Representations of the Moments of the Dickey-Fuller and Related Distributions

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Abstract
The purpose of this paper is to revisit the problem of finding closed forms for the moments of test statistics in Dickey-Fuller regressions. It is shown that in all cases the moments are expressible in terms of moments of only three distributions associated with the hyperbolic functions cosh, sinh and tanh, and their convolutions. The mean of the Dickey-Fuller distribution is expressed as a function of a 3F2(1) generalized hypergeometric function and it is shown there are only *four* independent Thomae relations connected to it (representing possible transformations of the series), none of which leads to a series corresponding to a known constant or function of known constants as listed in decimal expansion form either in Sloane's Online Encyclopaedia of Integer Sequences or through application of Plouffe's Inverter. Furthermore, the function is shown to belong to the same irreducible class of generalized hypergeometric function as Catalan's constant for which the problem of finding a closed form in terms of known mathematical constants, if such a closed form exists, is still open. The paper also shows that the hypergeometric identities that are applied in the case of the mean, and other relevant hypergeometric transformations, do not generalize to the higher moments. Each higher moment can be expressed, however, as a generalized hypergeometric function that allows it to be computed accurately; and it is shown the proposed method improves upon the method based on numerical integration and Simpson's rule used by Nabeya (1999). Various other expressions for the moments are provided including a double integral representation, a logarithmic integral representation and a fractional derivative representation. The paper extends and clarifies recent work by Tanaka (1996), Nielsen (1997), Gonzalo and Pitarakis (1998) and Nabeya (1999).

References

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