Representation Theory for Stochastic Integrals with Fractional Integrator Processes

James Davidson* and Nigar Hashimzade
University of Exeter

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Abstract

This paper considers large sample approximations to the covariances of a nonstationary fractionally integrated processes with the stationary increments of another such process – possibly, itself. Questions of interest include the relationship between the harmonic representation of these random variables, which we have analysed in a previous paper, and the construction derived from moving average representations in the time domain. The limiting integrals are shown to be expressible in terms of functionals of Itô integrals with respect to two distinct Brownian motions. They have an unexpectedly complex structure but possess the required characteristics, in particular an integration by parts formula. The advantages of our approach over the harmonic analysis include the facts that our formulas are valid for the full range of the long memory parameters, and extend to non-Gaussian processes.

*Corresponding author. School of Business and Economics, University of Exeter, Exeter EX4 4PU, UK. Email: james.davidson@exeter.ac.uk

1 Introduction

Let x_t and y_t be linear processes having the MA(∞) forms

$$x_t = \sum_{j=0}^{\infty} b_j u_{t-j} \tag{1.1}$$

$$y_t = \sum_{j=0}^{\infty} c_j w_{t-j} \tag{1.2}$$

where u_t, w_t are zero mean independent processes, and the coeffiicient sequences $\{b_j\}$ and $\{c_j\}$ decay hyperbolically. Fractional noise processes are a well-known simple case, in which

$$b_j = \frac{\Gamma(j+d_X)}{\Gamma(d_X)\Gamma(j+1)} \qquad c_j = \frac{\Gamma(j+d_Y)}{\Gamma(d_Y)\Gamma(j+1)} \tag{1.3}$$

for $-\frac{1}{2} < d_X, d_Y < \frac{1}{2}$. Considerably greater generality will be permitted but parameters d_X and d_Y, subject to these constraints, will in all cases index the rate of lag decay. Defining the partial sum processes X_n and Y_n on the unit interval by

$$X_n(\xi) = n^{-1/2-d_X} \sum_{t=1}^{[n\xi]} x_t, \quad Y_n(\xi) = n^{-1/2-d_Y} \sum_{t=1}^{[n\xi]} y_t; \quad 0 \le \xi \le 1 \tag{1.4}$$

it is known that $(X_n, Y_n) \xrightarrow{d} (X, Y)$ under fairly general assumptions, where the limit processes are fractional Brownian motions as defined by Mandelbrot and van Ness (1968). These results are given under the best known conditions by Davidson and de Jong (2000) (henceforth, DDJ). For representative case Y, the well-known formula is

$$Y(\xi) = \frac{1}{\Gamma(d_Y+1)} \left[\int_0^{\xi} (\xi-\tau)^{d_Y} dB_w(\tau) + \int_{-\infty}^{0} \left((\xi-\tau)^{d_Y} - (-\tau)^{d_Y} \right) dB_w(\tau) \right] \tag{1.5}$$

for $0 \le \xi \le 1$, where B_w is regular Brownian motion on $\mathbb{R}$.

Our interest here is in the limiting distribution of the stochastic process $G_n : [0,1] \longmapsto \mathbb{R}$ where [1]

$$G_n(\xi) = \frac{1}{K(n)} \sum_{t=1}^{[n\xi]-1} \sum_{s=1}^{t} x_s y_{t+1}, \quad 0 \le \xi \le 1. \tag{1.6}$$

and $K(n)$ is a function of sample size, which for the case of (1.3), at least, can set as $n^{1+d_X+d_Y}$. In applications we shall typically wish x_t and y_t to be respectively column and row vectors, and hence G_n to be a matrix, but these cases are notationally burdensome and it is convenient to derive the main results for the scalar case. The required extensions are obtainable by very straightforward generalizations, as required

Expressions of the form of G_n arise in the theory of cointegrating regressions. For example, in the standard case where x_t and y_t are I(0) processes, they appear (with $\xi = 1$) in the formulae for the Dickey-Fuller statistics and cointegrating regression errors-of-estimate, with y_t having the interpretation of a stationary error term, and x_t the differences of the trending regressor. In this paper we explore the form of the limit distributions. A companion paper (Davidson and

[1] The notation $[x]$ denotes the largest integer not exceeding x. Sums for which the lower bound exceeds the upper take the value 0 by convention.

Hashimzade 2007b) derives the weak convergence results that formally connect (1.6) with the the limit case.

These results are known for the harmonic representation of the variables, in the cases where where this is defined. In the fractional noise case,

$$x_t = \int_{-\pi}^{\pi} e^{it\lambda}(i\lambda)^{-d_X} W_u(d\lambda) \tag{1.7}$$

$$y_t = \int_{-\pi}^{\pi} e^{it\lambda}(i\lambda)^{-d_Y} W_w(d\lambda) \tag{1.8}$$

where i is the imaginary unit and (W_u, W_w) is a vector of complex-valued Gaussian random measures with the properties (for $j, k = w, u$)

$$W_j(-d\lambda) = \overline{W_j(d\lambda)} \tag{1.9a}$$

$$EW_j(d\lambda) = 0 \tag{1.9b}$$

$$EW_j(d\lambda)\overline{W_k(d\mu)} = \begin{cases} \omega_{jk}d\lambda, & \mu = \lambda \\ 0, & \text{otherwise.} \end{cases} \tag{1.9c}$$

Chan and Terrin (1995) is a well-known study that analyses the weak convergence of fractionally integrated processes under the harmonic representation. The model these authors analyse is different from the usual 'causal' (backward-looking) model considered here. However, Davidson and Hashimzade (2007a) extend their analysis, and apply it to this case in particular. The weak limits of the partial sum processes (1.4) take the form

$$X(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\lambda\xi} - 1}{i\lambda}(i\lambda)^{-d_X} W_u(d\lambda)$$

$$Y(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\lambda\xi} - 1}{i\lambda}(i\lambda)^{-d_Y} W_w(d\lambda)$$

and also that $G_n(\xi)$ has the weak limit[2]

$$\int_0^\xi XdY = \frac{1}{2\pi} \int_0^\xi dr \left[\int_{-\infty}^{\infty} \frac{e^{i\lambda r} - 1}{i\lambda}(i\lambda)^{-d_X} W_u(d\lambda) \int_{-\infty}^{\infty} e^{i\mu r}(i\lambda)^{-d_Y} W_w(d\mu) \right] \tag{1.10}$$

For the case $d_X + d_Y > 0$, the expected value of this random variable is derived (for $\xi = 1$) as

$$\begin{aligned} E\int_0^1 XdY &= \frac{\omega_{uw}}{2\pi} \int_0^1 \int_{-\infty}^{\infty} \frac{1 - e^{-i\lambda r}}{i\lambda} |\lambda|^{-d_X - d_Y} e^{-i\pi(d_X - d_Y)\,\mathrm{sgn}(\lambda)} d\lambda dr \\ &= \frac{\omega_{uw}\Gamma(1 - d_X - d_Y)}{\pi(1 + d_X + d_Y)(d_X + d_Y)} \sin \pi d_Y. \end{aligned} \tag{1.11}$$

In this paper, we explore the counterpart of this solution in the time domain. There are several reasons why this alternative approach provides an essential extension. The general weak convergence proofs given by Davidson and Hashimzade (2007a) are restricted to the case $d_X + d_Y > 0$, and the 'standard' case $d_X = d_Y = 0$ is especially intractable, because the harmonic representation of the integral breaks down (with undefined expectation) when the processes have summable covariances. While there is no difficulty in constructing more general dependence models than the fractional noise example given, there is the drawback that the harmonic representation requires Gaussian, identically distributed shocks – a restrictive requirement for econometric modelling. Working in the time domain allows all these limitations to be relaxed.

The specific assumptions to be adopted are as follows.

[2] Strictly, this is shown in the cited paper for the case $\xi = 1$, but the generalization is direct.

Assumption 1 *The collection $\{u_t, w_t; t \in \mathbb{Z}\}$ are identically and independently distributed with zero mean and covariance matrix*

$$E\begin{bmatrix} u_t \\ w_t \end{bmatrix}\begin{bmatrix} u_t & w_t \end{bmatrix} = \mathbf{\Omega} = \begin{bmatrix} \omega_{uu} & \omega_{uw} \\ \omega_{uw} & \omega_{ww} \end{bmatrix} \tag{1.12}$$

and $\mu_{uw}^4 = E(u_t^2 w_t^2) < \infty$. $u_t = w_t$ is an admissible case.

These random variables will define the filtered probability space on which our processes live, denoted $(\Omega, \mathcal{F}, P, \boldsymbol{F})$ where

$$\boldsymbol{F} = \{\mathcal{F}_t, t \in \mathbb{Z};\ \mathcal{F}_t \subseteq \mathcal{F} \text{ all } t, \text{ and } \mathcal{F}_t \subseteq \mathcal{F}_s \text{ iff } t \leq s\}.$$

The pair (u_t, w_t) are adapted to $\mathcal{F}_t$, and in this setup we may also use the notation $\mathcal{F}(r) = \mathcal{F}_{[nr]}$ for $0 \leq r \leq 1$ where n is sample size.

Assumption 2 *The sequences $\{b_j\}_0^\infty$ and $\{c_j\}_0^\infty$ depend on parameters $d_X \in (-\frac{1}{2}, \frac{1}{2})$ and $d_Y \in (-\frac{1}{2}, \frac{1}{2})$, respectively, and sequences $\{L_X(j)\}$ and $\{L_Y(j)\}$ that are at most slowly varying at infinity. These sequences satisfy one of the following conditions, stated for $\{b_j\}$ as representative case:*

(a) *If $0 < d_X < \frac{1}{2}$ then $b_j = \Gamma(d_X)^{-1} j^{d_X - 1} L_X(j)$.*

(b) *If $d_X = 0$ then $0 < |\sum_{j=0}^\infty b_j| < \infty$, and $b_j = O(j^{-1-\delta})$ for $\delta > 0$.*

(c) *If $-\frac{1}{2} < d_X < 0$ then $b_0 = a_0$ and $b_j = a_j - a_{j-1}$ for $j > 0$ where $a_j = \Gamma(1 + d_X)^{-1} j^{d_X} L_X(j)$ and $L_X(j+1)/L_X(j) = 1 + o(n^{-1})$.*

The condition on L_X in 2(c) is mild, noting that it holds in particular for the cases $(\log j)^\delta$, for any real δ.

Under these assumptions, we set

$$K(n) = n^{1+d_X+d_Y} L_X(n) L_Y(n) \tag{1.13}$$

in (1.6). While the 'pure fractional' cases represented by (1.3) satisfy Assumption 2, the assumption only controls the tail behaviour of the sequences, and allows arbitrary forms for a finite number of the lag coefficients. In particular, the x_t and y_t processes may be stable invertible ARFIMA(p, d, q) processes. Suppose more generally that

$$x_t = (1 - L)^{-d_X} \theta(L) u_t \tag{1.14}$$

where $\theta(L)$ is any lag polynomial with absolutely summable coefficients, specifically, where $\theta_j = O(j^{-1-\delta})$ for $\delta > 0$. Letting for $d_X > 0$ the identity $a(L) = (1 - L)^{-d_X}$ define the coefficients a_j, such that $a_j \sim \Gamma(d_X)^{-1} j^{d_X - 1}$,[3] and let

$$b(L) = a(L)\theta(L). \tag{1.15}$$

Then note the following fact.

Proposition 1.1 *If (1.15) holds then $b_j \sim \theta(1)\Gamma(d_X)^{-1} j^{d_X - 1}$ as $j \to \infty$.*

[3] The symbol '$\sim$' here denotes that the ratio of the connected sequences converges to 1 as $j \to \infty$.

(All proofs are given in the Appendix.) The slowly varying component can be defined to represent the ratio of b_j to the approximating sequence. Also, since $\mathbf{\Omega}$ is unrestricted, we could impose the normalization $\theta(1) = 1$, if desired, with no loss of generality.

The cases $d_X = 0$ and $d_Y = 0$ are deliberately restricted under Assumption 2(b) to rule out the 'knife-edge' non-summable case, to avoid complications of doubtful relevance. Be careful to note that δ is not a fractional differencing coefficient in this case. Also note that the pure fractional model, represented by (1.3) has $b_0 = 1$ and $b_j = 0$ for $j > 0$, in the case $d_X = 0$.

The case $d_X < 0$ under Assumption 2(c) has the 'overdifferenced' property, implying in particular that $|\sum_{k=0}^{j} b_k| = O(j^{d_X})$. The lag coefficients are summable, but more important, their sum is 0. In the pure fractional case these coefficients are all negative for $j > 0$.

A typical multivariate analysis would wish to consider VARFIMA models, which in the bivariate case would have a Wold representation of the form

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} (1-L)^{-d_Y} & 0 \\ 0 & (1-L)^{-d_X} \end{bmatrix} \begin{bmatrix} \theta_{YY}(L) & \theta_{YX}(L) \\ \theta_{XY}(L) & \theta_{XX}(L) \end{bmatrix} \begin{bmatrix} w_t \\ u_t \end{bmatrix}.$$

Note that the observed series are represented as sums of terms of the form (1.14),

$$\begin{aligned} y_t &= (1-L)^{-d_Y}\theta_{YY}(L)w_t + (1-L)^{-d_Y}\theta_{YX}(L)u_t \\ x_t &= (1-L)^{-d_X}\theta_{YY}(L)w_t + (1-L)^{-d_X}\theta_{XX}(L)u_t. \end{aligned}$$

In this example, (1.6) becomes a sum of four terms of the basic type we wish to analyse. Extending our results to general linear models of this type is therefore a simple application of the continuous mapping theorem to the limit distributions we explore in this paper.

2 Some Properties of G_n

The key step is the following decomposition of expression (1.6). First, expand it by substituting from (1.1) and (1.2) as

$$G_n(\xi) = \frac{1}{K(n)} \sum_{t=1}^{[n\xi]-1} \sum_{s=1}^{t} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_k c_j u_{s-k} w_{t+1-j}.$$

Decompose this sum into three components,

$$G_n = G_{1n} + G_{2n} + G_{3n}. \tag{2.1}$$

where

$$\begin{aligned} G_{1n}(\xi) &= \frac{1}{K(n)} \sum_{t=1}^{[n\xi]-1} \sum_{s=1}^{t} \sum_{k=0}^{\infty} \sum_{j=0}^{k+t-s} b_k c_j u_{s-k} w_{t+1-j} \\ &= \frac{1}{K(n)} \sum_{t=1}^{[n\xi]-1} \sum_{s=1}^{t} \sum_{j=0}^{\infty} \sum_{k=\max\{0,j+s-t\}}^{\infty} b_k c_j u_{s-k} w_{t+1-j} \end{aligned} \tag{2.2}$$

$$G_{2n}(\xi) = \frac{1}{K(n)} \sum_{t=1}^{[n\xi]-1} \sum_{s=1}^{t} \sum_{k=0}^{\infty} b_k c_{k+t-s+1} u_{s-k} w_{s-k} \tag{2.3}$$

and

$$G_{3n}(\xi) = \frac{1}{K(n)} \sum_{t=1}^{[n\xi]-1} \sum_{s=1}^{t} \sum_{k=0}^{\infty} \sum_{j=k+t-s+2}^{\infty} b_k c_j u_{s-k} w_{t+1-j}. \tag{2.4}$$

Thus, G_{1n} contains those terms, and only those terms, in which $s-k \leqslant t-j$, so that the time indices of w strictly exceed those of u, and hence $E(G_{1n}(\xi)) = 0$. In G_{2n}, $s-k = t+1-j$ such that the time indices of u and w match. In G_{3n}, $s-k > t+1-j$ such that the indices of u lead those of w, and $E(G_{3n}(\xi)) = 0$. In this section we consider the behaviour of the sequence $E(G_{2n})$. Broadly speaking, the nature of the limit depends on the sign of $d_X + d_Y$, and we consider the various cases in turn.

Proposition 2.1 *If $d_X + d_Y > 0$ then $E(G_{2n}(\xi)) \to \lambda_{XY}\xi^{1+d_X+d_Y}$ where*

$$\lambda_{XY} = \frac{\omega_{uw}}{\Gamma(d_X+1)\Gamma(d_Y+1)\,(d_X+d_Y)}\Bigg(\frac{d_Y}{(1+d_X+d_Y)}+ \int_0^\infty \Big[d_Y(1+\tau)^{d_X+d_Y} + d_X\tau^{d_X+d_Y} - (d_X+d_Y)(1+\tau)^{d_Y}\tau^{d_X}\Big]d\tau\Bigg) \tag{2.5}$$

Letting λ_{YX} denote the same limit with x_t and y_t interchanged, also note that

$$\begin{aligned}\lambda_{XY} + \lambda_{YX} &= \frac{\omega_{uw}}{\Gamma(d_X+1)\Gamma(d_Y+1)}\times \\ &\quad \Bigg(\frac{1}{(1+d_X+d_Y)} + \int_0^\infty \Big((1+\tau)^{d_X} - \tau^{d_X}\Big)\Big((1+\tau)^{d_Y} - \tau^{d_Y}\Big)\,d\tau\Bigg) \\ &= \psi_{XY}\end{aligned} \tag{2.6}$$

where

$$\psi_{XY} = \lim_{n\to\infty}\frac{1}{K(n)}E\Bigg(\sum_{t=1}^n x_t \sum_{t=1}^n y_t\Bigg).$$

This is the off-diagonal element of $\mathbf{\Psi}$, the long-run covariance matrix of the processes, according to equation (3.12) of DDJ. Considering the decomposition

$$E\Bigg(\sum_{t=1}^{[n\xi]} x_t \sum_{t=1}^{[n\xi]} y_t\Bigg) = \sum_{t=1}^{[n\xi]} E(x_t y_t) + \sum_{t=1}^{[n\xi]-1}\sum_{s=1}^{t} E(x_s y_{t+1}) + \sum_{t=1}^{[n\xi]-1}\sum_{s=1}^{t} E(y_s x_{t+1}) \tag{2.7}$$

where the second term on the right corresponds to $K(n)E(G_n(\xi))$, note that

$$E(x_t y_t) = \sigma_{XY} = \omega_{uw}\sum_{j=0}^{\infty} b_j c_j < \infty. \tag{2.8}$$

The first right-hand side term in (2.7) is $O(n)$, and hence this term is of small order under the normalization $K(n)$. The other two terms converge to λ_{XY} and λ_{YX} respectively under the same normalization, as indicated by (2.6).

Observe that λ_{XY} depends only on d_X, d_Y and ω_{uw} since any short-run parameters have been absorbed into the functions L_X and L_Y; compare Lemma 1.1 for example. The sign of λ_{XY} matches that of d_Y, and if $d_Y = 0$, then $\lambda_{XY} = 0$. When $d_X > 0$, the cases where y_t is i.i.d., and is merely short memory, are equivalent asymptotically. In the pure fractional model in which c_j is defined by (1.3), so that $c_j = 0$ for all $j > 0$ when $d_Y = 0$, note that $E(G_{2n}) = 0$ exactly, for all n. In the general case of weak dependence covered by Assumption 2(b), the proof of Lemma 2.1 shows that the expectation of the triple sum in (2.3) is of $O(n)$, so that with $d_X > 0$ the expression converges to zero.

We give these results in their most easily interpretable form, but for computational purposes, closed-form expressions are more useful. These are as follows.

Proposition 2.2 **(i)** $\lambda_{XY} = \dfrac{\omega_{uw}\Gamma(1 - d_X - d_Y)}{\pi\,(1 + d_X + d_Y)\,(d_X + d_Y)} \sin \pi d_Y.$

(ii) $\psi_{XY} = \dfrac{\omega_{uw}\Gamma(1 - d_X - d_Y)}{(1 + d_X + d_Y)} \left(\dfrac{\sin \pi d_X}{\pi d_X} + \dfrac{\sin \pi d_Y}{\pi d_Y} \right).$

Observe that the first of these formulae is the same as (1.11), indicating that the harmonic and moving average approaches to constructing fractional processes yield equivalent results.

Next, consider the cases where $d_X + d_Y$ is zero or negative. In the latter case, $E(G_{2n}(r))$ diverges.

Proposition 2.3 *If $d_X + d_Y \leq 0$ and $\omega_{wu} \neq 0$, then $E(G_{2n}(r)) = O(n/K(n))$.*

In this instance there is no decomposition of ψ_{YX} into components of the form λ_{XY}, and the three terms in (2.7) are each of $O(n)$. We may write $n^{-1}\sum_{t=1}^{[n\xi]} E(x_t y_t) = \sigma_{XY}\xi$, but also,

$$
\begin{aligned}
\frac{1}{n}\sum_{t=1}^{[n\xi]-1}\sum_{s=1}^{t} E(x_s y_{t+1}) &\to \lambda^*_{XY}\xi \\
\frac{1}{n}\sum_{t=1}^{[n\xi]-1}\sum_{s=1}^{t} E(y_s x_{t+1}) &\to \lambda^*_{YX}\xi
\end{aligned}
$$

where the limits are finite constants, denoted with a '$*$' to distinguish them from the cases with normalization $K(n)$. Unlike λ_{XY} and λ_{YX}, λ^*_{XY} and λ^*_{YX} may depend on the initial lag weights; in other words, dropping a finite number of terms from the front of the lag distribution does not affect the limit in Lemma 2.1, because the terms are non-summable, whereas dropping terms from the sums following Lemma 2.3 would change the limit. Note that $E(\sum_{t=1}^{n} x_t)^2 = O(n^{2d_X+1})$ and $E(\sum_{t=1}^{n} y_t)^2 = O(n^{2d_Y+1})$ (compare DDJ Lemmas 3.1 and 3.3), so for $d_X + d_Y < 0$, the left-hand side of (2.7) is necessarily $o(n)$ by the Cauchy-Schwarz inequality. Hence, in this case

$$
\sigma_{XY} + \lambda^*_{XY} + \lambda^*_{YX} = 0. \tag{2.9}
$$

All these conclusions assume $\omega_{uw} \neq 0$. If u_t and w_t are contemporaneously uncorrelated, which implies under Assumption 1 that the cross-correlogram is zero at all orders, then each of the terms in (2.7) is zero identically. Then, (2.6) holds trivially whatever the sign of $d_X + d_Y$, since $\lambda_{XY} = \lambda_{YX} = 0$.

The last result of this section shows that under our assumtions, G_{2n} is a consistent estimator of the mean – albeit not a feasible one. More important is the fact that that the limit distribution of $G_{1n} + G_{3n}$ matches that of the mean deviation of G_n, not forgetting that the mean diverges under the given normalization when $d_X + d_Y < 0$.

Theorem 2.1 *If Assumptions 1 and 2 hold, $G_{2n} - E(G_{2n}) \xrightarrow{L_2} 0$.*

3 Stochastic Integrals

In this section we construct limiting forms of the terms G_{1n} and G_{3n} using heuristic approximation arguments. Write, informally, the convergences $G_{1n}(\xi) \to \Xi_{1,XY}(\xi)$ and $G_{3n}(\xi) \to \Xi_{3,XY}(\xi)$ as $n \to \infty$, where these symbols will denote the (putative) respective limits. Letting $\Xi_{XY} = \Xi_{1,XY} + \Xi_{3,XY}$, we need to show that

$$
G_n - E(G_n) \to \Xi_{XY}
$$

In view of what is known about the orders of magnitude of G_{2n}, we conjecture the existence of limits for G_n of the following types, depending on the values of $d_X + d_Y$.

Proposition 3.1 *Let Assumptions 1 and 2 hold. For* $0 \leq \xi \leq 1$,

(i) If $d_X + d_Y > 0$,

$$G_n(\xi) \to \Xi_{XY}(\xi) + \lambda_{XY}\xi^{1+d_X+d_Y}.$$

(ii) If $d_X + d_Y = 0$,

$$G_n(\xi) \to \Xi_{XY}(\xi) + \lambda^*_{XY}\xi.$$

(iii) If $d_X + d_Y < 0$ and $\lambda^*_{XY} \neq 0$ then

$$\frac{K(n)}{n}G_n(\xi) \overset{L_2}{\to} \lambda^*_{XY}\xi.$$

(iv) If $-\frac{1}{2} < d_X + d_Y < 0$ and $\lambda^*_{XY} = 0$, and

$$G_n(\xi) \to \Xi_{XY}(\xi).$$

Here the arrows denote conjectured convergences, which in Davidson and Hashimzade (2007b), we show to exist as weak convergences in the space of cadlag functions equipped with the Skorokhod topology. However, note that case (iii) has been shown in Lemma 2.1 to hold in mean square, subject to the components G_{1n} and G_{3n} being of smaller order. In case (iv) there is an extra condition limiting the the joint degree of anti-persistence permitted, placing a lower bound on $d_X + d_Y$. We show in the paper cited that the condition violated without this extra assumption is the a.s. continuity of the integrand process appearing in parentheses in the representation of $\Xi_{1,XY}$ in (3.12) below. Note that this condition is contained in Assumption 2 except in cases where both $dY < 0$ and $d_X < 0$.

Consider the characterization of an integral with fractional integrator by first recalling the definition of a fractional Brownian motion Y in (1.5). Let $\{\mathcal{F}(\tau), \tau \in \mathbb{R}\}$ denote the filtration generated by Brownian motions B_u, B_w. A fundamental representation theorem is the following.

Theorem 3.1 *Y has the representation*

$$Y(\xi) = \int_0^\xi \delta Y, \qquad 0 \leq \xi \leq 1$$

where

$$\delta Y(r) = \frac{1}{\Gamma(d_Y)}\int_{-\infty}^r (r-\tau)^{d_Y-1}\, dB_w(\tau)\, dr \tag{3.1}$$

Be careful to note our choice of notation here, in which 'δY' – rather than 'dY' – is used to represent this specialized concept of differential. This is to avoid confusion with the usage already established, in expressions such as (1.10). In this context, the latter notation means something very different, as we shall see in the sequel. Equation (3.1) is formally equivalent to the Riemann-Liouville fractional integral of the integrable 'function' dB_w (see e.g. Samko et al 1993, Chapter 2). As is well known, the fractional Brownian motion has correlated increments, and this property is embodied precisely in formula (3.1).

A natural generalization suggests defining the integral with respect to the differential process δY of some general function F,

$$\int_0^\xi F\delta Y = \frac{1}{\Gamma(d_Y)}\int_0^\xi \int_{-\infty}^r F(\xi, r, \tau)(r-\tau)^{d_Y-1}\, dB_w(\tau)\, dr \tag{3.2}$$

for $0 \leq \xi \leq 1$, where for complete generality we let $F = F(\xi, r, \tau)$, for $\tau \leq r \leq \xi$. However, the dependence of F on r and ξ as well as τ introduces a generalization of the usual fractional integral formula. Although equation (3.2) can be rearranged formally as $\int_{-\infty}^{\xi} \bar{F}(\xi, \tau)\, dB_w(\tau)$ where

$$\bar{F}(\xi, \tau) = \frac{1}{\Gamma(d_Y)} \int_{\max\{0,\tau\}}^{\xi} F(\xi, r, \tau)(r - \tau)^{d_Y - 1}\, dr \tag{3.3}$$

it can be characterized as an Itô integral only in the case where $F(\xi, r, \tau)$, and hence also $\bar{F}(\xi, \tau)$, is adapted to $\mathcal{F}(\tau)$. In other cases, its status is undetermined. However, in the case where $F = F(\xi, \tau)$, not depending on r, the integral can be written equivalently using (1.5) as

$$\begin{aligned} \int_0^{\xi} F \delta Y &= \frac{1}{\Gamma(d_Y)} \int_0^{\xi} \int_{-\infty}^{r} F(\xi, \tau)(r - \tau)^{d_Y - 1}\, dB_w(\tau)\, dr \\ &= \frac{1}{\Gamma(d_Y + 1)} \left(\int_0^{\xi} (\xi - \tau)^{d_Y} F(\xi, \tau) dB_w(\tau) + \right. \\ &\qquad \left. \int_{-\infty}^{0} \left((\xi - \tau)^{d_Y} - (-\tau)^{d_Y} \right) F(\tau, \xi) dB_w(\tau) \right). \end{aligned} \tag{3.4}$$

If in addition F does not depend on ξ, and $F(\tau) = 0$ for $\tau < 0$, the formula further simplifies to

$$\int_0^{\xi} F \delta Y = \frac{1}{\Gamma(d_Y + 1)} \int_0^{\xi} (\xi - \tau)^{d_Y} F(\tau) dB_w(\tau). \tag{3.5}$$

A case in point would be where F represents another fractional Brownian motion on the unit interval, say X, as defined as in Lemma 3.1. This clearly denotes a well-defined random process $\int X \delta Y$ although, as already intimated, this is a different process from the one represented by (1.10). We now show by heuristic arguments that the limit of $G_{1n} + G_{3n}$ does *not* assume the form of (3.5).

Consider G_{n1} first. Replacing the summation over j in (2.2) by the summation over $m = t + 1 - j$, and the summation over k by the summation over $i = s - k$, rewrite $G_{n1}(\xi)$ as

$$\begin{aligned} G_{n1}(\xi) &= \frac{1}{K(n)} \sum_{t=1}^{[n\xi]-1} \sum_{s=1}^{t} \sum_{m=-\infty}^{t} \sum_{i=-\infty}^{\min\{s,m\}} b_{s-i} c_{t-m} u_i w_{m+1} \\ &= \frac{1}{K(n)} \sum_{t=1}^{[n\xi]-1} \sum_{m=-\infty}^{t} c_{t-m} w_{m+1} \left(\sum_{s=1}^{t} \sum_{i=-\infty}^{\min\{s,m\}} b_{s-i} u_i \right). \end{aligned} \tag{3.6}$$

A form for the limit random variable can then be constructed under Assumption 2 by making the substitutions $dB_u(\tau)$ for $u_{[n\tau]}/\sqrt{n}$, $dB_w(t)$ for $w_{[nt]}/\sqrt{n}$, $\Gamma(d_X)^{-1} s^{d_X - 1} ds$ for $(L_X([ns]) n^{d_X - 1})^{-1} b_{[ns]}$, and $\Gamma(d_Y)^{-1} r^{d_Y - 1} dr$ for $(L_Y([nr]) n^{d_Y - 1})^{-1} c_{[nr]}$. Replacing sums with the integrals in the limit as $n \to \infty$ and noting that $L_X([nr])/L_X(n) \to 1$ we obtain $G_{n1} \to \Xi_{1,XY}$ where

$$\Xi_{1,XY}(\xi) = \frac{1}{\Gamma(d_Y)} \int_0^{\xi} \left(\int_{-\infty}^{r} Z_X(r, t)(r - t)^{d_Y - 1}\, dB_w(t) \right) dr \tag{3.7}$$

where, for $r \in [0, \xi]$ and $t \in (-\infty, r]$,

$$Z_X(r, t) = \frac{1}{\Gamma(d_X)} \int_0^{r} \left(\int_{-\infty}^{\min\{t,s\}} (s - \tau)^{d_X - 1}\, dB_u(\tau) \right) ds$$

$$= \frac{1}{\Gamma(d_X)} \int_0^{\max\{0,t\}} \left(\int_{-\infty}^{s} (s-\tau)^{d_X-1} \, dB_u(\tau) \right) ds$$

$$+ \frac{1}{\Gamma(d_X)} \int_{\max\{0,t\}}^{r} \left(\int_{-\infty}^{t} (s-\tau)^{d_X-1} \, dB_u(\tau) \right) ds$$

$$= X(t) + \Phi_X(r,t) \tag{3.8}$$

(say). The last equality makes use of Theorem 3.1, and defines $\Phi_X(r,t)$. It follows that

$$\Xi_{1,XY}(\xi) = \int_0^{\xi} X\delta Y + \int_0^{\xi} \Phi_X \delta Y. \tag{3.9}$$

Note that $X(t)$ has the fractional integral representation corresponding to that of $Y(t)$ in Theorem 3.1, with $X(t) = 0$ for $t \le 0$. However, $\Phi_X(r,t)$ is not a comparable integral of 'increments' $\delta X(t)$, because $s \neq t$. From (3.6), $\Phi_X(r,t)$ can be viewed as the limiting case of

$$\Phi_{Xn}(r,t) = \frac{1}{n^{1/2+d_X} L_X(n)} \sum_{i=-\infty}^{[nt]-1} \left(\sum_{s=\max\{1,[nt]-1\}}^{[nr]} b_{s-i} \right) u_i \tag{3.10}$$

and it can be written in the equivalent form

$$\Phi_X(r,t) = \begin{cases} \dfrac{1}{\Gamma(1+d_X)} \int_{-\infty}^{t} [(r-\tau)^{d_X} - (t-\tau)^{d_X}] dB_u(\tau) & t > 0 \\ \dfrac{1}{\Gamma(1+d_X)} \int_{-\infty}^{t} [(r-\tau)^{d_X} - (-\tau)^{d_X}] dB_u(\tau) & t \le 0. \end{cases} \tag{3.11}$$

However, it is important to note that $\Phi_X(r,t)$ is adapted to $\mathcal{F}(t)$, and this means that it can be constructed as an Itô integral following (3.3). We have an alternate form of (3.7) as an Itô integral on $(-\infty, \xi]$ with respect to an $\mathcal{F}(t)$-adapted integrand process,

$$\Xi_{1,XY}(\xi) = \int_{-\infty}^{\xi} \left(\frac{1}{\Gamma(d_Y)} \int_{\max\{0,\tau\}}^{\xi} Z_X(r,t) (r-t)^{d_Y-1} \, dr \right) dB_w(t). \tag{3.12}$$

In this representation it is clear that $\Xi_{1,XY}(\xi)$ is adapted to $\mathcal{F}(t)$, and a martingale.

Next, consider G_{n3}. Proceeding to the limit in the same way as before, setting $m = t+1-j$ and $i = s-k$, we obtain from (2.4)

$$G_{n3}(\xi) = \frac{1}{K(n)} \sum_{s=1}^{[n\xi]-1} \sum_{t=s}^{n-1} \sum_{k=0}^{\infty} \sum_{j=k+t-s+2}^{\infty} b_k c_j u_{s-k} w_{t+1-j}$$

$$= \frac{1}{K(n)} \sum_{s=1}^{[n\xi]-1} \sum_{i=-\infty}^{s} b_{s-i} u_i \left(\sum_{t=s}^{[n\xi]-1} \sum_{m=-\infty}^{i-1} c_{t+1-m} w_m \right),$$

and hence

$$G_{n3}(\xi) \to \Xi_{3,XY}(\xi)$$

$$= \frac{1}{\Gamma(d_X)} \int_{-\infty}^{\xi} \left(\int_{-\infty}^{\xi} \Psi_Y(r,t,\xi) (r-t)^{d_X-1} \, dr \right) dB_u(t)$$

$$= \frac{1}{\Gamma(d_X)} \int_0^{\xi} \int_{-\infty}^{\xi} \Psi_Y(r,t,\xi) (r-t)^{d_X-1} \, dB_u(t) \, dr$$

$$= \int_0^\xi \Psi_Y \delta X \tag{3.13}$$

where, for $r \in [0, \xi]$ and $t \in (-\infty, r]$,

$$\begin{aligned}\Psi_Y(r,t,\xi) &= \frac{1}{\Gamma(d_Y)} \int_r^\xi \int_{-\infty}^t (s-\tau)^{d_Y-1}\, dB_w(\tau)\, ds \\ &= \frac{1}{\Gamma(1+d_Y)} \int_{-\infty}^t [(\xi-\tau)^{d_Y} - (r-\tau)^{d_Y}] dB_w(\tau) \end{aligned} \tag{3.14}$$

Note that $\Psi_Y(r,t,\xi)$ is adapted to $\mathcal{F}(t)$, and hence for given ξ and r can also be constructed as an Itô integral.

Thus, from (3.9) and (3.13) it emerges that the weak limit of $G_n - E(G_n)$ has three distinct components. All are adapted to $\mathcal{F}(t)$ so all have the Itô integral characterization, although in one case with respect to a different Brownian motion. This will be the key to establishing the status of these terms as weak limits of discrete sums, although it will be necessary to establish that the limiting integrand processes are stochastically bounded and almost surely continuous.

4 Integration by Parts

It appears natural, when $d_1 + d_2 > 0$ and under appropriate asssumptions such that both convergence results exist, to equate the process $\Xi_{XY}(\xi) + \lambda_{XY}\xi^{1+d_X+d_Y}$ with the one denoted $\int_0^\xi XdY$ in (1.10). As we show elsewhere, the existence of the former limit can be established under more general conditions that of the latter, but in view of the complex nature of these formulae, the 'stochastic integral' designation needs to be placed on firmer ground To this end, our purpose in this section is to establish the validity of the the integration by parts formula

$$\Xi_{XY}(\xi) + \Xi_{YX}(\xi) + \psi_{XY}\xi^{1+d_X+d_Y} = X(\xi)Y(\xi) \tag{4.1}$$

where ψ_{XY} is defined by the second equality of (2.6). The decomposition into components $\psi_{XY} = \lambda_{XY} + \lambda_{YX}$ is not defined for $d_X + d_Y \leq 0$, and $E(G_n)$ generally diverges when $d_X + d_Y < 0$. In this case a formula of the form (4.1) is not defined generally, but it does apply for the case $\psi_{XY} = 0$, such that the processes X and Y are independent.

Consider the random process complementary to the one we considered in the previous section, where the X and Y processes are everywhere interchanged. The next step is to establish the relationship between this process and the original. Define $\Psi_X(r,t,\xi)$ as the complementary case of (3.14) with X replacing Y, and consider (recalling $t \leq r$ in these formulae)

$$\begin{aligned}\tilde{X}(t,\xi) &= Z_X(r,t) + \Psi_X(r,t,\xi) \\ &= \frac{1}{\Gamma(d_X)} \int_0^r \left(\int_{-\infty}^{\min\{t,s\}} (s-\tau)^{d_X-1}\, dB_u(\tau) \right) ds \\ &\qquad + \frac{1}{\Gamma(d_X)} \int_r^\xi \left(\int_{-\infty}^{t} (s-\tau)^{d_X-1}\, dB_u(\tau) \right) ds \\ &= \frac{1}{\Gamma(d_X)} \int_0^\xi \left(\int_{-\infty}^{\min\{t,s\}} (s-\tau)^{d_X-1}\, dB_u(\tau) \right) ds \\ &= X(\xi) - \breve{X}(t,\xi) \end{aligned} \tag{4.2}$$

where

$$\breve{X}(t,\xi) = \frac{1}{\Gamma(d_X)} \int_{\max(t,0)}^\xi \left(\int_t^s (s-\tau)^{d_X-1}\, dB_u(\tau) \right) ds$$

$$= \begin{cases} \dfrac{1}{\Gamma(d_X+1)} \int_t^\xi (\xi-\tau)^{d_X} dB_u(\tau) & t \geq 0 \\ \dfrac{1}{\Gamma(d_X+1)} \begin{pmatrix} \int_0^\xi (\xi-\tau)^{d_X} dB_u(\tau) \\ \quad + \int_t^0 \left[(\xi-\tau)^{d_X} - (-\tau)^{d_X}\right] dB_u(\tau) \end{pmatrix} & t < 0. \end{cases} \tag{4.3}$$

Note the implication of (4.2) that

$$X(\xi)Y(\xi) = \int_0^\xi \tilde{X}\delta Y + \int_0^\xi \breve{X}\delta Y \tag{4.4}$$

where

$$\int_0^\xi \breve{X}\delta Y = \frac{1}{\Gamma(d_Y)} \int_0^\xi \int_{-\infty}^r \breve{X}(t,\xi)(r-t)^{d_Y-1} dB_w(t)\, dr.$$

The status of this latter term is not yet evident, although it exists by virtue of identity (4.4). However, since $E\left(\int_0^\xi \tilde{X}\delta Y\right) = 0$ by construction, we can deduce that

$$E\left(\int_0^\xi \breve{X}\delta Y\right) = EX(\xi)Y(\xi) = \psi_{XY}\xi^{1+d_X+d_Y}.$$

It further follows from (4.4) and its complementary case that

$$\begin{aligned} \Xi_{XY}(\xi) + \Xi_{YX}(\xi) &= \int_0^\xi \tilde{X}\delta Y + \int_0^\xi \tilde{Y}\delta X \\ &= 2X(\xi)Y(\xi) - \int_0^\xi \breve{X}\delta Y - \int_0^\xi \breve{Y}\delta X. \end{aligned} \tag{4.5}$$

Next, consider the following representations, using the rearrangements in (3.4) and (4.3) respectively. Write

$$\begin{aligned} \int_0^\xi \breve{X}\delta Y &= \frac{1}{\Gamma(d_Y+1)} \left(\int_0^\xi (\xi-t)^{d_Y} \breve{X}(t,\xi) dB_w(t) + \int_{-\infty}^0 \left[(\xi-t)^{d_Y} - (-t)^{d_Y}\right] \breve{X}(t,\xi) dB_w(t) \right) \\ &= \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)} \int_0^\xi \left(\int_t^\xi (\xi-\tau)^{d_X} dB_u(\tau) \right) (\xi-t)^{d_Y} dB_w(t) \\ &+ \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)} \int_{-\infty}^0 \left(\int_0^\xi (\xi-\tau)^{d_X} dB_u(\tau) \right) \left[(\xi-t)^{d_Y} - (-t)^{d_Y}\right] dB_w(t) \\ &+ \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)} \int_{-\infty}^0 \left(\int_t^0 \left[(\xi-\tau)^{d_X} - (-\tau)^{d_X}\right] dB_u(\tau) \right) \\ &\qquad \times \left[(\xi-t)^{d_Y} - (-t)^{d_Y}\right] dB_w(t) \end{aligned} \tag{4.6}$$

$$\begin{aligned} &= \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)} \int_0^\xi \left(\int_0^\tau (\xi-t)^{d_Y} dB_w(t) \right) (\xi-\tau)^{d_X} dB_u(\tau) \\ &+ \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)} \left(\int_{-\infty}^0 \left[(\xi-t)^{d_Y} - (-t)^{d_Y}\right] dB_w(t) \right) \left(\int_0^\xi (1-\tau)^{d_X} dB_u(\tau) \right) \\ &+ \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)} \int_{-\infty}^0 \left(\int_{-\infty}^\tau \left[(\xi-t)^{d_Y} - (-t)^{d_Y}\right] dB_w(t) \right) \\ &\qquad \times \left[(\xi-\tau)^{d_X} - (-\tau)^{d_X}\right] dB_u(\tau) + \psi_{YX}\xi^{1+d_X+d_Y}. \end{aligned} \tag{4.7}$$

Note how the third equality re-writes the integral in the form that separates the stochastic (zero mean) and non-stochastic components. In other words, in the terms of the fourth member, there is no intersection of the times of the Brownian motions B_w and B_u. The first term and third term of (4.7) can be constructed as Itô integrals with respect to $dB_u(\tau)$ of $\mathcal{F}(\tau)$-measurable processes, while the second term is the product of terms defined on $(-\infty, 0]$ and $(0, \xi]$ respectively. Since all these terms have mean 0, we can deduce that the mean of the third member of (4.6) has to appear explicitly in (4.7), as shown.

Further, consider the expression obtained by taking (4.7) and interchanging the pairs X, Y, u, w, and also t, τ. It is easily seen that the sum of this expression and (4.6) is simply $X(\xi)Y(\xi) + \psi_{XY}\xi^{1+d_X+d_Y}$. Clearly, the same equality holds if all the arguments are interchanged. It therefore becomes evident that

$$\int_0^\xi \breve{X}\delta Y + \int_0^\xi \breve{Y}\delta X = X(\xi)Y(\xi) + \psi_{XY}\xi^{1+d_X+d_Y}. \tag{4.8}$$

In combination with (4.5), (4.8) establishes the required integration by parts formula (4.1).

5 Discussion

There exists quite an extensive mathematical literature on the properties of integrals with respect to fractional Brownian motion. See, *inter alia*, Lin (1995), Dai and Heyde (1996), Zähle (1998), Decreusefond and Üstünel (1999), Decreusefond (2001), Pipiras and Taqqu (2000, 2001, 2002), Bender (2003) and the references therein. However, much of this literature is concerned with 'sample-path' definitions of the integrals, and with their existence for classes of deterministic integrand. There has been less emphasis on their properties as random variables, and as the limits of discrete sums. In this context our results appear to have some novel features.

In particular, we have drawn a distinction between the apparently 'natural' representation of an integral with fractional integrator that we have denoted by $\int X\delta Y$, and the limiting covariance process which, following previous literature, we should denote formally by $\int XdY$. For adapted integrands the former process may be given an Itô representation, as in (3.5), but it fails the criterion of satisfying an integration by parts formula, as follows implicitly from (4.1). Therefore it does not provide a satisfactory definition of stochastic integral. The limiting covariance process never has an Itô representation, and in fact decomposes into Itô-type terms in which the 'integrator' and 'integrand' change places, but it satisfies the integration by parts formula, and is the necessary time domain counterpart of the harmonic representation (1.10) for this case, as follows from (4.1), Another noteworthy feature of the results is the fact that, while the stochastic integral is both an adapted process and a martingale, the driving processes of 'integrator' and 'integrand' both play the role of Brownian integrators in the components. This is, we suggest, an illuminating way to view the implications of having an integrator process that is not a semimartingale.

6 Appendix: Proofs

6.1 Proof of Proposition 1.1

The coefficient of L^j in the expansion of $b(L) = \theta(L)a(L)$ is

$$b_j = \sum_{i=0}^{j} \theta_i a_{j-i}$$

$$\sim \frac{1}{\Gamma(d_X)} \sum_{i=0}^{j-1} \theta_i (j-i)^{d_X-1}. \tag{6.1}$$

Therefore, for any $\eta > 1$ note that

$$b_j \sim \frac{j^{d_X-1}}{\Gamma(d_X)} \Big(\frac{j-j^{1/\eta}}{j}\Big)^{d_X-1} \sum_{i=0}^{j-1} \theta_i \Big(\frac{j-i}{j-j^{1/\eta}}\Big)^{d_X-1}. \tag{6.2}$$

Write

$$\sum_{i=0}^{j-1} \theta_i \Big(\frac{j-i}{j-j^{1/\eta}}\Big)^{d_X-1} = A(j) + B(j)$$

where

$$A(j) = \sum_{i=0}^{[j^{1/\eta}]-1} \theta_i \Big(\frac{j-i}{j-j^{1/\eta}}\Big)^{d_X-1}$$

and

$$B(j) = \sum_{i=[j^{1/\eta}]}^{j-1} \theta_i \Big(\frac{j-i}{j-j^{1/\eta}}\Big)^{d_X-1}.$$

Since the θ_j are summable and

$$\frac{j}{j-j^{1/\eta}} \to 1$$

it is clear that $A(j) \to \theta(1)$ as $j \to \infty$. To show that $B(j) \to 0$, define $k = j - i$. since $\theta_i = O(i^{-1-\delta})$ for $\delta > 0$ by assumption,

$$\begin{aligned}
B(j) &\leq \sum_{i=[j^{1/\eta}]}^{j-1} |\theta_i| \Big(\frac{j-i}{j-j^{1/\eta}}\Big)^{d_X-1} \\
&= O\Bigg((j-j^{1/\eta})^{1-d_X} j^{-(1+\delta)/\eta} \sum_{k=1}^{j-[j^{1/\eta}]} \Big(\frac{j-k}{j^{1/\eta}}\Big)^{-1-\delta} k^{d_X-1}\Bigg) \\
&= O((j-j^{1/\eta}) j^{-(1+\delta)/\eta})
\end{aligned}$$

in view of the fact that $j - k \geq j^{1/\eta}$ for all the k. Since $\eta > 1$ is arbitrary, pick $\eta < 1 + \delta$ to complete the proof. ∎

6.2 Proof of Proposition 2.1

Under the independence assumption,

$$\begin{aligned}
E(G_{2n}) &= \frac{1}{K(n)} \sum_{k=0}^{\infty} b_k \sum_{j=k+1}^{k+n-1} c_j \sum_{i=1-k}^{n-j} E(u_i w_i) \\
&= \frac{\omega_{uw}}{K(n)} \sum_{k=0}^{\infty} b_k \sum_{t=1}^{n-1} (n-t) c_{k+t}. \qquad (6.3)
\end{aligned}$$

where the second equality makes the substitution $t = j - k$. It can be verified that

$$\sum_{k=0}^{\infty} b_k \sum_{t=1}^{n-1} (n-t) c_{k+t} = \sum_{t=1}^{n-1} \sum_{s=0}^{t-1} \Big(\sum_{k=0}^{s} b_k\Big) c_{s+1} + \sum_{t=1}^{n-1} \sum_{s=t}^{\infty} \Big(\sum_{k=s-t+1}^{s} b_k\Big) c_{s+1}$$

$$= \sum_{t=1}^{n-1} \sum_{s=0}^{t-1} a_{n,t-s}(t/n, 0) c_{s+1} + \sum_{t=1}^{n-1} \sum_{s=t}^{\infty} a_{n,t-s}(t/n, 0) c_{s+1} \tag{6.4}$$

where the expression

$$a_{nt}(s, s') = \sum_{j=\max\{0,[ns']-t+1\}}^{[ns]-t} b_j \tag{6.5}$$

is defined in DDJ, equation (3.2). According to a straightforward extension of DDJ Lemma 3.1,

$$a_{n,[ns]-[nx]}(s, 0) \sim \begin{cases} \dfrac{L_X(n)[nx]^{d_X}}{\Gamma(d_X+1)}, & 0 \le x \le s \\ L_X(n) \dfrac{[nx]^{d_X} - ([nx]-[ns])^{d_X}}{\Gamma(d_X+1)}, & x > s. \end{cases}$$

In the case $d_X + d_Y > 0$ we have, applying Assumption 2 and substituting $d_Y/\Gamma(d_Y+1)$ for $1/\Gamma(d_Y)$,

$$\begin{aligned} \frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=0}^{t-1} a_{n,t-s}(t/n, 0) c_{s+1} &\sim \frac{d_Y}{n^2 \Gamma(d_X+1)\Gamma(d_Y+1)} \sum_{t=1}^{n-1} \sum_{s=1}^{t} \left(\frac{s}{n}\right)^{d_X+d_Y-1} \\ &\to \frac{d_Y}{\Gamma(d_X+1)\Gamma(d_Y+1)} \int_0^1 \int_0^\tau \zeta^{d_X+d_Y-1} d\zeta d\tau \\ &= \frac{d_Y}{\Gamma(d_X+1)\Gamma(d_Y+1)(d_Y+d_X)(1+d_Y+d_X)}. \end{aligned} \tag{6.6}$$

Similarly,

$$\begin{aligned} &\frac{1}{K(n)} \sum_{t=1}^{n-1} \sum_{s=t}^{\infty} a_{n,t-s}(t/n, 0) c_{s+1} \\ &\sim \frac{d_Y}{n^2 \Gamma(d_X+1)\Gamma(d_Y+1)} \sum_{t=1}^{n-1} \sum_{s=0}^{\infty} \left(\left(\frac{s+t}{n}\right)^{d_X} - \left(\frac{s}{n}\right)^{d_X} \right) \left(\frac{s+t}{n}\right)^{d_Y-1} \\ &\to \frac{d_Y}{\Gamma(d_X+1)\Gamma(d_Y+1)} \int_0^\infty \int_0^1 ((\zeta+\tau)^{d_X} - \tau^{d_X})(\zeta+\tau)^{d_Y-1} d\zeta d\tau \\ &= \frac{1}{\Gamma(d_X+1)\Gamma(d_Y+1)\,(d_X+d_Y)} \\ &\quad \times \int_0^\infty [d_Y\,(1+\tau)^{d_X+d_Y} + d_X \tau^{d_X+d_Y} - (d_X+d_Y)\,(1+\tau)^{d_Y}\,\tau^{d_X}] d\tau. \end{aligned} \tag{6.7}$$

Combining these two limits completes the first part of the proof for the cases with $d_Y \neq 0$. If $d_Y = 0$, Assumption 2(b) does not permit the explicit representation used in (6.6) and (6.7). However, summability of the c_s coefficients implies that

$$\sum_{t=1}^{n-1} \sum_{s=0}^{\infty} a_{n,t-s}(t/n, 0) c_{s+1} = o(n^{1+d_X} L_X(n)) \tag{6.8}$$

and $E(G_{2n})$ vanishes in the limit. These expressions are therefore formally correct in all the cases. ∎

6.3 Proof of Proposition 2.2

Let

$$\mathcal{L}(d_X, d_Y) = \int_0^\infty \left[d_Y (1+\tau)^{d_X+d_Y} + d_X \tau^{d_X+d_Y} - (d_X+d_Y)(1+\tau)^{d_Y} \tau^{d_X} \right] d\tau.$$

Denote the integrand by $f(\tau)$. For $0 < d_X, d_Y < 1/2$, $\lim_{\tau\to\pm\infty} f(\tau) = 0$, $\lim_{\tau\to 0} f(\tau) = 1$, and the function is integrable for both positive and negative τ. For $-1/2 < d_X, d_Y < 0$, we have $\lim_{\tau\to\pm\infty} f(\tau) = 0$, $f(\tau)$ has a singularity at $\tau = 0$ with $\lim_{\tau\to 0} f(\tau)\,\tau^{-(d_X+d_Y)} = 1$, and $f(\tau)$ is integrable for $\tau \geq 0$. It also has a singularity at $\tau = -1$ with $\lim_{\tau\to -1} f(\tau)(\tau+1)^{-(d_X+d_Y)} = 1$, and so is also integrable.

Consider an auxiliary integral

$$\mathcal{L}^*(d_X, d_Y) = \int_{-\infty}^\infty f(\tau)\, d\tau.$$

Changing the variable of integration, $\tau + 1 = -t$, we obtain:

$$\begin{aligned}
\mathcal{L}^*(d_X, d_Y) &= \int_{-\infty}^\infty \left[d_Y (1+\tau)^{d_X+d_Y} + d_X \tau^{d_X+d_Y} - (d_X+d_Y)(1+\tau)^{d_Y} \tau^{d_X} \right] d\tau \\
&= \int_{-\infty}^\infty \left[d_Y (-t)^{d_X+d_Y} + d_X (-t-1)^{d_X+d_Y} - (d_X+d_Y)(-t)^{d_Y} (-t-1)^{d_X} \right] dt \\
&= (-1)^{(d_X+d_Y)} \int_{-\infty}^\infty \left[d_Y t^{d_X+d_Y} + d_X (t+1)^{d_X+d_Y} - (d_X+d_Y) t^{d_Y} (t+1)^{d_X} \right] dt \\
&= (-1)^{d_X+d_Y} \mathcal{L}^*(d_Y, d_X).
\end{aligned}$$

Note that by interchanging d_X and d_Y we obtain

$$\begin{aligned}
\mathcal{L}^*(d_X, d_Y) &= (-1)^{d_X+d_Y} \mathcal{L}^*(d_Y, d_X) \\
&= (-1)^{2(d_X+d_Y)} \mathcal{L}^*(d_X, d_Y) = 0
\end{aligned}$$

and hence also

$$\mathcal{L}^*(d_X, d_Y) = 0$$

unless $d_X + d_Y = 0, \pm 1, \pm 2, \ldots$. Next, divide the range of integration in $\mathcal{L}^*(d_Y, d_X)$ into $(-\infty, -1)$, $(-1, 0)$, and $(0, \infty)$. For the first interval change of variables $\tau = -t - 1$ gives

$$\begin{aligned}
&\int_{-\infty}^{-1} \left[d_Y t^{d_X+d_Y} + d_X (t+1)^{d_X+d_Y} - (d_X+d_Y) t^{d_Y} (t+1)^{d_X} \right] dt \\
&= \int_0^\infty \left[d_Y (-1-\tau)^{d_X+d_Y} + d_X (-\tau)^{d_X+d_Y} - (d_X+d_Y)(-1-\tau)^{d_Y} (-\tau)^{d_X} \right] d\tau \\
&= (-1)^{d_X+d_Y} \int_0^\infty \left[d_Y (1+\tau)^{d_X+d_Y} + d_X \tau^{d_X+d_Y} - (d_X+d_Y)(1+\tau)^{d_Y} \tau^{d_X} \right] d\tau \\
&= (-1)^{d_X+d_Y} \mathcal{L}(d_X, d_Y).
\end{aligned}$$

For the second interval using $\tau = -t$ we have

$$\begin{aligned}
&\int_{-1}^{0} \left[d_Y t^{d_X+d_Y} + d_X (t+1)^{d_X+d_Y} - (d_X+d_Y) t^{d_Y} (t+1)^{d_X} \right] dt \\
&= \int_0^1 \left[d_Y (-\tau)^{d_X+d_Y} + d_X (1-\tau)^{d_X+d_Y} - (d_X+d_Y)(-\tau)^{d_Y} (1-\tau)^{d_X} \right] d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \left[(-1)^{d_X+d_Y} d_Y \tau^{d_X+d_Y} + d_X (1-\tau)^{d_X+d_Y} - (-1)^{d_Y} (d_X+d_Y) \tau^{d_Y} (1-\tau)^{d_X}\right] d\tau \\
&= \frac{(-1)^{d_X+d_Y} d_Y}{d_X+d_Y+1}(1-0) - \frac{d_X}{d_X+d_Y+1}(0-1) - (-1)^{d_Y} (d_X+d_Y) B(d_X+1, d_Y+1) \\
&= \frac{(-1)^{d_X+d_Y} d_Y + d_X}{d_X+d_Y+1} - (-1)^{d_Y} (d_X+d_Y) B(d_X+1, d_Y+1)
\end{aligned}
$$

The integral over the third interval is simply $\mathcal{L}(d_Y, d_X)$. Adding the integrals over these three intervals we obtain

$$
\begin{aligned}
\mathcal{L}^*(d_Y, d_X) &= (-1)^{d_X+d_Y} \mathcal{L}(d_X, d_Y) + \frac{(-1)^{d_X+d_Y} d_Y + d_X}{d_X+d_Y+1} \\
&\quad - (-1)^{d_Y} (d_X+d_Y) B(d_X+1, d_Y+1) + \mathcal{L}(d_Y, d_X) \\
&= 0. \qquad (6.9)
\end{aligned}
$$

By symmetry,

$$
\begin{aligned}
\mathcal{L}^*(d_X, d_Y) &= (-1)^{d_X+d_Y} \mathcal{L}(d_Y, d_X) + \frac{(-1)^{d_X+d_Y} d_X + d_Y}{d_X+d_Y+1} \\
&\quad - (-1)^{d_X} (d_X+d_Y) B(d_X+1, d_Y+1) + \mathcal{L}(d_X, d_Y) \\
&= 0. \qquad (6.10)
\end{aligned}
$$

where we used $B(x, y) = B(y, x)$. Now we multiply (6.9) by $(-1)^{d_X+d_Y}$ and subtract from (6.10):

$$
\begin{aligned}
0 \quad = \quad & \left[1 - (-1)^{2(d_X+d_Y)}\right] \left[\mathcal{L}(d_X, d_Y) + \frac{d_Y}{d_X+d_Y+1}\right] \\
& - (-1)^{d_X} \left[1 - (-1)^{2d_Y}\right] (d_X+d_Y) B(d_X+1, d_Y+1).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{L}(d_X, d_Y) &= -\frac{d_Y}{d_X+d_Y+1} \\
&\quad + (-1)^{d_X} \frac{1 - (-1)^{2d_Y}}{1 - (-1)^{2(d_X+d_Y)}} (d_X+d_Y) B(d_X+1, d_Y+1).
\end{aligned}
$$

Finally, using $(-1)^x = e^{i\pi x}$ we rewrite in the second term

$$
\begin{aligned}
(-1)^{d_X} \frac{1 - (-1)^{2d_Y}}{1 - (-1)^{2(d_X+d_Y)}} &= e^{i\pi d_X} \frac{1 - e^{i\pi 2 d_Y}}{1 - e^{i\pi 2(d_X+d_Y)}} \\
&= e^{i\pi d_X} \frac{e^{i\pi d_Y}\left(e^{-i\pi d_Y} - e^{i\pi d_Y}\right)}{e^{i\pi(d_X+d_Y)}\left(e^{-i\pi(d_X+d_Y)} - e^{i\pi(d_X+d_Y)}\right)} \\
&= \frac{\sin \pi d_Y}{\sin \pi (d_X+d_Y)}
\end{aligned}
$$

and therefore

$$
\mathcal{L}(d_X, d_Y) = -\frac{d_Y}{d_X+d_Y+1} + (d_X+d_Y) B(d_X+1, d_Y+1) \frac{\sin \pi d_Y}{\sin \pi (d_X+d_Y)}.
$$

To complete proof of part (i), substitute this expression into (2.5) and rearrange using the identities $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, $\Gamma(1-x)\Gamma(x) = \pi/\sin\pi x$ and $\Gamma(x+1) = x\Gamma(x)$:

$$\begin{aligned}
\lambda_{XY} &= \frac{\omega_{uw}}{\Gamma(d_X+1)\,\Gamma(d_Y+1)\,(d_X+d_Y)} \times \\
&\left(\frac{d_Y}{1+d_X+d_Y} - \frac{d_Y}{d_X+d_Y+1} + (d_X+d_Y)\,B(d_X+1,d_Y+1)\,\frac{\sin\pi d_Y}{\sin\pi(d_X+d_Y)}\right) \\
&= \frac{\omega_{uw}}{\Gamma(d_X+1)\,\Gamma(d_Y+1)} B(d_X+1,d_Y+1)\,\frac{\sin\pi d_Y}{\sin\pi(d_X+d_Y)} \\
&= \frac{\omega_{uw}}{\Gamma(2+d_X+d_Y)} \frac{\sin\pi d_Y}{\sin\pi(d_X+d_Y)} \\
&= \frac{\omega_{uw}}{(1+d_X+d_Y)\,\Gamma(1+d_X+d_Y)} \frac{\Gamma(1-d_X-d_Y)\Gamma(d_X+d_Y)}{\Gamma(1-d_Y)\Gamma(d_Y)} \\
&= \frac{\omega_{uw}\Gamma(1-d_X-d_Y)}{\pi(1+d_X+d_Y)(d_X+d_Y)} \sin\pi d_Y.
\end{aligned}$$

Part (ii) follows immediately on summing this expression with the complementary case having d_X and d_Y interchanged. ∎

6.4 Proof of Proposition 2.3

In this case, note that if a_{nt} is defined by (6.5) then

$$a_{n,t-s}(t/n,0)c_{s+1} = O(s^{d_X+d_Y-1}L_X(s)L_Y(s))$$

so that these terms are summable by assumption. Considering expression (6.4), the lemma follows since

$$\sum_{t=1}^{n-1}\sum_{s=0}^{t-1} a_{n,t-s}(t/n,0)c_{s+1} = O(n)$$

and

$$\sum_{t=1}^{n-1}\sum_{s=t}^{\infty} a_{n,t-s}(t/n,0)c_{s+1} = o(n). \qquad ∎$$

6.5 Proof of Theorem 2.1

Without loss of generality we consider the case $\xi = 1$, and for simplicity of notation write G_{2n} for $G_{2n}(1)$. The extension to $\xi < 1$ is immediate.

Setting $i = s-k$, rewrite (2.3) as

$$\begin{aligned}
G_{2n} - E(G_{2n}) &= \frac{1}{K(n)} \sum_{t=1}^{n-1}\sum_{s=1}^{t}\sum_{k=0}^{\infty} b_k c_{k+t-s+1}(u_{s-k}w_{s-k} - \omega_{uw}) \\
&= \frac{1}{K(n)} \sum_{t=1}^{n-1}\sum_{s=1}^{t} P_{ts}
\end{aligned}$$

(say) where

$$P_{ts} = \sum_{i=-\infty}^{s} b_{s-i}c_{t+1-i}(u_i w_i - \omega_{uw}).$$

Hence note that

$$E(G_{2n}-E(G_{2n}))^2 \le \frac{2}{K(n)^2}\sum_{t=1}^{n-1}\sum_{s=1}^{t}\sum_{m=0}^{t-s}\sum_{k=0}^{s-1}E(P_{ts}P_{t-m,s-k}).$$

where, setting $j=s-i$ in the third member and letting C denote a generic finite constant,

$$\begin{aligned}
E(P_{ts}P_{t-m,s-k}) &= \frac{\mu_{uw}^4-\omega_{uw}^2}{K(n)^2}\sum_{i=-\infty}^{s-k} b_{s-i}b_{s-k-i}c_{t+1-i}c_{t-m+1-i} \\
&= \frac{\mu_{uw}^4-\omega_{uw}^2}{K(n)^2}\sum_{j=k}^{\infty} b_j b_{j-k}c_{t+1-s+j}c_{t-m+1-s+j} \\
&\le \frac{C}{n^{2(1+d_X+d_Y)}}\sum_{j=k}^{\infty} j^{d_X-1}(j-k)^{d_X-1}(j+t+1-s)^{d_Y-1}(j+t-m+1-s)^{d_Y-1} \\
&\le \frac{C}{n^{2(1+d_X+d_Y)}}k^{2d_X-1}(k+t+1-s)^{d_Y-1}(k+t-m+1-s)^{d_Y-1}.
\end{aligned}$$

Hence,

$$E(G_{2n}-E(G_{2n}))^2 \le \frac{C}{n^{2(1+d_X+d_Y)}}\sum_{t=1}^{n-1}\sum_{s=1}^{t}(s-1)^{2d_X}(t+1-s)^{d_Y-1}\sum_{m=0}^{t-s}(t-m+1-s)^{d_Y-1}.$$

These sums can be bounded by conventional summation arguments (Davidson 1994, Thm 2.27) as follows, also applying Lemma A.1 of DDJ in the case $d_X<0$.
Case $d_Y>0$:

$$\begin{aligned}
E(G_{2n}-E(G_{2n}))^2 &\le \frac{C}{n^{2(1+d_X+d_Y)}}\sum_{t=1}^{n-1}\sum_{s=1}^{t}(t+1-s)^{2d_Y-1}(s-1)^{2d_X} \\
&= O(n^{-1}).
\end{aligned}$$

Case $d_Y\le 0$:

$$\begin{aligned}
E(G_{2n}-E(G_{2n}))^2 &\le \frac{C}{n^{2(1+d_X+d_Y)}}\sum_{t=1}^{n-1}\sum_{s=1}^{t}(t+1-s)^{d_Y-1}(s-1)^{2d_X} \\
&= \left\{\begin{array}{ll} O(n^{-1}\log n), & d_Y=0 \\ O(n^{-1-2d_Y}), & d_Y<0 \end{array}\right. . \qquad \blacksquare
\end{aligned}$$

6.6 Proof of Theorem 3.1

A fractional Brownian motion Y is defined for $0\le t\le 1$ by

$$Y(t)=\frac{1}{\Gamma(d_Y+1)}\left[\int_0^t (t-\tau)^{d_Y}\,dB_w(\tau)+\int_{-\infty}^0\left((t-\tau)^{d_Y}-(-\tau)^{d_Y}\right)dB_w(\tau)\right].$$

However, note that for $0\le t\le 1$,

$$\begin{aligned}
&\frac{1}{\Gamma(d_Y+1)}\left[\int_0^t (t-\tau)^{d_Y}\,dB_w(\tau)+\int_{-\infty}^0\left((t-\tau)^{d_Y}-(-\tau)^{d_Y}\right)dB_w(\tau)\right] \\
&\quad=\frac{1}{\Gamma(d_Y)}\left[\int_0^t\left(\int_\tau^t (r-\tau)^{d_Y-1}\,dr\right)dB_w(\tau)\right.
\end{aligned}$$

$$
+ \int_{-\infty}^{0} \left(\int_{0}^{t} (r-\tau)^{d_Y-1}\, dr \right) dB_w(\tau) \Bigg]
$$

$$
= \frac{1}{\Gamma(d_Y)} \left[\int_{-\infty}^{t} \left(\int_{\max\{0,\tau\}}^{t} (r-\tau)^{d_Y-1}\, dr \right) dB_w(\tau) \right]
$$

$$
= \frac{1}{\Gamma(d_Y)} \int_{0}^{t} \left(\int_{-\infty}^{r} (r-\tau)^{d_Y-1}\, dB_w(\tau) \right) dr. \quad \blacksquare
$$

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