard KPSS test. Table 6 presents Monte Carlo evidence for a data generating process[9] made up of a random walk plus an AR(1) process with parameter $\phi = 0.5$. The disturbances are either $N(0,1)$ or scaled[10] Cauchy, with the random walk component scaled by a factor $\sigma_\eta = q = 0, .0001^{.5}, .001^{.5}, .01^{.5}, .1^{.5}, 1$. The sample size is $T = 200$ with the lag truncation parameter $m = int\left(4\left(T/100\right)^{1/4}\right) = 5$. With this choice of m, both tests are generally oversized, the worst case being the IQ test with Cauchy errors. Under Gaussianity, the rejection frequencies under the alternative largely mirror the IID case, with the IE test being more powerful. The results for the Cauchy distribution are rather surprising, however, since although the IE test is not valid, it is less oversized than the IQ test and the rejection frequencies are much higher than in the IID case.

Even the standard KPSS test is oversized with the above DGP. The oversizing of all tests becomes greater as ϕ moves towards one, though since the aim is to detect slowly near persistent movements this is not necessarily a bad thing.

Table 7 shows result for the stationary component following an MA(1) with negative first-order correlation of -0.4 (the parameter θ is -0.8). In contrast to the previous table the tests are now undersized, particularly for expectics. Nevertheless, in the Gaussian case, the IE test dominates the IQ test even for the smallest q considered. With (scaled) Cauchy disturbances, the rejection frequencies are well below those of the IQ test except when q is high.

In section 2 we showed that serial correlation tends to fall as τ moves

[9] The experimental design is as in DAS and our results for $\tau = 0.5$ correspond to those contained in their table 1 (lower part) and in their table 4 (except that they present size-adjusted power, while we give rejection frequencies).

[10] "Scaled Cauchy" means a Cauchy random variable multiplied by 0.1, so that the noise is not too small or large and hence rejection frequencies take meaningful values.

towards zero or one. This suggests that it might be useful to adopt a rule that gives lower values of m for such values.

5.4 Multivariate tests

Simulation evidence (not reported here) shows that there is no power gain from considering many quantiles (with respect to just considering the median) when the data is generated from a random walk plus noise. This is to be expected since all the quantiles and expectiles follow exactly the same path, differing only by a constant. On the other hand the power does not appear to fall as more tests are combined. Thus a failure to reject can be confidently interpreted as time invariance rather than low power against a time series that is nonstationary in the level.

To illustrate, consider the example from De Rossi and Harvey (2006a) where 500 observations were generated from a scale mixture of two Gaussian distributions with time-varying weights and variances chosen so as to keep the overall variance constant over time. Figure 2 shows the absolute values of the series and 98%, 90% and 50% quantiles, corresponding to 1%, 5% and 25% in the original series. The 98% and 90% quantiles diverge, reflecting the changing behaviour in the tails. The IQ test statistics for $\tau = .98$ and 0.9 were .214 and .594 respectively, so only the second rejects at the 5% level, though not at the 1% level. The joint test statistic is 1.265, which clearly rejects at the 1% level (the 5% critical value is .748, while the 1% is 1.074). The reason is that the correlation between the two quantics under the null is 0.44 but in this series the quantiles head off in different directions.

Adding the 50% quantics gives a joint test of 1.633. The individual test statistic for the 50% quantics is 0.311; this is not significant at the 10% level, which is hardly surprising given that the 50% quantile shows little movement. However, incorporating the 50% quantics into the joint test appears to have little adverse effect since the 1% critical value is 1.359.

6 Contrasts

A quantile contrast is a linear combination of quantiles. The contrasts between complementary quantiles, that is

$$D_{\xi}(\tau) = \xi_t(1-\tau) - \xi_t(\tau), \qquad \tau < 0.5, \qquad t = 1, .., T \tag{20}$$

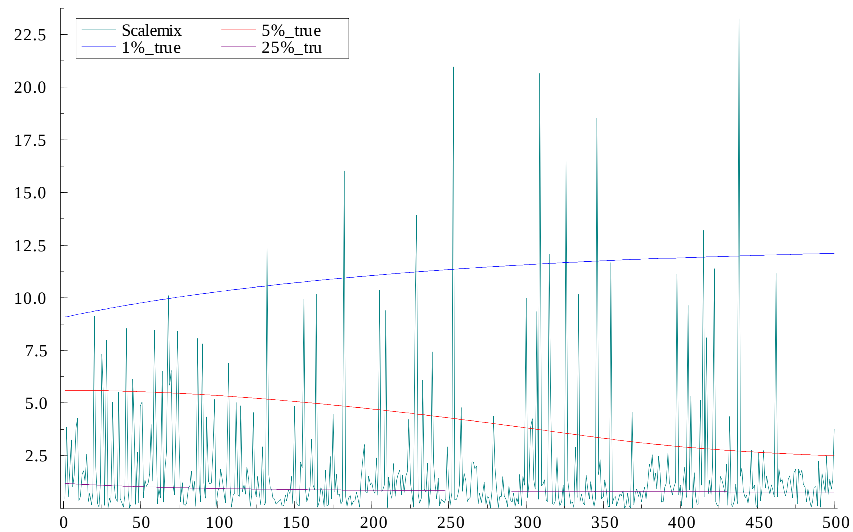

Figure 2: A tail of two quantics

yield measures of dispersion. For a symmetric distribution

$$S_{\xi}(\tau) = \xi_t(\tau) + \xi_t(1-\tau) - 2\xi_t(0.5) \tag{21}$$

is zero for all $t = 1, .., T$, so a plot of this contrast shows how the asymmetry captured by the complementary quantiles, $\xi_t(\tau)$ and $\xi_t(1-\tau)$, changes over time. Examples are given in De Rossi and Harvey (2006a).

A test based on a *quantic contrast*,

$$a.IQ(y_t - \widetilde{\xi}(\tau_1)) + b.IQ(y_t - \widetilde{\xi}(\tau_2)), \qquad t = 1, ..., T,$$

where a and b are constants, can be useful in pointing to specific departures from a time-invariant distribution. The corresponding population quantile indicator contrast has a mean of zero and a variance[11] that can be obtained by using the formula

$$cov(IQ(y_t - \xi(\tau_1)), IQ(y_t - \xi(\tau_2)) = \tau_1(1-\tau_2), \qquad \tau_2 > \tau_1, \tag{22}$$

[11] If neither $\widetilde{\xi}(\tau)$ nor $\widetilde{\xi}(1-\tau)$ coincides with an observation or observations, the sample and population covariances are exactly the same, as are the variances.

derived in appendix B. Tests statistics analogous to $\eta_\tau(Q)$ and $\eta_\tau(Q;m)$ can be constructed from the sample quantic contrasts. Under the conditions of assumption 1 applied to each quantile, they have the CvM distribution when the observations are, respectively, IID or strictly stationary. Similar tests can be constructed using expectics.

6.1 Dispersion

A test of constant dispersion can be based on the complementary quantic contrast

$$DIQ_t(\tau) = IQ(y_t - \widetilde{\xi}(1-\tau)) - IQ(y_t - \widetilde{\xi}(\tau)), \qquad \tau < 0.5 \qquad (23)$$

This mirrors the quantile contrast, $\widetilde{\xi}(1-\tau) - \widetilde{\xi}(\tau)$, used as a measure of dispersion. It follows from (22) that the variance of the population quantic contrast is $2\tau(1-2\tau)$. The test statistic will be denoted $\eta_\tau(DQ)$.

A complementary expectic contrast, denoted $DIE_t(\omega)$, can be constructed in a similar way. The contrast can be matched to a given τ and hence denoted $DIE_t(\tau)$. Values of ω and $1-\omega$ are obtained from (5); they could be constrained to add to one if desired. The statistic will be denoted $\eta_\tau(DE)$. Robustification can be carried out as before.

A test of constant dispersion can be constructed using the quantics from the $(1-2\tau)th$ quantile for the absolute values, $|y_t|$, when the median is zero. For known quantiles, it is shown in appendix F that $DIQ_t(\tau)$ is equal to $IQ(|y_t| - \xi(1-2\tau))$ when there is symmetry. Thus any difference in performance comes from enforcing the restriction that $\widetilde{\xi}(\tau) + \widetilde{\xi}(1-\tau) = 0$.

For known expectiles, $DIE_t(\omega)$ is not, in general, equal to $IE(|y_t| - \xi(1-2\tau))$. Forming expectics from the mean of absolute values gives the stationarity test of Nyblom and Mäkeläinen (1983) which Harvey and Streibel (1998) find to be more powerful than a test based on squared observations for detecting stochastic volatility. Note that for a Gaussian distribution, the mean of absolute values, $\sqrt{2/\pi}$, corresponds to the complementary quantiles at $\tau = 0.288$ in the original series.

6.1.1 Monte Carlo experiments for the stochastic volatility model

Here we consider the stochastic volatility model

$$\begin{aligned} y_t &= \varepsilon_t \exp(0.5h_t), \qquad \varepsilon_t \backsim NID(0,1) \\ h_t &= \phi h_{t-1} + \kappa_t, \qquad \kappa_t \backsim NID(0,\sigma_\kappa^2) \end{aligned} \qquad (24)$$

with $h_0 = 0$. Table 8 reports Monte Carlo simulated percentage rejections for tests, all with $m = 0$, based on the following statistics: (i) single quantile tests, $\eta_\tau(Q)$ and $\eta_\tau(E)$; dispersion tests $\eta_\tau(DQ)$ and $\eta_\tau(DE)$; $\eta_{1-2\tau}(Q)$ and $\eta_{1-2\tau}(E)$ computed from $|y_t|$.

The figures are for a sample size $T = 1000$, $\phi = 1$ (integrated volatility) and σ_κ set to 0, .005, .01, .02, .03, .05. The expectic tests are calculated with $\omega = \omega(\tau)$ as in (6). Because of symmetry we only report results for $\tau \leq 0.5$. As in the earlier tables, the number of replications is 50,000. The main findings are as follows.

(a) The $\eta_\tau(Q)$ tests reject the null hypothesis of constancy with increasing frequency as τ moves towards zero or one. The expectic tests, $\eta_\tau(E)$, display similar behaviour, except that the rejection frequencies are essentially the same for $\tau = 0.15$ and 0.05.

Although y_t is not strictly stationary when there is integrated stochastic volatility, the sizes of the quantic and expectic tests for $\tau = 0.5$ are affected very little; even with $\sigma_\kappa = 0.05$, they are still below 0.06. This is consistent with the theoretical analysis in Cavaliere and Taylor (2004).

(b) Power is monotonically increasing with σ_κ.

(c) The expectic test, $\eta_\tau(E)$, appears to be more powerful than the quantic test, $\eta_\tau(Q)$, except in the tails.

(d) As expected, the dispersion tests, $\eta_\tau(DQ)$ and $\eta_\tau(DE)$, reject more frequently than the corresponding single quantile tests, $\eta_\tau(Q)$ and $\eta_\tau(E)$. The expectic dispersion tests show a clearer domination over the dispersion quantic tests than do the corresponding single quantile tests. Furthermore, for the range of τ's that we consider, the power of $\eta_\tau(DQ)$ continues to increase as we move towards the tails, while that of $\eta_\tau(DE)$ falls off slightly. The best choice for the expectic test seems to be a value of ω corresponding to $\tau = 0.25$.

(e) The rejection probabilities for the quantic tests based on absolute values are very close to those of the $\eta_\tau(DQ)$ tests. The earlier theoretical analysis indicated that the statistics are identical for known quantiles and it appears that enforcing the symmetry restriction when estimating quantiles makes little or no difference. On the other hand, the expectic tests based on absolute values do behave somewhat differently from the $\eta_\tau(DE)$ tests. They are more powerful for $\tau = 0.15$ and $\tau = 0.05$, but less powerful for $\tau = 0.25$ and $\tau = 0.33$. The best tests against stochastic volatility appear to be the ones based on $\eta_\tau(DE)$ with τ equal to 0.25 and the absolute value expectic test with $\tau = 0.15$.

(f) We confirmed (in experiments not reported here) that all the tests display a tendency to reject the null hypothesis for very persistent but stationary volatility, for example, $\phi = 0.99$; see Harvey and Streibel (1998). However for $\phi = 0.95$, we found rejection frequencies close to the nominal size of 0.05.

The results for heavy-tails disturbances, generated by ε_t having a t-distribution with three degrees of freedom, are presented in Table 9. As with Gaussian disturbances, the test sizes are all close to the nominal. The main differences are as follows.

(a) The power of the single quantile test is now maximized at $\tau = 0.15$ rather than $\tau = 0.05$. The power of the expectic test is also maximized at $\tau = 0.15$.

(b) The quantic tests are more powerful than the corresponding expectic tests for all τ.Even though $\sigma_\varepsilon^2 = 3$,rather than one, the rejection frequencies for quantic tests are actually higher than those for Gaussian disturbances. On the other hand the rejection frequencies for expectic tests are lower.

(c) As regards the dispersion tests, the rejection frequencies for expectics are higher than those of quantics for $\tau = 0.33$, but smaller for $\tau = 0.15$ and $\tau = 0.05$.

(d) In contrast to the Gaussian case, the absolute value expectic test is now more powerful than the $\eta_\tau(DE)$ test for all quantiles. The best choice seems to be $\tau = 0.25$. However, the absolute value quantic test - and the very similar $\eta_\tau(DQ)$ test - has higher rejection frequencies for $\tau = 0.15$ and $\tau = 0.05$. If heavy tails are a possibility, a quantic dispersion test based on $\tau = 0.15$ might be the best option.

6.1.2 Financial time series

For GM, the test statistics $\eta_\tau(DQ)$ for the interquartile range and the 5%/95% range are 3.589 and 3.210 respectively. Thus both decisively reject. The corresponding expectic statistics, $\eta_\tau(DE)$, are 2.567 and 0.506. The failure to reject for $\tau = 0.05$ is consistent with the Monte Carlo results of the previous sub-section.

6.2 Asymmetry

A test of changing asymmetry may be carried out using the quantic contrast

$$SIQ_t(\tau) = IQ(y_t - \widetilde{\xi}(1-\tau)) + IQ(y_t - \widetilde{\xi}(\tau)), \qquad \tau < 0.5 \tag{25}$$

The variance of the population contrast is 2τ and it is uncorrelated with the population contrast corresponding to $DIQ(\tau)$. The sample values are -1 when $y_t < \widetilde{\xi}(\tau)$, 1 when $y_t > \widetilde{\xi}(1-\tau)$ and zero otherwise. The test statistic will be denoted $\eta_\tau(SQ)$.

For GM returns the asymmetry test statistics are 0.133 and 0.039 for $\tau = 0.25$ and $\tau = 0.05$ respectively. Thus neither rejects at the 10% level of significance. The same is true of the expectic tests where the test statistics are 0.061 and 0.069. However, changing skewness may be a feature of some financial time series, as shown in Harvey and Siddique (1999).

6.3 Combined tests

A multivariate test statistic for a pair of complementary quantics, denoted $\eta_{2(\tau)}(Q)$, contains information on changing dispersion and asymmetry. In fact since the population quantile indicator contrasts corresponding to $DIQ(\tau)$ and $SIQ(\tau)$ are mutually uncorrelated, it follows that, if the variances and covariances in the test statistics are set equal to population values, then

$$\eta_{2(\tau)}(Q) = \eta_\tau(DQ) + \eta_\tau(SQ), \qquad \tau < 0.5. \tag{26}$$

This will be approximately true when the long-run variance is estimated. For expectiles, the population variables corresponding to $DIQ(\tau)$ and $SIQ(\tau)$ are only uncorrelated if the distribution is symmetric.

For GM, the multivariate test statistic, $\eta_{2(\tau)}(Q)$, based on the 0.25 and 0.75 quantics is 3.722, which is exactly equal to the sum of the dispersion and asymmetry test statistics, 3.589 and 0.133. This clearly rejects as the 1% critical value is 1.074. The corresponding multivariate statistic for expectiles, $\eta_{2(\tau)}(E)$, is 2.669 which, although not equal to the sum of the dispersion and asymmetry test statistics, 2.567 and 0.061 respectively, is not too far away.

7 Relative time invariance

In looking at quantic and expectic tests so far it has been assumed that the location, that is the median and/or mean, is constant. Our main concern in this section is when the median and/or mean might be changing over time. The aim is to test time invariance of aspects of the distribution apart from

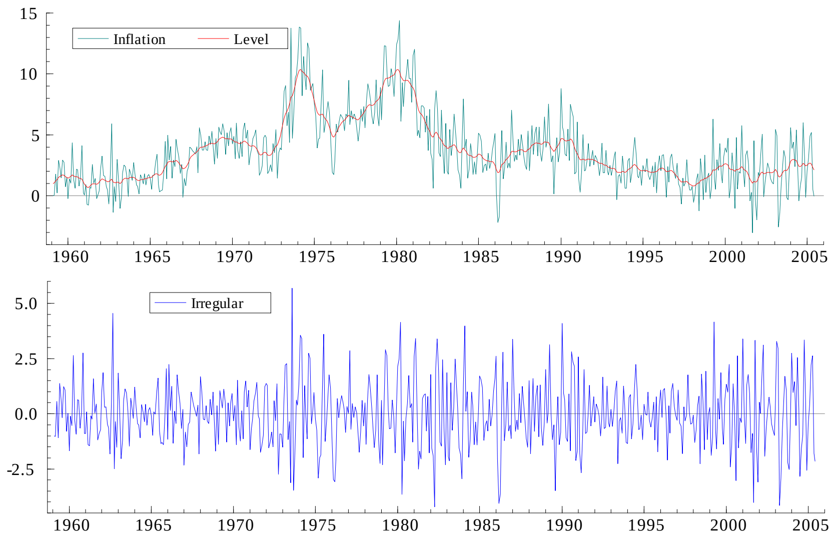

Figure 3: Monthly US inflation: smoothed level and residuals

its location. An example is provided by monthly U.S. inflation[12] in figure 3. The level was obtained by fitting a random walk plus noise using the STAMP 7 package of Koopman *et al* (2006). The signal-noise ratio was estimated by maximum likelihood as $q = 0.074$. The smoothed level is clearly not constant (indeed stationarity tests on all quantiles reject as one would expect), but there may be issues of changing dispersion, as highlighted recently by Stock and Watson (2007), and asymmetry.

7.1 Residuals and tests

We first consider testing the null hypothesis of the time-invariance of a given quantile with respect to the median, that is

$$\xi_t(\tau) = \alpha_\tau + \xi_t(0.5), \qquad t = 1, ..., T \tag{27}$$

where α_τ is a constant. Thus in (10) $\varepsilon_t(\tau) = \varepsilon_t(0.5) - \alpha_\tau$. The model therefore corresponds to the alternative hypothesis used for the investigation in section

[12] To be precise, the first difference of the logarithm of the personal consumer expenditure deflator (all) from 1959(1) to 2005(6) as given by Stock and Watson (2005). We have annualized in percentage terms by multiplying through by 1200.

5.

Quantic tests may be carried out[13] by estimating the time-varying median, $\widetilde{\xi}_t(0.5)$, as in De Rossi and Harvey (2006a), and then constructing the $IQ(y_t - \widehat{\xi}_t(\tau))'s$, where $\widehat{\xi}_t(\tau) = \widehat{\alpha}_\tau + \widetilde{\xi}_t(0.5)$ and $\widehat{\alpha}_\tau$ is the $\tau - th$ quantile of the residuals from the median residuals, $y_t - \widetilde{\xi}_t(0.5)$. Expectics, $IE(y_t - \widehat{\xi}_t(\tau))$ in the notation of (8), could be computed for given quantiles and the corresponding tests carried out as usual.

Quantic tests, and the corresponding $\tau-$expectic tests, can also be based on the quantiles computed from the residuals from the time-varying mean, that is $y_t - \widetilde{\mu}_t(0.5), t = 1, ..., T$. This raises an immediate issue, which is that when a signal plus noise model with a unit root is fitted, the residuals are strictly non-invertible. As a consequence, the long-run variance is zero. This non-invertibility does not carry over to the median indicators because as was shown in sub-section 2.5 the correlations are all closer to zero. As τ moves towards the tails, the correlations become even smaller in absolute value. The same is true for the expectics, but the reductions are smaller. This effect is more pronounced for negative autocorrelations and for a strictly non-invertible process, with spectrum at zero frequency, $\sum_{k=1}^{\infty} \rho(k) = -0.5$. For example, (in a doubly infinite sample) the residuals from a (correctly specified) random walk plus noise follow a strictly non-invertible ARMA(1,1) process with $\rho(1) = -(\theta + 1)/2$ and $\rho(\tau) = \theta^{\tau-1}\rho(1), \tau > 1$, where $\theta = (q+2)/2 - (q^2+4q)^{1/2}/2$ and so all the autocorrelations are negative. Note that because the operation used to estimate a time-varying median is a nonlinear one, it is difficult to determine the properties of the residuals, $y_t - \widetilde{\xi}_t(0.5)$. Indeed, the fact that they do not apparently follow a linear process when the original process is linear (after differencing) may be a cause for concern. But, of course, the linearity of the original is itself a convenient fiction.

In general the mean will be more efficient than the median. This is well documented when the mean and median are fixed, but it carries over to when they are time-varying; see De Rossi and Harvey (2006b). Although the mean is only equal to the median for symmetric distributions, symmetry is not needed since, under the null hypothesis of relative time-invariance, the time-varying mean will correspond to a time-varying quantile for some τ; the fact that this may not be the median does not matter since the quantiles move

[13] If the two quantiles were observed, one could be subtracted from the other and a non-stationarity test carried out; see Harvey and Bernstein (2003). This is not an option for a single series.

together over time.

Tests can also be based on the standardized innovations (one-step ahead prediction errors) from the Kalman filter. If the model is Gaussian and correctly specified, the standardized innovations will be normally distributed and serially independent with mean zero and constant variance. Hence the tests may be applied with no correction for serial correlation ($m = 0$). However, given model uncertainty, it might be better to correct for serial correlation, particularly if the procedure is robustified in the way suggested in sub-section 7.3. Tests constructed from innovations may have better size properties, but the question is whether a price is paid in terms of power.

7.2 Simulation evidence

For a Gaussian random walk plus noise model (17) with stochastic volatility in the noise, as in sub-section 6.1, we computed Monte Carlo rejection frequencies of the dispersion tests for relative invariance, that is $\eta_\tau(DQ; m)$ and $\eta_\tau(DE; m)$, based on the quantics and expectics constructed as $IQ(y_t - \widehat{\xi}_t(\tau))$ and $IE(y_t - \widehat{\xi}_t(\tau))$, where $\widehat{\xi}_t(\tau) = \widehat{\alpha}_\tau + \widetilde{\mu}_t(0.5)$, $\widetilde{\mu}_t(0.5)$ is the estimated time-varying mean and $\widehat{\alpha}_\tau$ is the $\tau - th$ quantile of the residuals $y_t - \widetilde{\mu}_t(0.5)$. The signal-noise ratio, q, was fixed at its value under the null. The estimated mean is obtained either by smoothing or (Kalman) filtering. Table 9 provides the percentage rejections for $\tau = 0.05, 0.25$, with $m = 6$, based on 10,000 replications.

Under the null, $\sigma_\kappa^2 = 0$, all quantiles, including the median are constant, and the set-up is similar to the one in table 8. When there is a random walk component, we find that the tests based on residuals from smoothing are undersized, particularly when $\tau = 0.25$. Using innovations seems to control size better (even if we set $m = 0$), but, unless σ_κ is close to zero, there are fewer rejections.

7.3 Robustness

Fitting a (time-varying) mean requires finite second moments for the disturbances in both the level and noise. A (time-varying) median requires finite second moments only for the disturbances in the level, together with assumption 1 on the noise. However, expectic tests computed from the residuals from a fitted median still need finite second moments for the noise.

In order to make the procedure robust we suggest proceeding as follows:

i) estimate the time-varying median;

ii) Winsorize the residuals (using a robust estimator of range);

iii) add the Winsorized residuals to the median and estimate the time-varying mean;

iv) compute quantiles from the residuals and carry out quantic and $\tau-$expectic tests.

Step (iii) is effectively a time series $M-$estimator; see Huber (1981). It could be iterated, with the range re-estimated at each iteration.

7.4 Misspecified detrending

The need to identify a suitable model for detrending prior to testing for relative time invariance may be source of unease. However, misspecification of such a model need not have serious consequences as the following proposition makes clear.

Proposition 7 *If the true model is a linear process after differencing d times, and the fitted model is a linear process after differencing $\overline{d}$ times, the modified test statistics, $\eta_\tau(Q;m)$ and $\eta_\tau(E;m)$, constructed from the residuals will still have an asymptotic CvM distribution under the null hypothesis provided that $\overline{d} \geq d$.*

The result may be shown by using the Wiener-Kolmogorov formula to find the properties of the residuals. Suppose that the true process is such that $\Delta^d y_t = \psi(L)\nu_t$, where $\psi(L)$ is a polynomial in the lag operator with all roots outside the unit circle and ν_t is white noise with variance σ^2. If the fitted UC model is of the form $y_t = \mu_t + \varepsilon_t$, where the estimated variance of the white noise process ε_t is σ_ε^2, and its reduced form is $\Delta^{\overline{d}} y_t = \overline{\psi}(L)\overline{\nu}_t$, where the variance of $\overline{\nu}_t$ is $\overline{\sigma}^2$, the process followed by the extracted residuals in a doubly infinite sample is

$$\widetilde{\varepsilon}_t = \frac{\sigma_\varepsilon^2(1-L)^{\overline{d}}(1-L^{-1})^{\overline{d}}}{\overline{\sigma}^2\overline{\psi}(L)\overline{\psi}(L^{-1})} y_t = \frac{\sigma_\varepsilon^2(1-L)^{\overline{d}-d}(1-L^{-1})^{\overline{d}}}{\overline{\sigma}^2\overline{\psi}(L)\overline{\psi}(L^{-1})}\psi(L)\nu_t$$

(Note the simplification if the model is correctly specified.) Clearly $\overline{d} \geq d$ is needed if the process is to be stationary. Although the process is strictly non-invertible, the weakening of the autocorrelation structure for indicators and expectics other than $\omega = 0.5$ means that the nonparametric correction in the test statistic can be legitimately employed. More generally, if ε_t is modeled as a stationary process, the same conclusions hold.

7.5 US inflation

Fitting a random walk plus noise model to US inflation, as in figure 3, results in some residual serial correlation in the innovation residuals, in particular the first-order autocorrelation is 0.14. The residual autocorrelation can be removed by fitting a stochastic cycle or low order autoregressive model. However, the result on misspecified trends in the previous sub-section indicates that this is not necessary. Indeed the first-order autocorrelation for the smoothed residuals from the random walk plus noise model is not strongly negative, presumably because of the misspecification.

A full set of results for the test statistics, as in table 2, is available on request. The main features are that the $\eta_\tau(DQ;8)$ test statistics for $\tau = 0.05$, 0.10 and 0.25 are 0.444, 0.636 and 0.273 respectively while the corresponding figures[14] for $\eta_\tau(DE;8)$ are 0.247, 0.469 and 0.612. There is clear evidence for changing dispersion and the statistics are consistent with the results of table 8 in that the quantic statistics are higher in the tails and lower for the interquartile range. There is no indication of changing asymmetry.

The tests based on innovations tell a similar story. The $\eta_\tau(DQ;8)$ test statistics for $\tau = 0.05$, 0.10 and 0.25 are 0.538, 0.393 and 0.417 respectively while the corresponding figures for $\eta_\tau(DE;8)$ are 0.247, 0.352 and 0.447. Again the asymmetry test statistics are small and statistically insignificant at the 10% level.

Finally we fitted an integrated random walk plus noise model and found the maximum likelihood estimate of the signal-noise ratio to be 0.0007. The $\eta_\tau(DQ;8)$ test statistics for $\tau = 0.05$, 0.10 and 0.25 were 0.598, 0.288 and 0.200 respectively while the corresponding figures for $\eta_\tau(DE;8)$ were 0.240, 0.425 and 0.550. These test statistics are not too different from those obtained with the random walk plus noise model.

8 Conclusion

We have shown how tests of time invariance may be constructed from what we call quantics or expectics. The former are based on weighted indicators obtained from an estimated quantile while the latter are based on residuals weighted differently according to whether the observation is above or below

[14] As expected from table 8 these figures are bigger than the corresponding figures for test statistic on individual quantics and expectics.

the quantile. The serial correlation structure is weakened by transforming to quantics or expectics with autocorrelations moving closer to zero as the quantile moves towards either of the tails. Stationarity test statistics constructed from quantics or expectics are shown to have the usual Cramér von Mises distribution, asymptotically, under the null hypothesis of time invariance. Our simulation results confirm certain prior expectations such as the higher power of expectic tests for Gaussian series for most quantiles, coupled with a tendency for the quantic tests to do better when there are heavy tails (where power is generally lower). A compromise is to robustify the expectic tests, but this does not always produce an improvement over the quantic tests. Multivariate tests, combining the information in a set of quantics or expectics, appear to be useful as general tests of time invariance.

Tests based on contrasts can be used to detect changing dispersion and asymmetry. Dispersion tests are based on complementary quantics or expectics. Alternatively tests can be based on absolute values (about the mean or median), though for quantics the tests are essentially the same when the distribution is symmetric. With expectics, a good test against changing volatility is the one based on the quartile contrast, though with heavy tails (but still finite variance) the corresponding test based on the mean of absolute values seems preferable. If there are concerns about the existence of moments, a quantic dispersion test constructed using the 15 and 85% quantiles is good choice.

Testing for relative time invariance introduces some new issues. Our recommendation is to estimate a stochastic trend or level as for a Gaussian model, possibly with prior treatment of potential outliers, and then to proceed to compute quantics and expectics by fitting quantiles to the residuals. Even if the fitted model is misspecified, the smoothed residuals are stationary and serial correlation not accounted for by the model will be accommodated by the nonparametric estimator of the long run variance; all that is required is that the number of unit roots in the fitted model is at least as great as in the true model.

There are a number of other open problems. For example, a test might be based on a combination of expectics and quantics. Also more work needs to be done on the effectiveness of methods for dealing with relative time invariance, including the treatment of time trends, and on robustification. Tests for movements in the tails, reflecting changing kurtosis, can be based on the outer quantiles, but an allowance needs to be made for changing dispersion and this requires some investigation. Finally we believe that our procedures can be modified to deal with irregular spacing, thereby providing tests for the linearity assumption in quantile or expectile regression against the alternative of nonlinear cubic splines.

APPENDICES

A Autocorrelations of quantile indicators

The autocorrelation of the τ-th quantile indicator corresponding to ρ_y, the autocorrelation of the covariance stationary process, y_t, at any lag $k = 1, 2, ...$, is

$$\rho_\tau(\rho_y) = \frac{E\left[IQ\left(y_t - \xi(\tau)\right)\ IQ\left(y_{t-k} - \xi(\tau)\right)\right]}{Var(IQ\left(y_t - \xi(\tau)\right))}.$$

The denominator is $\tau(1-\tau)$, while the numerator is

$$\begin{aligned}
&\tau^2 \Pr\left(y_t > \xi(\tau),\ y_{t-k} > \xi(\tau)\right) + (\tau-1)^2 \Pr\left(y_t < \xi(\tau),\ y_{t-k} < \xi(\tau)\right) \\
&+\tau(\tau-1) \Pr\left(y_t > \xi(\tau),\ y_{t-k} < \xi(\tau)\right) + \tau(\tau-1) \Pr\left(y_t < \xi(\tau),\ y_{t-k} > \xi(\tau)\right)
\end{aligned}$$

But

$$\begin{aligned}
&\Pr\left(y_t > \xi(\tau),\ y_{t-k} < \xi(\tau)\right) + \Pr\left(y_t < \xi(\tau),\ y_{t-k} > \xi(\tau)\right) \\
= &\ 1 - \Pr\left(y_t > \xi(\tau),\ y_{t-k} > \xi(\tau)\right) - \Pr\left(y_t > \xi(\tau),\ y_{t-k} > \xi(\tau)\right)
\end{aligned}$$

So the numerator becomes

$$\tau \Pr\left(y_t > \xi(\tau),\ y_{t-k} > \xi(\tau)\right) + (1-\tau) \Pr\left(y_t > -\xi(\tau),\ y_{t-k} > -\xi(\tau)\right) + \tau^2 - \tau$$

Hence (9) is obtained.

B Correlations between quantics

Tests involving more than one quantic need to take account of the correlation between them. To find the covariance of the τ_1 and τ_2 population quantics with $\tau_2 > \tau_1$, their product must be evaluated and weighted by (i) τ_1 when $y_t < \xi(\tau_1)$, (ii) $\tau_2 - \tau_1$ when $y_t > \xi(\tau_1)$ but $y_t < \xi(\tau_2)$, (iii) $1 - \tau_2$ when $y_t > \xi(\tau_2)$. This gives

$$(\tau_2 - 1)(\tau_1 - 1)\tau_1 + (\tau_2 - 1)\tau_1(\tau_2 - \tau_1) + \tau_2\tau_1(1 - \tau_2) = \tau_1(1 - \tau_2)$$

and on collecting terms we find that

$$cov(IQ(y_t - \xi(\tau_1)), IQ(y_t - \xi(\tau_2))) = \tau_1(1-\tau_2), \qquad \tau_2 > \tau_1. \tag{28}$$

It follows that the correlation between the population quantics for τ_1 and τ_2 is

$$\frac{\tau_1(1-\tau_2)}{\sqrt{\tau_1(1-\tau_1)\tau_2(1-\tau_2)}}, \qquad \tau_2 > \tau_1$$

The correlation between the complementary quantile indicators, $IQ(y_t - \xi(\tau))$ and $IQ(y_t - \xi(1-\tau))$, is simply $\tau/(1-\tau)$. This is 1/3 for the quartiles and 1/9 for the first and last deciles.

C Limiting null distribution of the IQ test

Let y_t be IID random variables, with $\xi(\tau)$ being the the unique population τ-th quantile. Assume that $y_t - \xi(\tau)$ has continuous density $f(x;\tau)$ in the neighborhood of $x = 0$ (Assumption 1) denote by $F(x;\tau)$ the corresponding cumulative distribution function. The limiting distribution of the sample τ-th quantile $\widehat{\xi}(\tau)$ is known to be given[15] by $T^{1/2}\left(\widetilde{\xi}(\tau) - \xi(\tau)\right) \xrightarrow{d} N\left(0, \tau(1-\tau)/f^2(0;\tau)\right)$; see, for example, Koenker (2005p 71-2). Now let

$$G_T(r,\phi) = T^{-1/2}\sum_{j=1}^{[rT]}\left(IQ\left(y_t - \xi(\tau) - \phi T^{-1/2}\right) - IQ\left(y_t - \xi(\tau)\right)\right),$$

for $r \in [0,1]$. On setting $\phi = T^{1/2}\left(\widetilde{\xi}(\tau) - \xi(\tau)\right)$ and rearranging the above expression we can write

$$T^{-1/2}\sum_{j=1}^{[rT]} IQ\left(y_t - \widetilde{\xi}(\tau)\right) = A + B + C, \tag{29}$$

[15] When extending results on testing the median in the presence of serial correlation, DAS note that the asymptotic variance of the median is multiplied by 2π times the spectrum at zero, that is the long-run variance. The same holds for other quantiles.

where

$$
\begin{aligned}
A &= G_T\left(r, T^{1/2}\left(\widetilde{\xi}(\tau)-\xi(\tau)\right)\right) - E\left[G_T\left(r, T^{1/2}\left(\widetilde{\xi}(\tau)-\xi(\tau)\right)\right)\right] \\
B &= T^{-1/2}\sum_{j=1}^{[rT]} IQ\left(y_t-\xi(\tau)\right), \\
C &= E\left[G_T\left(r, T^{1/2}\left(\widetilde{\xi}(\tau)-\xi(\tau)\right)\right)\right] \\
&= -\,T^{-1/2}[rT]\ \left(\widetilde{\xi}(\tau)-\xi(\tau)\right) f\left(\widetilde{\xi}(\tau)-\overline{\xi}(\tau)\right).
\end{aligned}
$$

The first term, A, is of $o_p(1)$ by the same type of arguments as in lemma 3 of DAS (uniform convergence in probability of $G_T(r,\phi)$ to its mean), while the expression for C is obtained by noting that $E\left[IQ\left(y_t-\xi(\tau)\right)\right]=0$ and that

$$
\begin{aligned}
E\left[IQ\left(y_t-\widetilde{\xi}(\tau)\right)\right] &= (\tau-1)\,F\left(\widetilde{\xi}(\tau)-\xi(\tau);\tau\right)+\tau\left(1-F\left(\widetilde{\xi}(\tau)-\xi(\tau);\tau\right)\right) \\
&= \tau-\left(\tau+\left(\widetilde{\xi}(\tau)-\xi(\tau)\right) f\left(\widetilde{\xi}(\tau)-\overline{\xi}(\tau)\right);\tau\right) \\
&= \quad -\left(\widetilde{\xi}(\tau)-\xi(\tau)\right) f\left(\widetilde{\xi}(\tau)-\overline{\xi}(\tau);\tau\right)
\end{aligned}
$$

where the second equality follows by a mean value theorem expansion of $F(.)$ and the consistency of $\widetilde{\xi}(\tau)$, with $\overline{\xi}(\tau)$ between $\xi(\tau)$ and $\widetilde{\xi}(\tau)$. Given that by the invariance principle $B \Rightarrow \sqrt{\tau(1-\tau)}W(r)$, where $W(r)$ is a standard Wiener process and the notation $\Rightarrow$ denotes weak convergence, expression (29) yields

$$
(\tau(1-\tau)T)^{-1/2}\sum_{j=1}^{[rT]} IQ\left(y_t-\widetilde{\xi}(\tau)\right) \Rightarrow W(r)-rW(1).
$$

Thus, by an application of the continuous mapping theorem,

$$
\eta_\tau(Q) \xrightarrow{d} \int_0^1 \left(W(r)-rW(1)\right)^2 dr \tag{30}
$$

which is the standard Cramér-von Mises distribution.

D Limiting null distribution of the IE test

Let y_t be IID random variables satisfying assumptions 1-4 of Newey and Powell (1987). In particular, the assumptions imply a continuous density function and the existence of moments of order slightly higher than four. Newey and Powell (1987, p 827) show that the limiting distribution of the sample ω-expectile, $\widetilde{\mu}(\omega)$, is given by

$$T^{1/2}\left(\widetilde{\mu}(\omega)-\mu(\omega)\right) \xrightarrow{d} N\left(0, E\left(IE_t^2\right)/\left(E\left(\left|\omega-I(y_t-\mu\left(\omega\right)<0)\right|\right)\right)^2,\right.$$

where IE_t denotes $IE\left(y_t-\mu\left(\omega\right)\right)$ as defined in (3).

As noted in proposition 2, IE_t has zero mean and finite variance. Thus the invariance principle holds, that is $T^{-\frac{1}{2}}\sum_{t=1}^{[rT]} IE_t \Rightarrow \sqrt{E\left(IE_t^2\right)}W\left(r\right),$ $r \in [0,1]$.

Now consider the case where $\widetilde{\mu}(\omega) < \mu(\omega)$; the same type of arguments would follow for $\widetilde{\mu}(\omega) \geq \mu(\omega)$. By simple re-arrangement one can write

$$IE\left(y_t-\widetilde{\mu}\left(\omega\right)\right)=IE_t-R_t, \qquad t=1,...,T,$$

where

$$R_t=\begin{cases}(\omega-1)\left(\widetilde{\mu}(\omega)-\mu(\omega)\right) & \text{for } y_t<\widetilde{\mu}(\omega)\\ \omega\left(\widetilde{\mu}(\omega)-\mu(\omega)\right)-\left(y_t-\mu(\omega)\right) & \text{for } \widetilde{\mu}(\omega)\leq y_t<\mu(\omega)\\ \omega\left(\widetilde{\mu}(\omega)-\mu(\omega)\right) & \text{for } y_t\geq\mu(\omega).\end{cases}$$

Given the limiting result for the sample expectile, it can be seen that $R_t = \left|\omega-I(y_t-\mu\left(\omega\right)<0)\right|\left(\widetilde{\mu}(\omega)-\mu(\omega)\right)+o_p(1)$ and so $T^{-\frac{1}{2}}\sum_{t=1}^{[rT]} R_t \Rightarrow r\sqrt{E\left(IE_t^2\right)}W\left(1\right),$ $r \in [0,1]$. Thus, by an application of the continuous mapping theorem, $\eta_\omega(E)$ converges to a CvM distribution, as in (30).

E Limiting distributions of test statistics under the alternative

Suppose that a functional central limit theorem holds for the scaled observations under the $I(1)$ alternative, i.e. $T^{-1/2}y_{[T\alpha]} \Rightarrow \lambda W(\alpha)$, $\alpha \in [0,1]$ for some $\lambda > 0$.

For quantic tests we will assume that $\lambda^{-1}T^{-1/2}\widetilde{\xi}(\tau) \xrightarrow{d} \overline{\xi}(\tau)$ for some (implicitly defined) random variable $\overline{\xi}(\tau)$. DAS prove that this is true for $\tau = 0.5$. We will find it convenient to write

$$IQ(y_t - \xi(\tau)) = \tau - 0.5 + 0.5sgn(y_t - \xi(\tau)).$$

Then, using the results of Park and Phillips (1999), for $\alpha \in [0, 1]$,

$$T^{-1}\sum_{j=1}^{[\alpha T]} IQ(y_t - \widetilde{\xi}(\tau)) \xrightarrow{d} \int_0^\alpha \{\tau - 0.5 + 0.5sgn(W(x) - \overline{\xi}(\tau))\}dx$$

and so

$$T^{-3}\sum_{t=1}^{T}\left(\sum_{j=1}^{t} IQ(y_t - \widetilde{\xi}(\tau))\right)^2 \xrightarrow{d} \int_0^1 \left(\int_0^\alpha \{\tau - 0.5 + 0.5sgn(W(x) - \overline{\xi}(\tau))\}dx\right)^2 d\alpha,$$

Hence we obtain (18).

For expectics write

$$IE(y_t - \mu(\omega)) = [0.5 + (\omega - 0.5)sgn(y_t - \mu(\omega))](y_t - \mu(\omega))$$

then

$$\begin{aligned} & T^{-3/2}\sum_{j=1}^{[\alpha T]} IE(y_t - \widetilde{\mu}(\omega)) \\ = \quad & T^{-3/2}\sum_{j=1}^{[\alpha T]}[0.5 + (\omega - 0.5)sgn(y_t - \widetilde{\mu}(\omega))](y_t - \widetilde{\mu}(\omega)) \\ \Rightarrow \quad & \lambda\int_0^\alpha \{0.5 + (\omega - 0.5)sgn(W(x) - \overline{\mu}(\omega)\}(W(x) - \overline{\mu}(\omega))dx \end{aligned}$$

where $\overline{\mu}(\omega)$ is such that $\lambda^{-1}T^{-1/2}\widetilde{\mu}(\omega) \to \overline{\mu}(\omega)$. (We assume this to be true for $\omega \neq 0.5$). Now

$$\begin{aligned} & T^{-4}\sum_{t=1}^{T}\sum_{j=1}^{t}\{0.5 + (\omega - 0.5)sgn(y_j - \widetilde{\mu}(\omega))(y_t - \widetilde{\mu}(\omega))\}^2 \\ & \xrightarrow{d} \lambda^2\int_0^1 \left(\int_0^\alpha \{0.5 + (\omega - 0.5)sgn(W(x) - \overline{\mu}(\omega))\}(W(x) - \overline{\mu}(\omega))dx\right)^2 d\alpha. \end{aligned}$$

Then, by the same arguments, we have in the denominator

$$
\begin{aligned}
T^{-1}\widehat{\sigma}^{2} &= T^{-2}\sum_{t=1}^{T}\{IE(y_t-\widetilde{\mu}\,(\omega))\}^{2} \\
&\rightarrow \lambda^{2}\int_{0}^{1}\{0.5+(\omega-0.5)sgn(W(x)-\overline{\mu}(\omega))\}^{2}(W(x)-\overline{\mu}(\omega))^{2}dx,
\end{aligned}
$$

and so we obtain (19).

When the expectile is matched to a given quantile, $\widetilde{\mu}(\omega)$ is replaced by $\widetilde{\xi}(\tau)$ and $\overline{\mu}(\omega)$ is replaced by $\overline{\xi}(\tau)$. However, this makes no difference to the general form of the asymptotic distribution.

F Dispersion and absolute values

Assume the quantiles are known. Let n_A be the number of observations greater than or equal to $\xi(1-\tau)$, n_B the number of observations less than or equal to $\xi(\tau)$ and n_C the number of observations less than $\xi(1-\tau)$ or greater than $\xi(\tau)$.Then

$$DIQ(\tau)=(1-\tau)n_A+(1-\tau)n_B-\tau(n_B+n_C)-\tau(n_A+n_C)$$

For a symmetric distribution and $\xi(1-\tau)-\xi(0.5)=-\xi(\tau)+\xi(0.5)$ and these are equal to the quantile at $1-2\tau$ for $|y_t-\xi(0.5)|$. Hence

$$IQ(1-2\tau)=(1-2\tau)(n_A+n_B)-2\tau n_C$$

which on re-arrangement is seen to be the same as the previous formula.

Acknowledgement *We wish Giuliano De Rossi for many discussions on this and related topics. Earlier versions of the paper were given at Hitotsubashi University (Tokyo), Oxford and Cambridge. We are grateful to Katsuto Tanaka, David Hendry, Steve Satchell, Robert Taylor and others for helpful comments. Any errors are our responsibility. We acknowledge financial support from the ESRC under the grant Time-Varying Quantiles, RES-062-23-0129.*

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Table 1. Autocorrelations for population quantics and expectics from a Gaussian time series.

	τ for quantics					$\omega(\tau)$ for expectics			
ρ_y	0.50	0.33	0.25	0.10	0.05	0.248(0.33)	0.153(0.25)	0.034(0.10)	0.012(0.05)
0.80	0.59	0.58	0.56	0.50	0.45	0.77	0.74	0.63	0.55
0.50	0.33	0.32	0.30	0.24	0.20	0.47	0.43	0.31	0.23
0.25	0.16	0.15	0.14	0.10	0.07	0.23	0.20	0.12	0.08
-0.25	-0.16	-0.15	-0.13	-0.07	-0.04	-0.22	-0.18	-0.08	-0.04
-0.50	-0.33	-0.29	-0.24	-0.10	-0.05	-0.42	-0.34	-0.14	-0.07
-0.80	-0.59	-0.45	-0.32	-0.11	-0.05	-0.66	-0.52	-0.20	-0.10

Table 2. Quantic, Expectic and Box-Ljung Indicator test for GM returns (2000 observations)

τ	m=0	m=8	m=17	m=25
Quantic test				
0.05	1.823	1.349	1.138	1.025
0.10	1.425	1.115	0.871	0.741
0.25	1.026	0.949	0.823	0.743
0.33	0.799	0.782	0.705	0.648
0.50	2.532	2.667	2.621	2.437
0.67	0.660	0.649	0.601	0.579
0.75	1.544	1.347	1.206	1.142
0.90	1.389	1.088	0.918	0.832
0.95	1.350	1.136	0.983	0.902
Expectic test				
0.05	0.116	0.081	0.077	0.077
0.10	0.301	0.209	0.187	0.180
0.25	0.350	0.295	0.251	0.232
0.33	0.203	0.197	0.171	0.160
0.50	0.074	0.090	0.085	0.083
0.67	0.462	0.533	0.501	0.487
0.75	0.715	0.751	0.682	0.649
0.90	0.906	0.811	0.698	0.648
0.95	0.578	0.509	0.440	0.410
Expectic-robust test				
0.05	1.047	0.877	0.787	0.769
0.10	1.189	0.968	0.813	0.747
0.25	0.749	0.718	0.604	0.544
0.33	0.394	0.416	0.366	0.337
0.50	0.064	0.075	0.072	0.070
0.67	0.489	0.523	0.490	0.475
0.75	0.819	0.802	0.727	0.690
0.90	1.412	1.262	1.091	1.007
0.95	1.241	1.137	0.997	0.922
Box-Ljung test				
0.05	3.547	40.694	54.174	79.925
0.10	1.539	38.747	64.821	92.909
0.25	0.123	7.927	22.389	35.324
0.33	0.045	6.865	17.686	33.029
0.50	0.526	8.776	13.888	24.229
0.67	0.785	10.523	20.706	24.217
0.75	2.116	11.809	19.042	24.523
0.90	4.843	19.019	35.425	39.768
0.95	8.117	16.496	35.239	41.228

Table 3. Percentage rejection frequencies. Errors **N(0,1)**. The random walk under the alternative is gaussian with innovation standard deviation equal to c/T. T=200.

	c=0				c=2.5				c=5				c=10				c=25				c=50			
τ	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)
0.05	4.9	4.9	7.0	6.3	6.2	8.6	7.3	6.5	11.2	19.0	8.4	8.0	25.8	43.4	12.1	13.8	57.8	78.9	29.9	43.0	77.0	93.4	58.7	77.1
0.15	4.9	4.9	2.9	4.5	7.6	10.6	2.8	4.9	17.2	25.8	3.5	6.2	40.0	54.2	7.8	14.7	76.3	87.2	40.8	62.3	91.4	96.9	82.6	94.0
0.25	5.2	5.1	5.1	5.1	9.4	11.6	5.3	5.0	19.9	28.5	6.0	6.8	45.6	57.8	11.3	17.0	81.3	89.1	55.5	71.3	94.4	97.8	92.7	97.2
0.33	5.1	5.2	4.4	5.0	9.4	12.2	4.4	5.0	21.3	29.9	5.3	6.6	48.1	59.2	11.2	18.8	83.2	89.8	58.7	74.8	95.3	98.1	94.5	98.2
0.50	5.0	5.3	4.5	5.2	9.6	12.6	4.8	5.4	22.3	30.7	5.3	6.5	49.4	60.2	12.8	20.8	84.3	90.5	63.5	78.1	95.9	98.3	96.1	98.7

Table 4. Percentage rejection frequencies. Errors **t(3)**. The random walk under the alternative is gaussian with innovation standard deviation equal to c/T. T=200

	c=0				c=2.5				c=5				c=10				c=25				c=50			
τ	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)
0.05	4.6	3.9	6.7	5.6	4.8	4.4	6.8	5.6	5.9	6.1	7.0	5.9	9.6	12.2	7.5	7.0	28.9	38.0	12.7	15.4	53.8	65.3	30.3	42.5
0.15	5.0	4.3	2.4	4.2	6.5	5.8	2.5	4.6	11.2	10.3	2.7	4.9	25.6	25.1	4.4	8.2	60.9	60.0	22.6	37.7	83.5	84.3	62.1	79.5
0.25	4.9	4.5	5.5	4.9	7.4	6.7	5.4	4.9	15.0	13.2	5.6	5.3	36.6	31.8	8.6	10.8	72.5	68.9	39.8	55.4	90.3	89.6	82.2	91.4
0.33	5.3	4.6	4.4	5.3	8.5	7.4	4.3	5.5	17.8	15.1	4.5	5.9	41.3	35.6	8.7	14.4	77.2	73.0	45.6	63.0	92.5	91.6	86.9	94.8
0.50	5.0	4.7	4.6	5.2	9.5	7.8	4.6	5.2	20.5	16.2	5.2	6.1	45.0	38.4	10.1	16.4	80.2	75.3	52.2	68.4	93.9	92.8	90.8	96.5

Table 5. Percentage rejection frequencies. Errors **Cauchy.** The random walk under the alternative is gaussian with innovation standard deviation equal to c/T. T=200

	c=0				c=2.5				c=5				c=10				c=25				c=50			
τ	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)	IQ	IE	r(1)	Q(1-5)
0.05	4.5	2.0	6.7	6.1	4.6	2.0	6.5	6.0	4.5	2.0	6.5	6.0	4.7	2.1	6.5	6.0	6.0	2.6	6.8	6.3	9.6	4.1	7.7	7.6
0.15	5.1	2.3	2.6	4.7	5.3	2.3	2.6	4.8	6.2	2.4	2.7	4.9	9.8	2.7	2.8	5.3	27.7	4.6	5.5	9.9	53.8	10.1	20.0	32.4
0.25	5.2	2.4	5.5	5.1	6.2	2.5	5.7	5.4	9.5	2.6	5.6	5.4	21.0	3.1	6.1	6.6	53.1	6.5	17.5	27.0	77.6	14.6	52.5	67.0
0.33	5.2	2.5	4.5	5.2	7.0	2.6	4.5	5.2	12.8	2.8	4.7	5.6	29.6	3.5	6.2	8.7	64.5	7.8	25.8	40.0	84.9	17.3	65.7	81.1
0.50	5.2	2.7	4.7	5.3	8.2	2.8	4.8	5.8	16.8	3.1	4.9	6.2	38.1	4.0	7.3	11.3	72.7	9.2	35.6	52.6	89.4	19.4	76.3	89.4

Table 6. T=200. The d.g.p. is Gaussian random walk plus AR(1) with ϕ=0.5. Disturbances driving the AR(1)are either N(0,1) or (scaled) Cauchy; q is the standard deviation of the AR(1) innovations.

	q=0				q=.0005^.5				q=.001^.5				q=.01^.5				q=.1^.5				q=1			
	Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy	
τ	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE
0.05	5.4	4.5	7.3	3.8	7.1	7.2	7.9	4.5	9.2	9.9	8.6	5.0	32.6	34.7	17.5	11.5	55.7	61.5	34.1	27.4	56.6	65.4	46.0	45.4
0.15	6.8	6.8	8.8	6.2	10.6	11.5	18.4	9.0	14.6	16.3	24.1	11.2	48.0	51.5	52.5	27.3	73.9	77.4	68.7	50.1	76.9	82.0	73.7	68.1
0.25	7.6	7.9	9.5	7.2	12.3	13.8	32.0	12.2	17.0	19.4	41.7	15.3	52.3	56.9	69.3	36.6	79.8	83.1	80.1	61.7	84.9	88.6	83.3	78.4
0.33	7.7	8.3	9.7	7.5	13.3	14.7	40.2	13.6	18.3	20.8	50.7	17.6	55.0	58.8	75.6	41.3	82.1	85.2	83.8	67.2	86.5	90.3	85.9	81.9
0.50	7.8	8.7	10.1	7.9	13.5	15.6	47.6	15.0	19.0	21.8	57.5	19.6	56.1	61.0	79.3	45.3	83.6	86.7	86.0	71.2	88.1	91.4	87.4	84.5

Table 7. T=200. The d.g.p. is Gaussian random walk plus MA(1) with θ=-0.8. Disturbances driving the MA(1)are either N(0,1) or (scaled) Cauchy; q is the standard deviation of the MA(1) innovations.

	q=0				q=.0005^.5				q=.001^.5				q=.01^.5				q=.1^.5				q=1			
	Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy		Normal		Cauchy	
τ	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE	IQ	IE
0.05	4.2	3.6	4.7	1.7	7.4	7.1	5.5	2.3	10.6	10.7	6.6	3.0	40.2	43.1	21.3	10.1	65.6	69.2	45.0	22.8	62.0	69.5	51.7	40.2
0.15	4.1	3.2	4.2	0.8	11.7	14.8	27.6	6.3	18.4	24.0	39.8	9.9	59.4	66.2	69.6	28.2	81.4	85.2	75.7	49.4	80.0	84.2	75.9	67.5
0.25	3.3	1.7	3.8	0.2	14.6	22.7	55.3	12.3	24.1	35.9	65.4	18.3	66.2	76.6	82.5	44.2	86.0	89.2	84.9	66.8	86.3	89.5	84.9	80.1
0.33	2.7	0.5	2.7	0.1	16.3	30.0	66.1	16.7	27.2	44.8	74.3	24.5	69.7	81.3	85.9	54.4	87.1	90.8	86.8	75.3	88.0	90.9	86.8	84.8
0.50	1.8	0.0	2.0	0.0	18.4	39.1	74.0	21.3	30.6	54.4	80.1	31.6	72.1	85.2	87.6	62.9	88.2	91.5	88.4	80.8	89.0	92.1	88.2	88.2

Table 8. Percentage rejections against Stochastic Volatility data generating process with Gaussian innovations. Sample size T=1000.

q:	0	0.005	0.01	0.02	0.03	0.05
Quantic test						
τ: 0.05	4.64	9.39	21.71	48.02	65.08	82.76
0.15	5.13	8.34	18.60	42.56	60.66	81.03
0.25	4.94	6.58	12.52	30.74	46.98	68.93
0.33	5.09	5.67	8.37	19.15	31.59	52.70
0.50	5.08	5.09	5.11	5.18	5.24	5.74
Expectic test		:				
τ: 0.05	4.69	9.19	21.65	47.26	63.34	80.08
0.15	4.67	9.26	21.27	47.20	63.90	81.81
0.25	4.57	7.43	14.91	36.16	52.43	72.21
0.33	4.60	6.08	9.93	23.05	36.74	55.76
0.50	4.82	4.85	4.87	5.00	5.01	4.89
Quantic dispersion test (DQ)						
τ: 0.05	5.02	14.60	35.65	65.31	79.99	92.64
0.15	5.07	13.87	33.15	63.29	79.16	92.58
0.25	5.01	10.91	25.75	54.33	71.74	88.51
0.33	5.08	8.77	19.08	43.17	61.82	81.69
Expectic dispersion test (DE)						
τ: 0.05	4.97	15.59	36.83	66.69	80.86	92.25
0.15	4.98	19.20	43.94	73.45	86.36	96.01
0.25	5.13	19.29	44.43	74.01	87.23	96.31
0.33	5.02	19.23	43.96	73.63	86.88	96.12
Dispersion test (IQ absolute deviations from median)						
τ: 0.05	4.94	14.58	35.43	64.94	80.10	92.56
0.15	5.14	13.96	33.02	63.28	79.54	92.52
0.25	4.79	10.64	25.65	54.43	71.50	88.53
0.33	5.26	8.92	19.22	43.38	61.44	81.71
Dispersion test (IE absolute deviations from mean)						
τ: 0.05	5.04	17.12	39.75	69.93	83.41	94.07
0.15	5.01	19.62	44.79	74.29	87.21	96.39
0.25	4.85	17.08	40.06	70.22	84.69	95.22
0.33	4.77	13.73	32.71	63.08	79.06	92.32

Table 9. Percentage rejections against Stochastic Volatility data generating process with t(3) innovations. Sample size T=1000.

q:	0	0.005	0.01	0.02	0.03	0.05
Quantic test						
τ: 0.05	4.92	9.86	23.86	51.22	68.11	83.84
0.15	4.65	11.29	28.94	58.01	73.89	88.82
0.25	4.33	8.98	21.87	48.75	66.21	83.58
0.33	4.99	7.15	14.31	35.10	51.92	72.73
0.50	5.01	5.03	5.03	5.21	5.52	6.87
Expectic test		:				
τ: 0.05	4.19	6.73	13.43	32.44	47.73	66.90
0.15	4.33	7.87	17.63	40.15	57.07	74.65
0.25	4.43	6.91	14.00	33.48	49.09	66.37
0.33	4.54	6.04	9.67	21.95	32.68	43.89
0.50	4.94	4.87	4.93	4.59	4.15	3.59
Quantic dispersion test (DQ)						
τ: 0.05	4.66	16.30	38.01	67.90	82.47	93.41
0.15	5.05	20.57	46.71	75.84	88.05	96.53
0.25	4.89	18.24	42.09	72.62	85.91	95.56
0.33	5.07	14.16	34.75	64.87	80.29	92.92
Expectic dispersion test (DE)						
τ: 0.05	4.50	9.86	23.46	49.18	65.72	82.64
0.15	4.22	15.67	37.36	66.49	81.33	92.82
0.25	4.46	18.96	42.58	72.15	85.28	94.98
0.33	4.48	20.09	44.53	74.39	86.79	95.74
Dispersion test (IQ absolute deviations from median)						
τ: 0.05	4.55	16.37	38.16	68.26	82.52	93.40
0.15	4.84	20.81	46.74	75.94	88.11	96.52
0.25	5.01	18.22	42.16	72.71	85.81	95.49
0.33	4.90	14.56	34.95	64.71	80.40	92.78
Dispersion test (IE absolute deviations from mean)						
τ: 0.05	4.39	11.56	28.74	56.04	72.25	87.51
0.15	4.44	19.77	44.19	74.12	86.63	95.76
0.25	4.81	22.02	48.27	78.02	89.06	96.90
0.33	4.75	20.60	45.45	75.45	87.81	96.24

Table 10a. Percentage rejections of the quantic test of relative invariance (T=500, m=6)

IQ TEST		σ_κ:	0	0.025	0.05	0.1	0.25
SMOOTHED RESIDUALS							
σ_η=0	τ=.05		4.8	26.9	52.9	74.1	86.4
	τ=.25		4.9	15.8	37.0	64.5	87.6
σ_η=.1	τ=.05		2.9	25.1	51.9	74.3	86.5
	τ=.25		0.6	10.1	34.0	62.5	79.6
σ_η=1	τ=.05		4.1	17.6	38.8	61.8	76.1
	τ=.25		2.4	10.1	27.1	49.8	67.4
FILTERED RESIDUALS							
σ_η=0	τ=.05		4.8	26.6	52.2	73.7	86.2
	τ=.25		4.9	15.7	37.3	64.5	87.4
σ_η=.1	τ=.05		4.8	25.3	50.5	72.4	84.4
	τ=.25		4.8	14.8	34.4	59.5	77.1
σ_η=1	τ=.05		4.7	12.9	29.6	51.0	69.0
	τ=.25		4.9	8.8	18.8	36.5	57.3

Table 10b. Percentage rejections of the expectic test of relative invariance (T=500, m=6)

IE TEST		σ_κ:	0	0.025	0.05	0.1	0.25
SMOOTHED RESIDUALS							
σ_η=0	τ=.05		4.4	25.4	49.1	67.0	74.8
	τ=.25		4.7	18.4	40.4	62.9	58.4
σ_η=.1	τ=.05		2.4	23.8	48.8	67.6	76.5
	τ=.25		0.0	11.1	38.8	64.2	69.3
σ_η=1	τ=.05		3.9	17.0	37.8	59.1	71.2
	τ=.25		0.8	14.3	38.9	63.3	75.7
FILTERED RESIDUALS							
σ_η=0	τ=.05		4.5	24.7	48.3	66.4	74.5
	τ=.25		4.8	18.2	39.9	62.6	58.5
σ_η=.1	τ=.05		4.6	23.8	47.2	66.0	74.6
	τ=.25		5.0	17.4	38.2	60.6	64.4
σ_η=1	τ=.05		4.6	12.8	28.5	48.8	63.8
	τ=.25		4.9	10.0	22.8	42.3	61.1