Panel cointegration rank test with cross-section dependence

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Abstract

In this paper we propose a test statistic to determine the cointegration rank of VAR processes in panel data allowing for cross-section dependence among the time series in the panel data. The cross-section dependence is accounted for through the specification of an approximate common factor model, which allows to cover situations where there is cross-cointegration, i.e. cointegration among the cross-section dimension. The framework that is considered in the paper embedes as a particular case the situation where time series in the panel data set are cross-section independent. Finite sample sizes and powers are simulated via a Monte Carlo experiment.

JEL Classification: C12, C22

Keywords: Panel data cointegration, cointegrating rank, cross-section dependence, cross-cointegration

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1 Introduction

Since the pioneering work of Engel and Granger (1987) and Johansen (1988, 1995), the literature on cointegration grew at a rapid pace. Even though this topic has been covered extensively in time series, the analysis of cointegration in panel data is in early stages of development. The two main approaches to the analysis of cointegration in panels are the residual-based and the system-based approaches. Regarding the residual-based approximation, we can find in the econometric literature proposals that specify either the null hypothesis of no cointegration – see Pedroni (1995, 1999, 2004) and Kao (1999), among others – or the null hypothesis of cointegration – see McCoskey and Kao (1998) and Westerlund (2005). The system-based approach has relied on the panel extension of Johansen's methodology. Thus, based on the vector autoregressive model (VAR) framework, Larsson, Lyhagen and Löthgren (2001) propose the panel Johansen's test analog to determine the rank of the cointegrating space.

The proposals mentioned above define the so-called first generation of panel cointegration tests, in which the time series (individuals) that define the panel data set are assumed to be cross-section independent. Unfortunately, this assumption is crucial for the limiting distributions that are obtained in these papers and, in most cases, it is not satisfied from an empirical point of view. Violation of the cross-section independence assumption implies that Central Limit Theorems (CLT) cannot be applied and, hence, the panel data based statistics do not converge to the standard normal distribution. Provided that in most cases the economic time series of different sectors, cities, regions or countries are closely related, the use of these panel data statistics to analyse the presence of cointegration can lead to misleading conclusions. The challenge to overcome this limitation has given rise to the so-called second generation of panel cointegration tests.

Proposals that consider the presence of cross-section dependence among the time series that define the panel data set include Bai and Carrion-i-Silvestre (2005), Banerjee and Carrion-i-Silvestre (2006), and Gengenbach, Palm and Urbain (2006) for the single equation framework, and Groen and Kleinberger (2003) and Breitung (2005) for the vector error correction (VECM) framework. For a more detailed literature review on panel cointegration see Breitung and Pesaran (2007).

The aim of this paper is to solve some limitations of the existing literature on panel cointegration analysis. To this end, we propose a test statistic to determine the cointegrating rank in a panel system of equations allowing for the presence of cross-section dependence across the systems of variables in the panel set-up. We deal with cross-section dependence by means of approximate common factor models as proposed in Bai and Ng (2004), Bai and Carrion-i-Silvestre (2005), Banerjee and Carrion-i-Silvestre (2006), and Gengenbach, Palm and Urbain (2006), among others. The novelty of our approach is that it takes into account the possibility that there might be more than one cointegrating relationship among the variables that define the system for each individual, at the time that controls for the presence of cross-section dependence among the different systems in a parsimonious way through the use of common factors. To the best of our knowledge, this has not been addressed in the literature.

Our proposal relates to other test statistics that are available in the literature to determine the number of stochastic trends in individual systems. Thus, the statistic determines the number of stochastic trends using the principal component approximation as in Stock and Watson (1988) and Bai and Ng (2004). Other statistics that base on the degeneration of the moment matrix that involves the time series in the system are the ones in Phillips and Ouliaris (1990), Shintani (2001), Harris and Poskitt (2004) and Cai and Shintani (2006). However, none of these approaches considers the case where the variables in the model are affected by global stochastic trends, which in our set-up are captured by the common factors.

In order to investigate the small-sample properties of the panel cointegration rank test that we propose, we conduct a Monte Carlo simulation. We estimate two different models depending on the form of the deterministic component. One model where the deterministic term consists of only the constant and another where the deterministic term consists of the constant and the linear trend. The results of the simulation study indicate that, in general, panel data statistic performs better than the univariate one.

The remainder of the paper is organized as follows. Section 2 presents the model, assumptions and the test statistic that is used to determine the cointegrating rank. In addition, we discuss the way in which the individual statistics can be combined to specify a panel data cointegrating rank statistic. Section 4 analyses the finite sample of our approach, both in an individual-by-individual framework and in a panel set-up, by means of Monte Carlo simulation. Finally, some concluding remarks are presented in Section 5. The Appendix collects all the proofs.

2 Model and assumptions

Let $Y_{i,t}$ be a $(k \times 1)$ vector of stochastic process with the data generating process (DGP) defined as:

$$\begin{aligned} Y_{i,t} &= D_{i,t} + u_{i,t} & (1) \\ u_{i,t} &= \lambda_i F_t + e_{i,t} & (2) \\ (I - L) F_t &= C(L) w_t & (3) \\ (I - L) e_{i,t} &= G_i(L) \varepsilon_{i,t}, & (4) \end{aligned}$$

where $D_{i,t}$ denotes the deterministic component, which in this paper can be either $D_{i,t} = \mu_i$ – henceforth, this specification is denoted as the only constant case – or $D_{i,t} = \mu_i + \delta_i t$ – hereafter, the linear time trend case – $t = 1, \ldots, T$ and $i = 1, \ldots, N$. Note that the case of non-deterministic components $D_{i,t} = 0$ is also covered in our framework as a particular case of the only constant case. The component F_t denotes a $(q \times 1)$ vector of common factors and λ_i is a $(k \times q)$ matrix of factor loadings. Finally, $e_{i,t}$ is a $(k \times 1)$ vector that collects the idiosyncratic stochastic component. It is worth mentioning that the presentation of the model is done in a general way, so that the case where there are no common factors at all, i.e. $\lambda_i = 0\ \forall i$, can be embeded in our framework – further specific comments on this concern are given below. Let $M < \infty$ be a generic positive number, not depending on T and N. Throught the paper, we use $\|A\|$ to denote the Euclidean norm $tr\,(A'A)^{1/2}$ of matrix A. The stochastic processes that participate on the definition of the DGP are assumed to satisfy the following assumptions:

Assumption A: (i) for non-random λ_i, $\|\lambda_i\| \leq M$; for random λ_i, $E\|\lambda_i\|^4 \leq M$, (ii) $\frac{1}{N}\sum_{i=1}^{N} \lambda_i' \lambda_i \xrightarrow{p} \Sigma_\Lambda$, a $(q \times q)$ positive definite matrix.

Assumption B: (i) $w_t \sim iid\,(0, \Sigma_w)$, $E\|w_t\|^4 \leq M$, and (ii) $Var\,(\Delta F_t) = \sum_{j=0}^{\infty} C_j \Sigma_w C_j' > 0$, (iii) $\sum_{j=0}^{\infty} j\,\|C_j\| < M$; and (iv) $C\,(1)$ has rank q_1, $0 \leq q_1 \leq q$.

Assumption C: (i) for each i, $\varepsilon_{i,t} \sim iid\,(0, \Sigma_{\varepsilon_i})$, $E\|\varepsilon_{i,t}\|^4 \leq M$, (ii) $Var\,(\Delta\varepsilon_{i,t}) = \sum_{j=0}^{\infty} G_{i,j}\Sigma_{\varepsilon_i} G_{i,j}' > 0$, (iii) $\sum_{j=0}^{\infty} j\,\|G_{i,j}\| < M$; and (iv) $G\,(1)$ has rank r.

Assumption D: The errors $\varepsilon_{i,t}$, w_t, and the loadings λ_i are three mutually independent groups.

Assumption E: $E\|F_0\| \leq M$, and for every $i = 1, \ldots, N$, $E\|e_{i,0}\| \leq M$.

The definition of the $(k \times q)$ loading matrix λ_i is given by

$$\lambda_i = \begin{bmatrix} \lambda_{i,1,1} & \lambda_{i,1,2} \\ \lambda_{i,2,1} & \lambda_{i,2,2} \end{bmatrix}$$

so that we can impose restrictions on how the factors affect the elements of $Y_{i,t}$ in (1). Thus, some of the factors can only affect one subset of the variables, say, the series that defines the cointegrating space, but not the other variables, and the other way round. Therefore, situations where $\lambda_{i,1,2} = 0$ and/or $\lambda_{i,2,1} = 0$ are covered in this set-up. Note that it is possible that both $\lambda_{i,1,2} \neq 0$ and $\lambda_{i,2,1} \neq 0$, which is the general situation that is assumed henceforth.

The unobservable common factors are estimated using the principal component approach suggested in Bai and Ng (2002, 2004). Let us consider the general deterministic component given by $D_{i,t} = \mu_i + \delta_i t$. Taking the first difference of the model we have

$$\Delta Y_{i,t} = \delta_i + \lambda_i \Delta F_t + \Delta e_{i,t}. \tag{5}$$

We can define the idempotent matrix $M = I_{T-1} - \iota\,(\iota'\iota)^{-1}\,\iota'$, with ι a $(T-1) \times 1$ vector of ones. Then,

$$\begin{aligned} M\Delta Y_i &= M\Delta F \lambda_i' + M\Delta e_i. \\ y_i &= f\lambda_i' + z_i. \end{aligned}$$

Note that when the deterministic component is $D_{i,t} = \mu_i$, taking first differences removes the constant term, so that in this case we can define $M = I_{T-1}$ and the rest of our discussion applies without sole modification. The common factors are extracted as the q eigenvectors corresponding to the q largest eigenvalues of the $(T-1) \times (T-1)$ matrix yy', where $y = [y_1, \ldots, y_N]$ is a $(T-1) \times Nk$ matrix that is defined using the $(T-1) \times k$ matrices y_i, $i = 1, \ldots, N$. The matrix of estimated weights, $\hat{\Lambda} = \left(\hat{\lambda}_1, \ldots, \hat{\lambda}_N\right)$, is given by $\hat{\Lambda} = y'\hat{f}$. We can obtain an estimate of z_i from $\hat{z}_i = y_i - \hat{f}\hat{\lambda}_i'$, as in Bai and Ng (2004). Note that we can recover the common factors as $\hat{F}_t = \sum_{j=2}^{t} \hat{f}_j$ and the idiosyncratic component as $\hat{e}_{i,t} = \sum_{j=2}^{t} \hat{z}_{i,j}$. The determination of the cointegrating rank can be devised using $\hat{e}_{i,t}$ in the usual VECM representation.

The model given by (1) and (2) can be written as

$$Y_{i,t} = D_{i,t} + \lambda_i F_t + e_{i,t}. \tag{6}$$

Note that if we assume the simplest VAR model for the idiosyncratic disturbance term, we have

$$e_{i,t} = B_i e_{i,t-1} + \varepsilon_{i,t},$$

and defining $B_i - I_p = \Pi_i = \alpha_i \beta_i'$, we get $e_{i,t} = \left[I_p - \left(I_p + \alpha_i \beta_i'\right) L\right]^{-1} \varepsilon_{i,t}$, which substituted in (6), for $D_{i,t} = \mu_i + \delta_i t$, gives

$$\begin{aligned}
Y_{i,t} &= D_{i,t} + \lambda_i F_t + \left[I_p - \left(I_p + \alpha_i \beta_i'\right) L\right]^{-1} \varepsilon_{i,t} \\
\Delta Y_{i,t} &= \left[I_p - \left(I_p + \alpha_i \beta_i'\right) L\right] D_{i,t} + \left[I_p - \left(I_p + \alpha_i \beta_i'\right) L\right] \lambda_i F_t + \varepsilon_{i,t} \\
&= \left[-\alpha_i \beta_i' \mu_i + \left(I_p + \alpha_i \beta_i'\right) \delta_i\right] - \alpha_i \beta_i' \delta_i t + \alpha_i \beta_i' Y_{i,t-1} \\
&\quad + \lambda_i \Delta F_t - \alpha_i \beta_i' \lambda_i F_{t-1} + \varepsilon_{i,t}.
\end{aligned}$$

The model can be expressed as:

$$\begin{aligned}
\Delta\left(Y_{i,t} - D_{i,t} - \lambda_i F_t\right) &= \alpha_i \beta_i' \left(Y_{i,t-1} - D_{i,t-1} - \lambda_i F_{t-1}\right) + \varepsilon_{i,t} \\
\Delta e_{i,t} &= \alpha_i \beta_i' e_{i,t-1} + \varepsilon_{i,t}.
\end{aligned} \tag{7}$$

We can see that the cointegrating rank can be obtained from the analysis of the idiosyncratic stochastic component. Instead of using the Johansen LR statistic as in Larsson, Lyhagen and Löthgren (2001), in this paper we propose to determine the rank with a test statistic that is based on the multivariate version of the square of the modified Sargan-Bhargava (MSB) statistic proposed in Stock (1999).

The definition of the testing procedure builds upon the different rates of convergence of the elements on the $Q_{\hat{e}_i \hat{e}_i}$ matrix under the null hypothesis. Without loss of generality, let us assume that the rank of the cointegrating space is $0 < r < k$. We can define the orthogonal matrix $A = [A_1 : A_2]$ with A_1 a $(k \times r)$ matrix and A_2 a $(k \times m)$ matrix, $m = k - r$, such that the first r elements of the rotated vector $e_{i,t}^A = A' e_{i,t} = \left(\left(A_1' e_{i,t}\right)', \left(A_2' e_{i,t}\right)'\right)'$ are I(0) and the other m elements are I(1). Accordingly, we define the partition of the long-run variance matrix as

$$\Omega_{\Delta e_i^A \Delta e_i^A} = \begin{bmatrix} \Omega_{11,i} & \Omega_{12,i} \\ \Omega_{21,i} & \Omega_{22,i} \end{bmatrix}.$$

Furthermore, note that $\pi\left(T^{-1} Q_{\hat{e}_i \hat{e}_i} \hat{\Omega}_{\Delta \hat{e}_i \Delta \hat{e}_i}^{-1}\right) = \pi\left(T^{-1} Q_{\hat{e}_i^A \hat{e}_i^A} \hat{\Omega}_{\Delta \hat{e}_i^A \Delta \hat{e}_i^A}^{-1}\right) = \pi\left(T^{-1} \hat{\Omega}_{\Delta \hat{e}_i^A \Delta \hat{e}_i^A}^{-1/2} Q_{\hat{e}_i^A \hat{e}_i^A} \hat{\Omega}_{\Delta \hat{e}_i^A \Delta \hat{e}_i^A}^{-1/2}\right)$, where $\pi(\cdot)$ denotes the eigenvalues of the matrix between parenthesis. The determination of the number of stochastic trends in the system relies on the following sequential testing procedure:

1. First, assume that the cointegrating rank is zero, i.e. set $m = k$.

2. Specify the null hypothesis that there are $l = m$ common stochastic trends $(H_0 : l = m)$ against the alternative hypothesis that there are $l < m$ common stochastic trends $(H_1 : l < m)$.

3. Estimate A_2 as the m eigenvectors that corresponds with the m largest eigenvalues of $T^{-1} Q_{\hat{e}_i \hat{e}_i}$.

4. Define the univariate MSB statistic as

$$
\begin{aligned}
MSB_{j,i}(m) &= \pi^{\min}\left(T^{-1}Q_{\hat{e}_i^{A_2}\hat{e}_i^{A_2}}\hat{\Omega}^{-1}_{\Delta\hat{e}_i^{A_2}\Delta\hat{e}_i^{A_2}}\right) \\
&= \pi^{\min}\left(T^{-1}\hat{\Omega}^{-1/2}_{\Delta\hat{e}_i^{A_2}\Delta\hat{e}_i^{A_2}}Q_{\hat{e}_i^{A_2}\hat{e}_i^{A_2}}\hat{\Omega}^{-1/2}_{\Delta\hat{e}_i^{A_2}\Delta\hat{e}_i^{A_2}}\right) \\
&= \hat{\pi}_1, \qquad (8)
\end{aligned}
$$

where the subscript $j = \{\mu, \tau\}$ refers to the deterministic component that is used in the model – μ for the $D_{i,t} = \mu_i$ deterministic specification and τ for the $D_{i,t} = \mu_i + \delta_i t$ one – being $\pi_1 < \cdots < \pi_m$ the eigenvalues of $T^{-1}Q_{\hat{e}_i^{A_2}\hat{e}_i^{A_2}}\hat{\Omega}^{-1}_{\Delta\hat{e}_i^{A_2}\Delta\hat{e}_i^{A_2}}$ sorted in ascending order, and $\pi^{\min}(\cdot)$ denoting the minimum eigenvalue operator.

5. Compare the value of the $MSB_{j,i}(m)$ statistic with the corresponding critical values from the left tail of the distribution – i.e. the null hypothesis is rejected if $MSB_{j,i}(m)$ is smaller than the critical value.

6. If the null hypothesis of $H_0 : l = m$ common stochastic trends is rejected, specify $l = m - 1$ and return to step 2. The process continues till either the null hypothesis is not rejected or when $l = 0$ is achieved.

The estimation of $\Omega_{\Delta e_i^{A_2}\Delta e_i^{A_2}}$ can be obtained in a parametric way from the estimation of the VECM model specification. Expressed in matrix notation, we have

$$
\begin{aligned}
\Delta e_i^{A_2} &= e_{i,-1}^{A_2}\Pi_i + \Delta e_i^{A_2}\Gamma_{i,p_i}(L) + \varepsilon_i \\
\Delta e_i^{A_2}\left(I - \Gamma_{i,p_i}(L)\right) &= e_{i,-1}^{A_2}\Pi_i + \varepsilon_i \\
\Delta e_i^{A_2} &= e_{i,-1}^{A_2}\Pi_i\left(I - \Gamma_{i,p_i}(L)\right)^{-1} + \varepsilon_i\left(I - \Gamma_{i,p_i}(L)\right)^{-1},
\end{aligned}
$$

where p_i denotes the number of lags of $\Delta e_{i,i}^{A_2}$ that are considered. Following Ng and Perron (2001), we define $\hat{\Omega}^{VAR}_{\Delta\hat{e}_i^{A_2}\Delta\hat{e}_i^{A_2}} = \left(\left(I - \hat{\Gamma}_{i,p_i}(1)\right)^{-1}\right)' T^{-1}\hat{\varepsilon}_i'\hat{\varepsilon}_i\left(I - \hat{\Gamma}_{i,p_i}(1)\right)^{-1}$, where the lag order of the model p_i is estimated using the modified information criterion in Qu and Perron (2006) assuming that the cointegrating rank is zero – note that under the null hypothesis we assume that there are m stochastic trends in the system defined by the m variables of $e_{i,t}^{A_2}$.

The limiting distribution of the $MSB_{j,i}(m)$ statistic, $j = \{\mu, \tau\}$, is established in the following Theorem.

Theorem 1 *Let $Y_{i,t}$, $i = 1, \ldots, N$, $t = 1, \ldots, T$, be a $(k \times 1)$ vector of stochastic processes with the DGP given by (1) to (4). Under the null hypothesis that there are $m = k - r$ common stochastic trends, with $p_i \to \infty$ and $p_i^3/T \to 0$ as $T \to \infty$, the MSB statistic given in (8) converges to:*

(a) For the only constant model: $MSB_{\mu,i}(m) \Rightarrow \pi^{\min}\left(\int_0^1 W_i(s)W_i(s)'ds\right)$

(a) For the linear trend model: $MSB_{\tau,i}(m) \Rightarrow \pi^{\min}\left(\int_0^1 V_i(s)V_i(s)'ds\right)$,

where $\Rightarrow$ denotes weak convergence, $W_i(s)$ is an $(m \times 1)$ vector of independent standard Brownian motions, and $V_i(s) = W_i(s) - sW_i(1)$ is an $(m \times 1)$ vector of independent Brownian bridges.

The proof of Theorem 1 is given in the Appendix. It has to be stressed that our framework treats as a special case the situation in which the time series are assumed to be cross-section independent. Thus, note that if we impose $\lambda_i = 0\ \forall i$, then equation (6) is given by $Y_{i,t} = D_{i,t} + e_{i,t}$. The estimation of the parameters of the deterministic component can be done specifying the model in first differences so that $y_i = z_i$, where $y_i = M\Delta Y_i$ and $z_i = M\Delta e_i$ – as before, $M = I_{T-1}$ for the constant and $M = I_{T-1} - \iota(\iota'\iota)^{-1}\iota'$ for the time trend deterministic specifications. Then, defining $\hat{e}_{i,t} = \sum_{j=2}^{t} y_{i,j}$ the computation of the MSB statistic proceeds as above, giving rise to test statistics with the same limiting distribution as the ones reported in Theorem 1. The critical values for the $MSB_j(m)$ statistic, $j = \{\mu, \tau\}$, are reported in Table 1 for different sample sizes. These finite sample critical values are computed using the autoregressive spectral density estimator $\hat{\Omega}^{VAR}_{\Delta\hat{e}_i^{A_2}\Delta\hat{e}_i^{A_2}}$, where the lag order of the model p_i is estimated using the modified information criterion in Qu and Perron (2006) assuming that the cointegrating rank is zero. Following Ng and Perron (2001) and Qu and Perron (2006), we have specified the upper bound for p_i as $int\left(12\,(T/100)^{1/4}\right)$.

We need to give further details on the way in which the maximum lag order is specified... to be specific, we need to inform about:

- The maximum number of lags that were used to obtain the critical values for the different T:

    1. Do we used the rule $p_{i,\max} = int\left[12\,(T/100)^{1/4}\right]$? If so, in which cases?

- We need to report "asymptotic critical values" which can be computed assuming that the disturbance terms are *iid* and using $T = 1000$

- We can report the estimated response surfaces to approximate the p-values, or mention that they are available upon request

The MSB statistic that is presented in the paper is consistent under the alternative hypothesis that there are less common stochastic trends than the ones specified under the null hypothesis. The following Theorem presents the rate at which the MSB statistic diverges under the alternative hypothesis.

Theorem 2 *Let $Y_{i,t}$, $i = 1, \ldots, N$, $t = 1, \ldots, T$, be a $(k \times 1)$ vector of stochastic processes with the DGP given by (1) to (4). Under the alternative hypothesis that there are $l < m$ common stochastic trends we have that $MSB_{j,i}(m) = O_p(T^{-1})$, $j = \{\mu, \tau\}$.*

The proof is given in the Appendix. The result in Theorem 2 shows that the MSB statistic diverges under the alternative hypothesis at a faster rate than the cointegrating rank test statistics proposed, for instance, in Phillips and Ouliaris (1990), Shintani (2001), and Cai and Shintani (2006). Note that this faster rate of convergence derives from the use of the autoregressive spectral density estimator $\hat{\Omega}^{VAR}_{\Delta\hat{e}_i^{A}\Delta\hat{e}_i^{A}}$.

3 Panel data cointegrating rank tests

The individual MSB statistics can be pooled to define panel data statistics, which are expected to increase the performance of the statistical inference when estimating the cointegrating rank. In this section we define up to four different panel data statistics depending on the way in which the individual information is combined. The performance of these four alternatives is analyzed by simulation in the next section.

The first panel data MSB (PMSB) statistic is based on the standardized mean of the individual statistics

$$PMSB_j^Z(m) = \frac{\sqrt{N}(\overline{MSB}_j(m) - E(MSB_j(m)))}{\sqrt{Var(MSB_j(m))}}, \tag{9}$$

where $\overline{MSB}_j(m) = N^{-1}\sum_{i=1}^{N} MSB_{j,i}(m)$, and $E(MSB_j(m))$ and $Var(MSB_j(m))$ are the mean and the variance of the $MSB_j(m)$ statistic computed from (8). The limiting distribution of the PMSB statistic is given in the following Theorem.

Theorem 3 *Let $Y_{i,t}$, $i = 1, \ldots, N$, $t = 1, \ldots, T$, be a $(k \times 1)$ vector of stochastic processes with the DGP given by (1) to (4). Under the null hypothesis that there are $m = k - r$ common stochastic trends, with $p_i \to \infty$ and $p_i^3/T \to 0$ as $T \to \infty$, the PMSB statistic given in (9) converges to:*

$$PMSB_j^Z(m) \Rightarrow N(0,1).$$

As in Pedroni (2004), in order to prove Theorem 3 we require only the assumption of finite second moments of the random variables characterized as Brownian motion functionals $\Upsilon \equiv \left(\pi^{\min}\left(\int_0^1 W_i(s) W_i(s)' ds\right), \pi^{\min}\left(\int_0^1 V_i(s) V_i(s)' ds\right)\right)'$, which will allow us to apply the Lindberg-Levy Central Limit Theorem as $N \to \infty$. The mean and the variance of the $MSB_j(m)$ statistic, $j = \{\mu, \tau\}$, that have been computed by simulation are presented in Table 2.

It is possible to define panel data statistics based on the combination of the individual p-values. Maddala and Wu (1999) defines the panel data Fisher-type statistic $PMSB_j^F(m) = -2\sum_{i=1}^{N} \ln \varphi_i \sim \chi_{2N}^2$, where φ_i denotes the p-value of the $MSB_{j,i}(m)$ statistic, $j = \{\mu, \tau\}$. Although the $PMSB_j^F(m)$ statistic is valid for finite N, Choi (2001) suggests to compute the following tests when $N \to \infty$:

$$\begin{aligned} PMSB_j^{C1}(m) &= \frac{-2\sum_{i=1}^{N} \ln \varphi_i - 2N}{\sqrt{4N}} \Rightarrow N(0,1) \\ PMSB_j^{C2}(m) &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Phi^{-1}(\varphi_i) \Rightarrow N(0,1), \end{aligned}$$

where $\Phi(\cdot)$ denotes the standard cumulative distribution function.

4 Monte Carlo simulation

We now analyze the small sample performance of the MSB panel cointegration rank test for the two deterministic specifications that are considered in this paper. The DGP that is used in this section is based on the ones in Toda (1995) and Saikkonen and Lütkepohl (2000) and has the following form:

$$
\begin{aligned}
Y_{i,t} &= D_{i,t} + \lambda_i F_t + e_{i,t} && (10)\\
\begin{pmatrix} e_{1,i,t} \\ e_{2,i,t} \end{pmatrix} &= \begin{pmatrix} \psi_i & 0 \\ 0 & I_{k-r} \end{pmatrix} \begin{pmatrix} e_{1,i,t-1} \\ e_{2,i,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,i,t} \\ \varepsilon_{2,i,t} \end{pmatrix} && (11)\\
F_j &= \rho F_{j-1} + \sigma_F w_t && (12)\\
\varepsilon_{i,t} &= iid\ N\left(0, \begin{pmatrix} I_r & \theta_i \\ \theta_i & I_{k-r} \end{pmatrix}\right), && (13)
\end{aligned}
$$

where $i = 1, \ldots, N$, $t = 1, \ldots, T$ and $j = 1, \ldots, q$. Note that when there is no cointegration among any of the time series ($r = 0$) equation (11) reduces to $e_{i,t} = e_{i,t-1} + \varepsilon_{i,t}$ with $\varepsilon_{i,t} = iid\ N(0, I_k)$. Besides, when all time series are stationary ($r = k$) equation (11) reduces to $e_{i,t} = \psi_i e_{i,t-1} + \varepsilon_{i,t}$.

The parameter sets are defined as follows. We consider a system defined by $k = 3$ variables. For the deterministic component, we have considered $D_{i,t} = \mu_i + \delta_i t$, with $\mu_i \sim U[-1, 1]$ and $\delta_i \sim U[-0.5, 0.5]$, where U denotes the uniform distribution. The idiosyncratic cointegrating rank is investigated using $\psi_i = aI_r$ with $a = \{0.5, 0.8, 0.9, 1\}$. Furthermore, we define $\theta_i = b1_{r\times(k-r)}$ where $1_{r\times(k-r)}$ denotes a $r \times (k - r)$ matrix of ones, and $b = \{0, 0.4, 0.8\}$. Note that θ_i controls the correlation among the $\varepsilon_{i,t}$ disturbance terms. As for the common factor component, we specify $\lambda_i \sim N(1, 1)$, $\rho = \{0.9, 0.95, 1\}$, $\sigma_F^2 = \{0.5, 1, 10\}$ and $w_t \sim N(0, 1)$, $j = 1, \ldots, q$, with $q = \{1, 3\}$ common factors. The number of common factors is estimated using the panel Bayesian information criterion (BIC) in Bai and Ng (2002) using $q_{\max} = 6$ as the maximum number of common factors.

The simulations are performed in GAUSS using the COINT 2.0 library. The empirical size and power of the statistics are obtained using 1000 replications with the level of significance set at the 5% level for all different combinations of individuals $N = \{1, 20, 40\}$ and number of time series observations $T = \{50, 100, 200\}$. For conciseness, we report only the results for $N = 1$ and for $N = 20$. The results for $N = 40$ are very similar to those for $N = 20$ and they are available upon request. The simulations have used either the critical values or the mean and the variance in Tables 1 and 2.

4.1 Unit-by-unit analysis

In this section we investigate the performance of the statistic when $N = 1$, provided that this is the first time that the multivariate MSB statistic is used to estimate the number of common idiosyncratic stochastic trends.

Tables 3 to 6 present the results for the only constant and the time trend cases. First, we can see that the results do not depend on the stochastic properties of the common factor, provided that the performance of the MSB statistics is similar regardless of whether the common factor is I(0) or I(1), and regardless of the magnitude of the disturbance variance that participates in the generation of the common factors (σ_F^2). As for the estimation of the common factors, the procedure always detects the true number of common factors. The performance of the MSB statistic depends on how close is the autoregressive parameter a

to one. Thus, for $a = 0.5$ (Tables 3 and 5) we can see that the statistical procedure that has been proposed in this paper selects the correct number of stochastic trends in most of cases, especially for $T > 50$. As expected, the behaviour of the procedure worsens for $a = 0.8$ (Tables 4 and 6), although it tends to select the correct number of stochastic trends as T increases. It is worth mentioning that these conclusions are reached regardless of the deterministic component that is used.

4.2 Panel data analysis

The multivariate results for both the only constant and the time trend cases when $N = 20$ are presented in Tables 7 to A.2. The definition of the MSB panel data statistic helps to increase the ability of the statistical inference to select the correct number of stochastic trends. This improvement is noticeable for the smaller sample size that we have considered, where the use of the panel data based statistic reduces the tendency shown by the individual MSB statistic to overestimate the number of common stochastic trends. Thus, if we compare the results in Tables 4 and A.2 we can see that the number of stochastic trends that is selected using the individual MSB test tends to be greater than the true one, whereas this overestimation feature is not so marked when using the panel data statistic. Examination of Tables 6 and A.2 reveals a similar pattern for the case of linear trend.

The performance of the statistic improves as the sample size increases. The power of the panel statistic is one or almost one, especially for $T > 50$. Another conclusion that arises from the examination of the results is that the statistic is slightly undersized regardless of the deterministic specification. As in the univariate case, the procedure always detects the correct number of common factors. Another similarity with the individual test statistic is that the multivariate statistic does not depend on the stochastic properties of the common factor and the magnitude of the disturbance variance that participates in the generation of the common factors.

5 Conclusion

In this paper we propose a new test statistic to estimate the cointegrating rank both in a unit-by-unit analysis and in a panel data framework. Our proposal covers the case of cross-section dependence, which is a relevant situation from both theoretical and empirical point of views. The set-up that is considered in the paper allows to cover wild cross-section dependence cases, i.e. cases where the time series of one individual systems are cointegrated with times series of other individual systems. This situation can be considered as the multivariate extension of the cross-cointegration concept defined earlier in the literature.

The performance of the proposal is investigated with Monte Carlo simulations. In general, the panel data based MSB statistic provides with better estimation of the number of stochastic trends that are present in each individual system.

A Mathematical appendix

A.1 Proof of Theorem 1

A.1.1 The only constant case

Note that the estimated difference of the idiosyncratic stochastic term is:

$$\hat{z}_{i,t} = z_{i,t} + \lambda_i f_t - \hat{\lambda}_i \hat{f}_t.$$

Following Bai and Ng (2003), we can express the model as

$$\begin{aligned} \hat{z}_{i,t} &= z_{i,t} + \lambda_i H^{-1} H f_t - \lambda_i H^{-1} \hat{f}_t + \lambda_i H^{-1} \hat{f}_t - \hat{\lambda}_i \hat{f}_t \\ &= z_{i,t} + \lambda_i H^{-1} \left(H f_t - \hat{f}_t \right) - \left(\hat{\lambda}_i - \lambda_i H^{-1} \right) \hat{f}_t \\ &= z_{i,t} + \lambda_i H^{-1} v_t - d_i \hat{f}_t, \end{aligned} \tag{14}$$

where $v_t = \left(H f_t - \hat{f}_t \right)$ and $d_i = \left(\hat{\lambda}_i - \lambda_i H^{-1} \right)$. Let us define the partial sum process using the estimated residuals as $\hat{e}_{i,t} = \sum_{j=2}^{t} \hat{z}_{i,j} = \sum_{j=2}^{t} \left([M \ \Delta \hat{e}_i]_j \right)'$, where $[\cdot]_j$ denotes the file j of the matrix between brackets. By Lemma 3 and C1 in Bai and Ng (2004), $T^{-1/2} \left\| \sum_{s=j}^{t} \lambda_i H^{-1} v_j \right\| = o_p(1)$ and $T^{-1/2} \left\| \sum_{j=2}^{t} d_i \hat{f}_j \right\| = o_p(1)$, so that

$$T^{-1/2} \hat{e}_{i,t} = T^{-1/2} \sum_{j=2}^{t} \left([M \Delta e_i]_j \right)' + o_p(1), \tag{15}$$

with

$$\sum_{j=2}^{t} [M \Delta e_i]_j = \sum_{j=2}^{t} \left(\Delta e'_{i,j} - [P \Delta e_i]_j \right),$$

where here P is a matrix of zeros provided that for the only constant case we have $M = I_{T-1}$. The cumulated process is equal to $T^{-1/2} \hat{e}_{i,t} = T^{-1/2} e_{i,t} - T^{-1/2} e_{i,1} + o_p(1)$. If we rotate the vector $\hat{e}_{i,t}$ and define $\hat{e}^A_{i,t} = A' \hat{e}_{i,t} = \left((A'_1 \hat{e}_{i,t})', (A'_2 \hat{e}_{i,t})' \right)'$ we can see that $T^{-1/2} A'_1 \hat{e}_{i,t} = o_p(1)$ provided that $A'_1 \hat{e}_{i,t}$ defines the stationary relationships and $T^{-1/2} A'_2 \hat{e}_{i,t} = O_p(1)$ given that $A'_2 \hat{e}_{i,t}$ defines the I(1) stochastic trends. Therefore,

$$\begin{aligned} T^{-1/2} A' \hat{e}_{i,t} &= \left(\left(T^{-1/2} A'_1 (e_{i,t} - e_{i,1}) \right)', \left(T^{-1/2} A'_2 (e_{i,t} - e_{i,1}) \right)' \right)' \\ &\Rightarrow \left(0'_r, \Omega^{1/2}_{22,i} \left(W_i(s) - W_i(0) \right)' \right)' \\ &\equiv \left(0'_r, \Omega^{1/2}_{22,i} W_i(s)' \right)' \end{aligned}$$

where 0_r is an r vector of zeros, $W_i(s)$ denotes a $k - r$ vector of independent standard Brownian motions, and $W_i(0) = 0$. Then, we can see that

$$\begin{aligned} T^{-1} Q_{\hat{e}^A_i \hat{e}^A_i} &= T^{-2} \hat{e}^{A\prime}_i \hat{e}^A_i \\ &\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & \Omega^{1/2}_{22,i} \int_0^1 W_i(s) W_i(s)' ds \ \Omega^{1/2}_{22,i} \end{bmatrix}, \end{aligned}$$

given that $T^{-2}A_1'e_i'e_iA_1 = o_p(1)$, $T^{-2}A_1'e_i'e_iA_2 = o_p(1)$ and $T^{-2}A_2'e_i'e_iA_2 = O_p(1)$. Therefore, using these elements the limiting distribution of the multivariate MSB statistic is given by:

$$\begin{aligned} MSB_{\mu,i}(m) &= \pi^{\min}\left(T^{-1}Q_{\hat{e}_i\hat{e}_i}\hat{\Omega}^{-1}_{\Delta\hat{e}_i\Delta\hat{e}_i}\right) \\ &= \pi^{\min}\left(T^{-1}Q_{\hat{e}_i^A\hat{e}_i^A}\hat{\Omega}^{-1}_{\Delta\hat{e}_i^A\Delta\hat{e}_i^A}\right) \\ &= \pi^{\min}\left(T^{-1}\hat{\Omega}^{-1/2}_{\Delta\hat{e}_i^A\Delta\hat{e}_i^A}Q_{\hat{e}_i^A\hat{e}_i^A}\hat{\Omega}^{-1/2}_{\Delta\hat{e}_i^A\Delta\hat{e}_i^A}\right) \\ &\Rightarrow \pi^{\min}\left(\int_0^1 W_i(s)\,W_i(s)'\,ds\right), \end{aligned}$$

provided that $\hat{\Omega}_{\Delta\hat{e}_i^A\Delta\hat{e}_i^A} \xrightarrow{p} \Omega_{\Delta e_i^A\Delta e_i^A}$, where $\xrightarrow{p}$ denotes convergence in probability, and where $W_i(s)$ denotes an $m = k - r$ vector of independent standard Brownian motions.

A.1.2 The linear time trend case

The proof for this deterministic component follows the one for the constant case, but where the projection matrix $M = I_{T-1} - \iota(\iota'\iota)^{-1}\iota'$. As above, $T^{-1/2}\hat{e}_{i,t}$ is given by (15) where now

$$\begin{aligned} \hat{e}_{i,t} &= \sum_{j=2}^{t}[M\ \Delta e_i]_j = \sum_{j=2}^{t}\left(\Delta e_{i,j}' - [P\Delta e_i]_j\right) = \sum_{j=2}^{t}\left(\Delta e_{i,j}' - \left[\iota(\iota'\iota)^{-1}\iota'\Delta e_i\right]_j\right) \\ &= \sum_{j=2}^{t}\Delta e_{i,j}' - \frac{t-1}{T-1}\sum_{t=2}^{T}\Delta e_{i,t}. \end{aligned}$$

As before, we define $\hat{e}_{i,t}^A = A'\hat{e}_{i,t} = \left((A_1'\hat{e}_{i,t})', (A_2'\hat{e}_{i,t})'\right)'$, with $T^{-1/2}A_1'\hat{e}_{i,t} = o_p(1)$ and $T^{-1/2}A_2'\hat{e}_{i,t} = O_p(1)$ provided that $A_2'\hat{e}_{i,t}$ defines the I(1) stochastic trends. Then,

$$T^{-1/2}A'\hat{e}_{i,t} \Rightarrow \left(0_r', \Omega_{22,i}^{1/2}(W_i(s) - sW_i(1))'\right)',$$

which implies that

$$\begin{aligned} T^{-1}Q_{\hat{e}_i^A\hat{e}_i^A} &= T^{-2}\hat{e}_i^{A\prime}\hat{e}_i^A \\ &\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & \Omega_{22,i}^{1/2}\int_0^1 V_i(s)\,V_i(s)'\,ds\ \Omega_{22,i}^{1/2} \end{bmatrix}, \end{aligned}$$

where $V_i(s) = W_i(s) - sW_i(1)$ is a vector of independent Brownian bridges. Therefore,

$$MSB_{\tau,i}(m) \Rightarrow \pi^{\min}\left(\int_0^1 V_i(s)\,V_i(s)'\,ds\right),$$

given that $\hat{\Omega}_{\Delta\hat{e}_i^A\Delta\hat{e}_i^A} \xrightarrow{p} \Omega_{\Delta e_i^A\Delta e_i^A}$.

A.2 Proof of Theorem 2

Let us consider the null hypothesis that there are $m = k - r$ stochastic trends. From the proof of Theorem 1 we have that $T^{-1}A_1'e_i'e_iA_1 = O_p(1)$, $T^{-1}A_1'e_i'e_iA_2 = O_p(1)$ and $T^{-2}A_2'e_i'e_iA_2 = O_p(1)$, so that $T^{-2}A_1'e_i'e_iA_1 = O_p(T^{-1})$ and $T^{-2}A_1'e_i'e_iA_2 = O_p(T^{-1})$. Consequently, under the alternative hypothesis that there are $l < m$ stochastic trends the rank of the matrix $T^{-1}Q_{\hat{e}_i^A\hat{e}_i^A}$ will be $l < m$. Using these elements, we can see that the cross-products involving I(0) stochastic processes in $T^{-1}Q_{\hat{e}_i^A\hat{e}_i^A}$ tend to zero at rate $O_p(T^{-1})$.

Let us now focus on the estimate of the long-run covariance matrix. Note that under both the null and the alternative hypotheses $T^{-1}\hat{\varepsilon}_i'\hat{\varepsilon}_i = O(1)$, with $T^{-1}\hat{\varepsilon}_i'\hat{\varepsilon}_i \xrightarrow{p} \Sigma_{\varepsilon_i}$. Since all roots of the determinant of $\left(I - \hat{\Gamma}_{i,p_i}(L)\right)$ lie outside the unit circle interval, we can define $\hat{\Xi}_{i,\infty}(L) = \left(I - \hat{\Gamma}_{i,p_i}(L)\right)^{-1}$, with $\Xi_{i,\infty}(L) = (I + \Xi_{i,1}L + \Xi_{i,2}L^2 + \cdots)$ and where the sequence of matrix coefficients $\{\Xi_{i,s}\}_{s=0}^{\infty}$ is absolutely summable. Then, $\Xi_{i,\infty}(1) < \infty$ so that $\hat{\Omega}^{VAR}_{\Delta\hat{e}_i^A\Delta\hat{e}_i^A} \xrightarrow{p} \Xi_{i,\infty}'(1)\Sigma_{\varepsilon_i}\Xi_{i,\infty}(1)$. Therefore, the long-run covariance matrix estimator converges to a positive definite matrix under both the null and the alternative hypotheses – note that this result can be seen as the generalization of the one in Perron and Ng (1998) and Stock (1999). Finally, note that under the alternative hypothesis that there are $l\,(< m)$ stochastic trends $rank\left(T^{-1}Q_{\hat{e}_i^A\hat{e}_i^A}\hat{\Omega}^{-1}_{\Delta\hat{e}_i^A\Delta\hat{e}_i^A}\right) = l$, where the elements that cause rank deficiency tend to zero at rate $O_p(T^{-1})$. This proves the consistency of the MSB statistic under the alternative hypothesis.

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Table 1: Critical values for the MSB_μ and MSB_τ statistics

	MSB_μ statistic								
	$T = 50$			$T = 100$			$T = 200$		
k	1%	5%	10%	1%	5%	10%	1%	5%	10%
1	0.0472	0.0697	0.0939	0.0399	0.0613	0.0832	0.0365	0.0595	0.0806
2	0.0307	0.0380	0.0436	0.0246	0.0326	0.0382	0.0217	0.0290	0.0348
3	0.0247	0.0289	0.0318	0.0186	0.0230	0.0259	0.0163	0.0205	0.0232
4	0.0219	0.0244	0.0262	0.0158	0.0182	0.0201	0.0131	0.0159	0.0177
5	0.0203	0.0220	0.0232	0.0140	0.0159	0.0171	0.0113	0.0132	0.0145
6	0.0191	0.0205	0.0213	0.0127	0.0142	0.0151	0.0100	0.0114	0.0124

	MSB_τ statistic								
	$T = 50$			$T = 100$			$T = 200$		
k	1%	5%	10%	1%	5%	10%	1%	5%	10%
1	0.0365	0.0495	0.0600	0.0296	0.0425	0.0527	0.0272	0.0391	0.0487
2	0.0274	0.0331	0.0371	0.0212	0.0266	0.0305	0.0181	0.0240	0.0277
3	0.0240	0.0270	0.0291	0.0172	0.0204	0.0225	0.0146	0.0177	0.0199
4	0.0214	0.0236	0.0249	0.0150	0.0170	0.0185	0.0121	0.0143	0.0158
5	0.0199	0.0215	0.0225	0.0134	0.0150	0.0160	0.0106	0.0123	0.0134
6	0.0189	0.0202	0.0209	0.0123	0.0135	0.0143	0.0096	0.0108	0.0116

k denotes the number of stochastic trends under the null hypothesis. Simulations are based on 10,000 replications.

Table 2: Simulated mean and variance of the MSB_μ and MSB_τ statistics

	MSB_μ statistic					
	$T = 50$		$T = 100$		$T = 200$	
k	Mean	Variance	Mean	Variance	Mean	Variance
1	0.54081116	0.37677437	0.51126334	0.33774980	0.49284473	0.30372427
2	0.10188155	0.00443026	0.09426914	0.00392838	0.08896918	0.00377839
3	0.05342706	0.00047627	0.04615793	0.00040768	0.04340242	0.00040678
4	0.03763585	0.00012361	0.03105959	0.00010498	0.02839736	0.00009889
5	0.03007539	0.00004001	0.02391362	0.00003820	0.02113698	0.00003563
6	0.02597402	0.00001818	0.01973492	0.00001666	0.01705470	0.00001597

	MSB_τ statistic					
	$T = 50$		$T = 100$		$T = 200$	
k	Mean	Variance	Mean	Variance	Mean	Variance
1	0.19866958	0.02784316	0.17762218	0.02165034	0.17079184	0.02038170
2	0.07007464	0.00121271	0.06216263	0.00110966	0.05870114	0.00106729
3	0.04458992	0.00022282	0.03705674	0.00019089	0.03399447	0.00017578
4	0.03378692	0.00006865	0.02703102	0.00005969	0.02409582	0.00005412
5	0.02817940	0.00002666	0.02162892	0.00002485	0.01890827	0.00002410
6	0.02498843	0.00001314	0.01825715	0.00001165	0.01566494	0.00001202

k denotes the number of stochastic trends under the null hypothesis. Simulations are based on 10,000 replications.

Table 3: Proportions for MSB_μ with $a = 0.5$ and $N = 1$

			Panel A: $r = 0$				Panel B: $r = 1$				Panel C: $r = 2$				Panel D: $r = 3$			
T	$\sigma^2_{F,j}$	ρ_j	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
50	0.5	0.90	0.819	0.063	0.028	0.090	0.042	0.601	0.164	0.193	0.001	0.046	0.531	0.422	0.000	0.001	0.038	0.961
		0.95	0.729	0.141	0.045	0.085	0.042	0.603	0.161	0.194	0.000	0.030	0.546	0.424	0.000	0.000	0.040	0.960
		1	0.502	0.325	0.063	0.110	0.035	0.478	0.237	0.250	0.000	0.035	0.492	0.473	0.000	0.000	0.023	0.977
	1	0.90	0.819	0.074	0.028	0.079	0.059	0.607	0.158	0.176	0.000	0.032	0.541	0.427	0.000	0.001	0.041	0.958
		0.95	0.750	0.119	0.038	0.093	0.048	0.575	0.157	0.220	0.001	0.045	0.495	0.459	0.000	0.000	0.044	0.956
		1	0.502	0.321	0.078	0.099	0.038	0.469	0.245	0.248	0.000	0.025	0.519	0.456	0.000	0.001	0.033	0.966
	10	0.90	0.831	0.053	0.036	0.080	0.038	0.611	0.160	0.191	0.001	0.027	0.553	0.419	0.000	0.000	0.034	0.966
		0.95	0.734	0.129	0.045	0.092	0.038	0.572	0.184	0.206	0.000	0.038	0.510	0.452	0.000	0.000	0.035	0.965
		1	0.522	0.300	0.069	0.109	0.027	0.513	0.230	0.230	0.000	0.032	0.475	0.493	0.000	0.002	0.041	0.957
100	0.5	0.90	0.996	0.004	0.000	0.000	0.056	0.943	0.001	0.000	0.000	0.047	0.952	0.001	0.000	0.000	0.041	0.959
		0.95	0.988	0.012	0.000	0.000	0.051	0.948	0.001	0.000	0.003	0.054	0.940	0.003	0.000	0.000	0.050	0.950
		1	0.737	0.263	0.000	0.000	0.060	0.893	0.047	0.000	0.001	0.048	0.928	0.023	0.000	0.000	0.038	0.962
	1	0.90	0.998	0.002	0.000	0.000	0.049	0.951	0.000	0.000	0.003	0.047	0.950	0.000	0.000	0.000	0.050	0.950
		0.95	0.988	0.012	0.000	0.000	0.060	0.934	0.006	0.000	0.001	0.041	0.954	0.004	0.000	0.001	0.030	0.969
		1	0.743	0.257	0.000	0.000	0.057	0.884	0.059	0.000	0.000	0.052	0.931	0.017	0.000	0.001	0.045	0.954
	10	0.90	0.994	0.006	0.000	0.000	0.070	0.930	0.000	0.000	0.000	0.049	0.950	0.001	0.000	0.000	0.048	0.952
		0.95	0.985	0.015	0.000	0.000	0.058	0.939	0.003	0.000	0.002	0.040	0.955	0.003	0.000	0.000	0.040	0.960
		1	0.771	0.229	0.000	0.000	0.046	0.907	0.047	0.000	0.000	0.042	0.942	0.016	0.000	0.000	0.042	0.958
200	0.5	0.90	1.000	0.000	0.000	0.000	0.053	0.947	0.000	0.000	0.001	0.051	0.948	0.000	0.000	0.000	0.045	0.955
		0.95	1.000	0.000	0.000	0.000	0.086	0.914	0.000	0.000	0.001	0.049	0.950	0.000	0.000	0.000	0.067	0.933
		1	0.814	0.186	0.000	0.000	0.061	0.929	0.010	0.000	0.000	0.039	0.959	0.002	0.000	0.001	0.056	0.943
	1	0.90	1.000	0.000	0.000	0.000	0.068	0.932	0.000	0.000	0.001	0.057	0.942	0.000	0.000	0.000	0.050	0.950
		0.95	1.000	0.000	0.000	0.000	0.058	0.942	0.000	0.000	0.001	0.044	0.955	0.000	0.000	0.000	0.042	0.958
		1	0.803	0.197	0.000	0.000	0.047	0.936	0.017	0.000	0.000	0.050	0.944	0.006	0.000	0.000	0.054	0.946
	10	0.90	1.000	0.000	0.000	0.000	0.062	0.938	0.000	0.000	0.000	0.047	0.953	0.000	0.000	0.001	0.049	0.950
		0.95	1.000	0.000	0.000	0.000	0.061	0.939	0.000	0.000	0.002	0.050	0.948	0.000	0.000	0.000	0.046	0.954
		1	0.830	0.170	0.000	0.000	0.040	0.948	0.012	0.000	0.000	0.047	0.950	0.003	0.000	0.001	0.054	0.945

Note: MSB_τ is the statistic for the linear time trend model for $N = 1$ and r denotes the number of stochastic trends. Simulations are based on 1,000 replications using the 5% critical values from Table 1. For *Panel A*, $\psi_i = diag\,(0.8, 0.5, 0.5)$, for *Panel B*, $\psi_i = diag\,(0.8, 0.5)$ and for *Panel C*, $\psi_i = diag\,(0.8)$.

Table 4: Proportions for MSB_μ with $a = 0.8$ and $N = 1$

			Panel A: $r=0$				Panel B: $r=1$				Panel C: $r=2$				Panel D: $r=3$			
T	$\sigma^2_{F,j}$	ρ_j	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
50	0.5	0.90	0.446	0.254	0.120	0.180	0.007	0.135	0.454	0.404	0.000	0.003	0.094	0.903	0.000	0.001	0.038	0.961
		0.95	0.422	0.234	0.151	0.193	0.004	0.149	0.427	0.420	0.000	0.005	0.104	0.891	0.000	0.000	0.040	0.960
		1	0.268	0.262	0.233	0.237	0.006	0.119	0.397	0.478	0.000	0.007	0.091	0.902	0.000	0.000	0.023	0.977
	1	0.90	0.438	0.253	0.132	0.177	0.012	0.140	0.457	0.391	0.000	0.004	0.094	0.902	0.000	0.001	0.041	0.958
		0.95	0.402	0.226	0.161	0.211	0.017	0.141	0.401	0.441	0.000	0.005	0.100	0.895	0.000	0.000	0.044	0.956
		1	0.241	0.283	0.246	0.230	0.008	0.111	0.422	0.459	0.000	0.001	0.080	0.919	0.000	0.001	0.033	0.966
	10	0.90	0.452	0.246	0.128	0.174	0.012	0.137	0.441	0.410	0.000	0.001	0.097	0.902	0.000	0.000	0.034	0.966
		0.95	0.388	0.267	0.136	0.209	0.005	0.125	0.442	0.428	0.000	0.002	0.080	0.918	0.000	0.000	0.035	0.965
		1	0.261	0.268	0.242	0.229	0.006	0.106	0.427	0.461	0.000	0.003	0.081	0.916	0.000	0.002	0.041	0.957
100	0.5	0.90	0.980	0.020	0.000	0.000	0.036	0.745	0.219	0.000	0.000	0.023	0.399	0.578	0.000	0.000	0.041	0.959
		0.95	0.967	0.032	0.001	0.000	0.052	0.732	0.216	0.000	0.000	0.027	0.395	0.578	0.000	0.000	0.050	0.950
		1	0.778	0.205	0.017	0.000	0.040	0.655	0.283	0.022	0.000	0.022	0.396	0.582	0.000	0.000	0.038	0.962
	1	0.90	0.978	0.022	0.000	0.000	0.039	0.732	0.229	0.000	0.001	0.024	0.440	0.535	0.000	0.000	0.050	0.950
		0.95	0.972	0.027	0.001	0.000	0.046	0.732	0.221	0.001	0.000	0.018	0.434	0.548	0.000	0.001	0.030	0.969
		1	0.770	0.213	0.017	0.000	0.041	0.675	0.263	0.021	0.000	0.024	0.385	0.591	0.000	0.001	0.045	0.954
	10	0.90	0.979	0.021	0.000	0.000	0.049	0.741	0.209	0.001	0.001	0.017	0.413	0.569	0.000	0.000	0.048	0.952
		0.95	0.974	0.025	0.001	0.000	0.047	0.742	0.208	0.003	0.000	0.014	0.446	0.540	0.000	0.000	0.040	0.960
		1	0.800	0.184	0.016	0.000	0.034	0.675	0.280	0.011	0.000	0.018	0.411	0.571	0.000	0.000	0.042	0.958
200	0.5	0.90	1.000	0.000	0.000	0.000	0.050	0.950	0.000	0.000	0.000	0.044	0.945	0.011	0.000	0.000	0.045	0.955
		0.95	1.000	0.000	0.000	0.000	0.077	0.922	0.001	0.000	0.001	0.041	0.945	0.013	0.000	0.000	0.067	0.933
		1	0.892	0.108	0.000	0.000	0.058	0.926	0.016	0.000	0.000	0.032	0.939	0.029	0.000	0.001	0.056	0.943
	1	0.90	1.000	0.000	0.000	0.000	0.067	0.933	0.000	0.000	0.000	0.054	0.935	0.011	0.000	0.000	0.050	0.950
		0.95	1.000	0.000	0.000	0.000	0.065	0.935	0.000	0.000	0.001	0.040	0.949	0.010	0.000	0.000	0.042	0.958
		1	0.883	0.117	0.000	0.000	0.046	0.930	0.024	0.000	0.000	0.045	0.926	0.029	0.000	0.000	0.054	0.946
	10	0.90	1.000	0.000	0.000	0.000	0.059	0.941	0.000	0.000	0.000	0.045	0.941	0.014	0.000	0.001	0.049	0.950
		0.95	1.000	0.000	0.000	0.000	0.057	0.942	0.001	0.000	0.001	0.044	0.943	0.012	0.000	0.000	0.046	0.954
		1	0.904	0.096	0.000	0.000	0.046	0.933	0.021	0.000	0.000	0.042	0.938	0.020	0.000	0.001	0.054	0.945

Note: MSB_τ is the statistic for the linear time trend model for $N = 1$ and r denotes the number of stochastic trends. Simulations are based on $1,000$ replications using the 5% critical values from Table 1. For *Panel A*, $\psi_i = diag\,(0.8, 0.5, 0.5)$, for *Panel B*, $\psi_i = diag\,(0.8, 0.5)$ and for *Panel C*, $\psi_i = diag\,(0.8)$.

Table 5: Proportions for MSB_τ with $a = 0.5$ and $N = 1$

			Panel A: $r = 0$				Panel B: $r = 1$				Panel C: $r = 2$				Panel D: $r = 3$			
T	$\sigma^2_{F,j}$	ρ_j	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
50	0.5	0.90	0.285	0.276	0.150	0.289	0.015	0.258	0.303	0.424	0.000	0.027	0.345	0.628	0.000	0.002	0.033	0.965
		0.95	0.254	0.290	0.177	0.279	0.025	0.260	0.289	0.426	0.001	0.030	0.322	0.647	0.000	0.002	0.047	0.951
		1	0.217	0.298	0.195	0.290	0.017	0.208	0.330	0.445	0.001	0.025	0.308	0.666	0.000	0.000	0.032	0.968
	1	0.90	0.251	0.300	0.182	0.267	0.019	0.243	0.346	0.392	0.000	0.034	0.323	0.643	0.000	0.001	0.037	0.962
		0.95	0.250	0.270	0.183	0.297	0.018	0.238	0.290	0.454	0.000	0.028	0.295	0.677	0.000	0.002	0.038	0.960
		1	0.208	0.297	0.185	0.310	0.019	0.234	0.300	0.447	0.001	0.021	0.321	0.657	0.000	0.001	0.028	0.971
	10	0.90	0.283	0.277	0.188	0.252	0.015	0.258	0.288	0.439	0.002	0.027	0.335	0.636	0.000	0.002	0.042	0.956
		0.95	0.251	0.280	0.196	0.273	0.013	0.252	0.308	0.427	0.001	0.021	0.315	0.663	0.000	0.000	0.044	0.956
		1	0.242	0.283	0.203	0.272	0.020	0.238	0.309	0.433	0.000	0.025	0.317	0.658	0.000	0.002	0.035	0.963
100	0.5	0.90	0.826	0.174	0.000	0.000	0.057	0.872	0.071	0.000	0.002	0.045	0.911	0.042	0.000	0.001	0.041	0.958
		0.95	0.806	0.193	0.001	0.000	0.050	0.836	0.113	0.001	0.001	0.040	0.898	0.061	0.000	0.001	0.046	0.953
		1	0.753	0.245	0.002	0.000	0.049	0.835	0.114	0.002	0.000	0.053	0.878	0.069	0.000	0.001	0.043	0.956
	1	0.90	0.799	0.201	0.000	0.000	0.049	0.862	0.089	0.000	0.000	0.045	0.904	0.051	0.000	0.000	0.038	0.962
		0.95	0.801	0.196	0.003	0.000	0.060	0.855	0.084	0.001	0.000	0.038	0.893	0.069	0.000	0.001	0.040	0.959
		1	0.731	0.268	0.001	0.000	0.061	0.817	0.120	0.002	0.002	0.058	0.859	0.081	0.000	0.000	0.042	0.958
	10	0.90	0.802	0.198	0.000	0.000	0.056	0.870	0.074	0.000	0.000	0.048	0.895	0.057	0.000	0.000	0.034	0.966
		0.95	0.792	0.207	0.001	0.000	0.043	0.860	0.097	0.000	0.001	0.034	0.903	0.062	0.000	0.001	0.034	0.965
		1	0.756	0.241	0.003	0.000	0.055	0.843	0.099	0.003	0.000	0.047	0.892	0.061	0.000	0.001	0.035	0.964
200	0.5	0.90	0.903	0.097	0.000	0.000	0.049	0.949	0.002	0.000	0.000	0.046	0.952	0.002	0.000	0.000	0.044	0.956
		0.95	0.912	0.088	0.000	0.000	0.047	0.946	0.007	0.000	0.000	0.048	0.952	0.000	0.000	0.000	0.043	0.957
		1	0.859	0.141	0.000	0.000	0.055	0.932	0.013	0.000	0.000	0.048	0.950	0.002	0.000	0.000	0.042	0.958
	1	0.90	0.918	0.082	0.000	0.000	0.062	0.935	0.003	0.000	0.001	0.057	0.942	0.000	0.000	0.002	0.046	0.952
		0.95	0.900	0.100	0.000	0.000	0.052	0.944	0.004	0.000	0.000	0.042	0.958	0.000	0.000	0.000	0.049	0.951
		1	0.860	0.140	0.000	0.000	0.065	0.921	0.014	0.000	0.002	0.050	0.945	0.003	0.000	0.002	0.039	0.959
	10	0.90	0.922	0.078	0.000	0.000	0.064	0.930	0.006	0.000	0.000	0.049	0.947	0.004	0.000	0.001	0.053	0.946
		0.95	0.916	0.084	0.000	0.000	0.055	0.941	0.004	0.000	0.002	0.047	0.950	0.001	0.000	0.001	0.046	0.953
		1	0.865	0.135	0.000	0.000	0.054	0.939	0.007	0.000	0.001	0.046	0.950	0.003	0.000	0.001	0.055	0.944

Note: MSB_τ is the statistic for the linear time trend model for $N = 1$ and r denotes the number of stochastic trends. Simulations are based on 1,000 replications using the 5% critical values from Table 1. For *Panel A*, $\psi_i = diag\,(0.8, 0.5, 0.5)$, for *Panel B*, $\psi_i = diag\,(0.8, 0.5)$ and for *Panel C*, $\psi_i = diag\,(0.8)$.

Table 6: Proportions for MSB_τ with $a = 0.8$ and $N = 1$

			Panel A: $r = 0$				Panel B: $r = 1$				Panel C: $r = 2$				Panel D: $r = 3$			
T	$\sigma^2_{F,j}$	ρ_j	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
50	0.5	0.90	0.060	0.247	0.271	0.422	0.002	0.054	0.319	0.625	0.000	0.002	0.058	0.940	0.000	0.002	0.033	0.965
		0.95	0.054	0.217	0.298	0.431	0.002	0.057	0.310	0.631	0.000	0.004	0.075	0.921	0.000	0.002	0.047	0.951
		1	0.044	0.214	0.296	0.446	0.000	0.049	0.282	0.669	0.000	0.003	0.063	0.934	0.000	0.000	0.032	0.968
	1	0.90	0.056	0.235	0.281	0.428	0.003	0.055	0.341	0.601	0.000	0.002	0.063	0.935	0.000	0.001	0.037	0.962
		0.95	0.068	0.226	0.280	0.426	0.002	0.057	0.285	0.656	0.000	0.003	0.075	0.922	0.000	0.002	0.038	0.960
		1	0.048	0.228	0.291	0.433	0.000	0.044	0.316	0.640	0.000	0.002	0.054	0.944	0.000	0.001	0.028	0.971
	10	0.90	0.066	0.245	0.291	0.398	0.004	0.043	0.313	0.640	0.000	0.002	0.075	0.923	0.000	0.002	0.042	0.956
		0.95	0.066	0.222	0.278	0.434	0.004	0.050	0.302	0.644	0.000	0.000	0.062	0.938	0.000	0.000	0.044	0.956
		1	0.052	0.224	0.309	0.415	0.004	0.052	0.303	0.641	0.000	0.005	0.057	0.938	0.000	0.002	0.035	0.963
100	0.5	0.90	0.624	0.345	0.031	0.000	0.029	0.403	0.529	0.039	0.000	0.009	0.224	0.767	0.000	0.001	0.041	0.958
		0.95	0.575	0.378	0.044	0.003	0.023	0.381	0.556	0.040	0.000	0.011	0.211	0.778	0.000	0.001	0.046	0.953
		1	0.569	0.374	0.053	0.004	0.017	0.380	0.547	0.056	0.000	0.012	0.221	0.767	0.000	0.001	0.043	0.956
	1	0.90	0.585	0.375	0.040	0.000	0.018	0.358	0.577	0.047	0.000	0.011	0.252	0.737	0.000	0.000	0.038	0.962
		0.95	0.600	0.365	0.035	0.000	0.024	0.363	0.577	0.036	0.000	0.005	0.232	0.763	0.000	0.001	0.040	0.959
		1	0.562	0.370	0.068	0.000	0.028	0.374	0.530	0.068	0.000	0.012	0.231	0.757	0.000	0.000	0.042	0.958
	10	0.90	0.610	0.361	0.029	0.000	0.020	0.392	0.547	0.041	0.000	0.006	0.242	0.752	0.000	0.000	0.034	0.966
		0.95	0.607	0.355	0.037	0.001	0.017	0.406	0.530	0.047	0.000	0.005	0.255	0.740	0.000	0.001	0.034	0.965
		1	0.569	0.382	0.047	0.002	0.029	0.386	0.535	0.050	0.000	0.010	0.213	0.777	0.000	0.001	0.035	0.964
200	0.5	0.90	0.933	0.067	0.000	0.000	0.047	0.867	0.086	0.000	0.000	0.037	0.803	0.160	0.000	0.000	0.044	0.956
		0.95	0.920	0.080	0.000	0.000	0.040	0.872	0.088	0.000	0.000	0.031	0.800	0.169	0.000	0.000	0.043	0.957
		1	0.911	0.089	0.000	0.000	0.050	0.857	0.093	0.000	0.002	0.035	0.802	0.161	0.000	0.000	0.042	0.958
	1	0.90	0.940	0.060	0.000	0.000	0.064	0.859	0.076	0.001	0.002	0.043	0.814	0.141	0.000	0.002	0.046	0.952
		0.95	0.913	0.087	0.000	0.000	0.051	0.862	0.087	0.000	0.000	0.031	0.801	0.168	0.000	0.000	0.049	0.951
		1	0.908	0.092	0.000	0.000	0.055	0.841	0.104	0.000	0.002	0.043	0.777	0.178	0.000	0.002	0.039	0.959
	10	0.90	0.938	0.062	0.000	0.000	0.056	0.873	0.070	0.001	0.001	0.040	0.804	0.155	0.000	0.001	0.053	0.946
		0.95	0.946	0.054	0.000	0.000	0.050	0.864	0.086	0.000	0.002	0.035	0.809	0.154	0.000	0.001	0.046	0.953
		1	0.912	0.088	0.000	0.000	0.051	0.855	0.094	0.000	0.001	0.045	0.776	0.178	0.000	0.001	0.055	0.944

Note: MSB_τ is the statistic for the linear time trend model for $N = 1$ and r denotes the number of stochastic trends. Simulations are based on $1,000$ replications using the 5% critical values from Table 1. For *Panel A*, $\psi_i = diag\,(0.8, 0.5, 0.5)$, for *Panel B*, $\psi_i = diag\,(0.8, 0.5)$ and for *Panel C*, $\psi_i = diag\,(0.8)$.

Table 7: Proportions for $PMSB_{\mu}$ with $a = 0.5$ and $N = 20$

			Panel A: $r = 0$				Panel B: $r = 1$				Panel C: $r = 2$				Panel D: $r = 3$			
T	$\sigma^2_{F,j}$	ρ_j	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
50	0.5	0.90	1.000	0.000	0.000	0.000	0.126	0.874	0.000	0.000	0.005	0.013	0.982	0.000	0.000	0.000	0.012	0.988
		0.95	0.998	0.002	0.000	0.000	0.152	0.848	0.000	0.000	0.003	0.012	0.985	0.000	0.000	0.000	0.012	0.988
		1	0.864	0.136	0.000	0.000	0.129	0.870	0.001	0.000	0.002	0.013	0.985	0.000	0.000	0.000	0.016	0.984
	1	0.90	1.000	0.000	0.000	0.000	0.122	0.878	0.000	0.000	0.000	0.019	0.981	0.000	0.000	0.000	0.013	0.987
		0.95	0.994	0.006	0.000	0.000	0.115	0.885	0.000	0.000	0.001	0.012	0.987	0.000	0.000	0.000	0.013	0.987
		1	0.877	0.123	0.000	0.000	0.137	0.863	0.000	0.000	0.000	0.010	0.990	0.000	0.000	0.000	0.011	0.989
	10	0.90	1.000	0.000	0.000	0.000	0.119	0.881	0.000	0.000	0.000	0.008	0.992	0.000	0.000	0.000	0.010	0.990
		0.95	0.998	0.002	0.000	0.000	0.154	0.846	0.000	0.000	0.001	0.017	0.982	0.000	0.000	0.000	0.004	0.996
		1	0.872	0.128	0.000	0.000	0.121	0.879	0.000	0.000	0.002	0.014	0.984	0.000	0.000	0.000	0.016	0.984
100	0.5	0.90	1.000	0.000	0.000	0.000	0.016	0.984	0.000	0.000	0.000	0.031	0.969	0.000	0.000	0.000	0.027	0.973
		0.95	1.000	0.000	0.000	0.000	0.019	0.981	0.000	0.000	0.000	0.030	0.970	0.000	0.000	0.000	0.014	0.986
		1	0.952	0.048	0.000	0.000	0.012	0.988	0.000	0.000	0.000	0.036	0.964	0.000	0.000	0.000	0.015	0.985
	1	0.90	1.000	0.000	0.000	0.000	0.021	0.979	0.000	0.000	0.000	0.026	0.974	0.000	0.000	0.000	0.015	0.985
		0.95	1.000	0.000	0.000	0.000	0.021	0.979	0.000	0.000	0.000	0.024	0.976	0.000	0.000	0.000	0.030	0.970
		1	0.953	0.047	0.000	0.000	0.010	0.990	0.000	0.000	0.000	0.021	0.979	0.000	0.000	0.000	0.026	0.974
	10	0.90	1.000	0.000	0.000	0.000	0.019	0.981	0.000	0.000	0.000	0.025	0.975	0.000	0.000	0.000	0.015	0.985
		0.95	1.000	0.000	0.000	0.000	0.022	0.978	0.000	0.000	0.000	0.027	0.973	0.000	0.000	0.000	0.026	0.974
		1	0.959	0.041	0.000	0.000	0.016	0.984	0.000	0.000	0.000	0.030	0.970	0.000	0.000	0.000	0.023	0.977
200	0.5	0.90	1.000	0.000	0.000	0.000	0.024	0.976	0.000	0.000	0.000	0.022	0.978	0.000	0.000	0.000	0.028	0.972
		0.95	1.000	0.000	0.000	0.000	0.023	0.977	0.000	0.000	0.000	0.031	0.969	0.000	0.000	0.000	0.028	0.972
		1	0.998	0.002	0.000	0.000	0.023	0.977	0.000	0.000	0.000	0.023	0.977	0.000	0.000	0.000	0.024	0.976
	1	0.90	1.000	0.000	0.000	0.000	0.021	0.979	0.000	0.000	0.000	0.015	0.985	0.000	0.000	0.000	0.038	0.962
		0.95	1.000	0.000	0.000	0.000	0.030	0.970	0.000	0.000	0.000	0.018	0.982	0.000	0.000	0.000	0.023	0.977
		1	0.994	0.006	0.000	0.000	0.025	0.975	0.000	0.000	0.000	0.031	0.969	0.000	0.000	0.000	0.025	0.975
	10	0.90	1.000	0.000	0.000	0.000	0.027	0.973	0.000	0.000	0.000	0.025	0.975	0.000	0.000	0.000	0.030	0.970
		0.95	1.000	0.000	0.000	0.000	0.029	0.971	0.000	0.000	0.000	0.023	0.977	0.000	0.000	0.000	0.031	0.969
		1	0.994	0.006	0.000	0.000	0.015	0.985	0.000	0.000	0.000	0.025	0.975	0.000	0.000	0.000	0.022	0.978

Table 8: Proportions for $PMSB_{\mu}$ with $a = 0.8$ and $N = 20$

			Panel A: $r=0$				Panel B: $r=1$				Panel C: $r=2$				Panel D: $r=3$			
T	$\sigma^2_{F,j}$	ρ_j	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
50	0.5	0.90	1.000	0.000	0.000	0.000	0.457	0.543	0.000	0.000	0.001	0.000	0.792	0.207	0.000	0.000	0.012	0.988
		0.95	1.000	0.000	0.000	0.000	0.486	0.514	0.000	0.000	0.000	0.000	0.798	0.202	0.000	0.000	0.012	0.988
		1	0.959	0.041	0.000	0.000	0.490	0.504	0.006	0.000	0.003	0.000	0.741	0.256	0.000	0.000	0.016	0.984
	1	0.90	1.000	0.000	0.000	0.000	0.480	0.520	0.000	0.000	0.004	0.000	0.803	0.193	0.000	0.000	0.013	0.987
		0.95	1.000	0.000	0.000	0.000	0.479	0.520	0.001	0.000	0.000	0.000	0.782	0.218	0.000	0.000	0.013	0.987
		1	0.949	0.051	0.000	0.000	0.506	0.491	0.003	0.000	0.001	0.000	0.766	0.233	0.000	0.000	0.011	0.989
	10	0.90	1.000	0.000	0.000	0.000	0.471	0.529	0.000	0.000	0.002	0.000	0.783	0.215	0.000	0.000	0.010	0.990
		0.95	1.000	0.000	0.000	0.000	0.509	0.491	0.000	0.000	0.002	0.000	0.786	0.212	0.000	0.000	0.004	0.996
		1	0.950	0.050	0.000	0.000	0.526	0.463	0.011	0.000	0.000	0.000	0.755	0.245	0.000	0.000	0.016	0.984
100	0.5	0.90	1.000	0.000	0.000	0.000	0.016	0.984	0.000	0.000	0.007	0.012	0.981	0.000	0.000	0.000	0.027	0.973
		0.95	1.000	0.000	0.000	0.000	0.012	0.988	0.000	0.000	0.005	0.012	0.983	0.000	0.000	0.000	0.014	0.986
		1	0.982	0.018	0.000	0.000	0.016	0.984	0.000	0.000	0.011	0.011	0.978	0.000	0.000	0.000	0.015	0.985
	1	0.90	1.000	0.000	0.000	0.000	0.023	0.977	0.000	0.000	0.003	0.009	0.988	0.000	0.000	0.000	0.015	0.985
		0.95	1.000	0.000	0.000	0.000	0.017	0.983	0.000	0.000	0.010	0.016	0.974	0.000	0.000	0.000	0.030	0.970
		1	0.978	0.022	0.000	0.000	0.021	0.979	0.000	0.000	0.005	0.010	0.985	0.000	0.000	0.000	0.026	0.974
	10	0.90	1.000	0.000	0.000	0.000	0.011	0.989	0.000	0.000	0.007	0.007	0.986	0.000	0.000	0.000	0.015	0.985
		0.95	1.000	0.000	0.000	0.000	0.018	0.982	0.000	0.000	0.005	0.008	0.987	0.000	0.000	0.000	0.026	0.974
		1	0.978	0.022	0.000	0.000	0.012	0.988	0.000	0.000	0.010	0.015	0.975	0.000	0.000	0.000	0.023	0.977
200	0.5	0.90	1.000	0.000	0.000	0.000	0.022	0.978	0.000	0.000	0.000	0.016	0.984	0.000	0.000	0.000	0.028	0.972
		0.95	1.000	0.000	0.000	0.000	0.029	0.971	0.000	0.000	0.000	0.019	0.981	0.000	0.000	0.000	0.028	0.972
		1	0.999	0.001	0.000	0.000	0.023	0.977	0.000	0.000	0.000	0.026	0.974	0.000	0.000	0.000	0.024	0.976
	1	0.90	1.000	0.000	0.000	0.000	0.024	0.976	0.000	0.000	0.000	0.018	0.982	0.000	0.000	0.000	0.038	0.962
		0.95	1.000	0.000	0.000	0.000	0.027	0.973	0.000	0.000	0.000	0.022	0.978	0.000	0.000	0.000	0.023	0.977
		1	0.997	0.003	0.000	0.000	0.023	0.977	0.000	0.000	0.000	0.027	0.973	0.000	0.000	0.000	0.025	0.975
	10	0.90	1.000	0.000	0.000	0.000	0.023	0.977	0.000	0.000	0.000	0.025	0.975	0.000	0.000	0.000	0.030	0.970
		0.95	1.000	0.000	0.000	0.000	0.019	0.981	0.000	0.000	0.000	0.021	0.979	0.000	0.000	0.000	0.031	0.969
		1	0.999	0.001	0.000	0.000	0.020	0.980	0.000	0.000	0.000	0.021	0.979	0.000	0.000	0.000	0.022	0.978

Table 9: Proportions for $PMSB_{\tau}$ with $a = 0.5$ and $N = 20$

			Panel A: $r = 0$				Panel B: $r = 1$				Panel C: $r = 2$				Panel D: $r = 3$			
T	$\sigma^2_{F,j}$	ρ_j	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
50	0.5	0.90	0.938	0.062	0.000	0.000	0.613	0.387	0.000	0.000	0.006	0.003	0.991	0.000	0.000	0.000	0.010	0.990
		0.95	0.911	0.089	0.000	0.000	0.593	0.407	0.000	0.000	0.013	0.001	0.986	0.000	0.000	0.000	0.016	0.984
		1	0.881	0.119	0.000	0.000	0.626	0.373	0.001	0.000	0.008	0.002	0.990	0.000	0.000	0.000	0.011	0.989
	1	0.90	0.953	0.047	0.000	0.000	0.569	0.431	0.000	0.000	0.010	0.004	0.986	0.000	0.000	0.000	0.011	0.989
		0.95	0.913	0.087	0.000	0.000	0.593	0.406	0.001	0.000	0.010	0.002	0.988	0.000	0.000	0.000	0.017	0.983
		1	0.887	0.113	0.000	0.000	0.604	0.396	0.000	0.000	0.014	0.003	0.983	0.000	0.000	0.000	0.012	0.988
	10	0.90	0.934	0.066	0.000	0.000	0.598	0.402	0.000	0.000	0.014	0.004	0.982	0.000	0.000	0.000	0.013	0.987
		0.95	0.903	0.097	0.000	0.000	0.601	0.399	0.000	0.000	0.011	0.001	0.988	0.000	0.000	0.000	0.010	0.990
		1	0.894	0.106	0.000	0.000	0.601	0.399	0.000	0.000	0.006	0.004	0.990	0.000	0.000	0.000	0.013	0.987
100	0.5	0.90	1.000	0.000	0.000	0.000	0.024	0.976	0.000	0.000	0.000	0.018	0.982	0.000	0.000	0.000	0.021	0.979
		0.95	0.999	0.001	0.000	0.000	0.016	0.984	0.000	0.000	0.000	0.025	0.975	0.000	0.000	0.000	0.021	0.979
		1	0.964	0.036	0.000	0.000	0.008	0.992	0.000	0.000	0.000	0.029	0.971	0.000	0.000	0.000	0.024	0.976
	1	0.90	1.000	0.000	0.000	0.000	0.017	0.983	0.000	0.000	0.000	0.019	0.981	0.000	0.000	0.000	0.017	0.983
		0.95	0.997	0.003	0.000	0.000	0.018	0.982	0.000	0.000	0.000	0.031	0.969	0.000	0.000	0.000	0.020	0.980
		1	0.968	0.032	0.000	0.000	0.012	0.988	0.000	0.000	0.000	0.016	0.984	0.000	0.000	0.000	0.022	0.978
	10	0.90	1.000	0.000	0.000	0.000	0.018	0.982	0.000	0.000	0.000	0.018	0.982	0.000	0.000	0.000	0.018	0.982
		0.95	0.996	0.004	0.000	0.000	0.020	0.980	0.000	0.000	0.000	0.028	0.972	0.000	0.000	0.000	0.021	0.979
		1	0.977	0.023	0.000	0.000	0.013	0.987	0.000	0.000	0.000	0.026	0.974	0.000	0.000	0.000	0.022	0.978
200	0.5	0.90	1.000	0.000	0.000	0.000	0.029	0.971	0.000	0.000	0.000	0.038	0.962	0.000	0.000	0.000	0.027	0.973
		0.95	1.000	0.000	0.000	0.000	0.025	0.975	0.000	0.000	0.000	0.032	0.968	0.000	0.000	0.000	0.022	0.978
		1	0.998	0.002	0.000	0.000	0.027	0.973	0.000	0.000	0.000	0.023	0.977	0.000	0.000	0.000	0.027	0.973
	1	0.90	1.000	0.000	0.000	0.000	0.013	0.987	0.000	0.000	0.000	0.015	0.985	0.000	0.000	0.000	0.027	0.973
		0.95	1.000	0.000	0.000	0.000	0.027	0.973	0.000	0.000	0.000	0.023	0.977	0.000	0.000	0.000	0.030	0.970
		1	1.000	0.000	0.000	0.000	0.029	0.971	0.000	0.000	0.000	0.026	0.974	0.000	0.000	0.000	0.026	0.974
	10	0.90	1.000	0.000	0.000	0.000	0.018	0.982	0.000	0.000	0.000	0.029	0.971	0.000	0.000	0.000	0.019	0.981
		0.95	1.000	0.000	0.000	0.000	0.038	0.962	0.000	0.000	0.000	0.026	0.974	0.000	0.000	0.000	0.023	0.977
		1	0.999	0.001	0.000	0.000	0.018	0.982	0.000	0.000	0.000	0.030	0.970	0.000	0.000	0.000	0.029	0.971

Table 10: Proportions for $PMSB_\tau$ with $a = 0.8$ and $N = 20$

			Panel A: $r = 0$				Panel B: $r = 1$				Panel C: $r = 2$				Panel D: $r = 3$			
T	$\sigma^2_{F,j}$	ρ_j	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
50	0.5	0.90	0.991	0.009	0.000	0.000	0.687	0.083	0.230	0.000	0.001	0.000	0.318	0.681	0.000	0.000	0.010	0.990
		0.95	0.972	0.028	0.000	0.000	0.687	0.068	0.245	0.000	0.000	0.000	0.282	0.718	0.000	0.000	0.016	0.984
		1	0.986	0.014	0.000	0.000	0.686	0.070	0.244	0.000	0.000	0.000	0.280	0.720	0.000	0.000	0.011	0.989
	1	0.90	0.980	0.020	0.000	0.000	0.684	0.078	0.238	0.000	0.000	0.000	0.319	0.681	0.000	0.000	0.011	0.989
		0.95	0.979	0.021	0.000	0.000	0.649	0.077	0.274	0.000	0.000	0.000	0.304	0.696	0.000	0.000	0.017	0.983
		1	0.972	0.028	0.000	0.000	0.663	0.069	0.268	0.000	0.000	0.000	0.279	0.721	0.000	0.000	0.012	0.988
	10	0.90	0.985	0.015	0.000	0.000	0.688	0.070	0.242	0.000	0.000	0.000	0.302	0.698	0.000	0.000	0.013	0.987
		0.95	0.985	0.015	0.000	0.000	0.672	0.083	0.245	0.000	0.000	0.000	0.313	0.687	0.000	0.000	0.010	0.990
		1	0.982	0.018	0.000	0.000	0.687	0.056	0.257	0.000	0.000	0.000	0.246	0.754	0.000	0.000	0.013	0.987
100	0.5	0.90	1.000	0.000	0.000	0.000	0.035	0.965	0.000	0.000	0.001	0.000	0.999	0.000	0.000	0.000	0.021	0.979
		0.95	1.000	0.000	0.000	0.000	0.039	0.961	0.000	0.000	0.001	0.000	0.998	0.001	0.000	0.000	0.021	0.979
		1	0.989	0.011	0.000	0.000	0.033	0.967	0.000	0.000	0.005	0.000	0.994	0.001	0.000	0.000	0.024	0.976
	1	0.90	1.000	0.000	0.000	0.000	0.026	0.974	0.000	0.000	0.005	0.000	0.994	0.001	0.000	0.000	0.017	0.983
		0.95	1.000	0.000	0.000	0.000	0.029	0.971	0.000	0.000	0.007	0.000	0.992	0.001	0.000	0.000	0.020	0.980
		1	0.992	0.008	0.000	0.000	0.032	0.968	0.000	0.000	0.004	0.000	0.995	0.001	0.000	0.000	0.022	0.978
	10	0.90	1.000	0.000	0.000	0.000	0.028	0.972	0.000	0.000	0.003	0.000	0.997	0.000	0.000	0.000	0.018	0.982
		0.95	1.000	0.000	0.000	0.000	0.031	0.969	0.000	0.000	0.003	0.000	0.997	0.000	0.000	0.000	0.021	0.979
		1	0.996	0.004	0.000	0.000	0.031	0.969	0.000	0.000	0.006	0.000	0.994	0.000	0.000	0.000	0.022	0.978
200	0.5	0.90	1.000	0.000	0.000	0.000	0.020	0.980	0.000	0.000	0.000	0.012	0.988	0.000	0.000	0.000	0.027	0.973
		0.95	1.000	0.000	0.000	0.000	0.028	0.972	0.000	0.000	0.000	0.025	0.975	0.000	0.000	0.000	0.022	0.978
		1	1.000	0.000	0.000	0.000	0.017	0.983	0.000	0.000	0.000	0.012	0.988	0.000	0.000	0.000	0.027	0.973
	1	0.90	1.000	0.000	0.000	0.000	0.025	0.975	0.000	0.000	0.000	0.018	0.982	0.000	0.000	0.000	0.027	0.973
		0.95	1.000	0.000	0.000	0.000	0.023	0.977	0.000	0.000	0.000	0.019	0.981	0.000	0.000	0.000	0.030	0.970
		1	0.999	0.001	0.000	0.000	0.028	0.972	0.000	0.000	0.002	0.013	0.985	0.000	0.000	0.000	0.026	0.974
	10	0.90	1.000	0.000	0.000	0.000	0.020	0.980	0.000	0.000	0.000	0.023	0.977	0.000	0.000	0.000	0.019	0.981
		0.95	1.000	0.000	0.000	0.000	0.017	0.983	0.000	0.000	0.000	0.024	0.976	0.000	0.000	0.000	0.023	0.977
		1	1.000	0.000	0.000	0.000	0.021	0.979	0.000	0.000	0.000	0.012	0.988	0.000	0.000	0.000	0.029	0.971