Bridging the Gap to Analytical and Computational Foundations

Naomi Bedeau, John Harknett, Katherine Watson

June 2021

Analytical and Computational Foundations (MATH1005) is a first year Core module. This document will help to bridge the gap from A level to University Mathematics. In these chapters, we will go over preliminary details in order to prepare you for some of the topics you will face this academic year. There are questions throughout this document with video links explaining the solutions.

Contents

1 Sets and Set Notation

Knowledge of **sets** is very important, especially for 'ACF' (Analytical and Computational Foundations) and 'FPM' (Foundations of Pure Mathematics) this year, but also your entire study of Mathematics. In this chapter, we will detail some of the basic definitions and properties of sets. This will prepare you for the early topics in a few of your modules.

1.1 Definitions

Since university mathematics can sometimes throw students in at the deep end, in terms of notation and different processes, this section aims to provide some context to make your first few weeks a bit easier.

Definition of a Set: A set is a collection of distinct mathematical objects. These objects/elements can be numbers, people or even other sets!

Elements: Elements are what a set contains. If x is an element of the set A, we write $x \in A$. The symbol '$\in$' simply means is an element of.

1.2 Examples

Now that the basic notation has been discussed, we will get into some examples (these should give you a very basic understanding of what a set is).

Example 1

$$A = \{1, 2, 3, 4, 5\} \tag{1}$$

The above line, another notation, means that 1,2,3,4 and 5 are elements of the set A. Writing the curly brackets is simply a convention. Here, 1 is an element of A, but 11 is not (for instance).

The mathematical way of writing this is as follows: $1 \in A$, $11 \notin A$.

Example 2

$$\text{B} = \text{`The set of all fruits'} \tag{2}$$

This example is a bit odd, but B is still a set; sets really can contain anything! For instance, apple $\in B$, but pizza $\notin B$.

Now we progress to more complicated, specific examples.

Example 3

$$C = \{x \text{ is a positive integer} : x \text{ is divisible by } 2\} \tag{3}$$

Here, ignoring the second half of the bracketed text, C could contain all positive integers. However, this second half holds a great deal of information. The colon : denotes the phrase 'such that'. Therefore, equation (3) reads 'C is the set containing all positive even integers'. x represents a general element of the set; we could write $C = \{2, 4, 6, 8, ...\}$, which essentially means the same thing as equation (3).

Note that **divisibility** means having the capacity to be divided without remainder (i.e. division gives an integer result). For instance, 15 is divisible by 5, but 21 is not.

Example 4

$$D = \{(x, y) \in \mathbb{R}^2 : x < y\} \tag{4}$$

Some new notation is introduced here: $\mathbb{R}$ simply denotes the set of all real numbers (i.e. all numbers without an imaginary part (see Linear Mathematics (MATH1007) preliminary document if confused), and raising this $\mathbb{R}$ to the second power simply refers to the x-y plane (which you should be familiar with from GCSE/A-Level).

We can now assess D itself. Equation (4) reads 'D is the set of points (x, y) in the x-y plane such that $x < y$'. Therefore, we now know that any choice of x and y, provided x is smaller than y, satisfies the inequality. We can graph this, but that will be shown in the 4th chapter 'Inequalities' below.

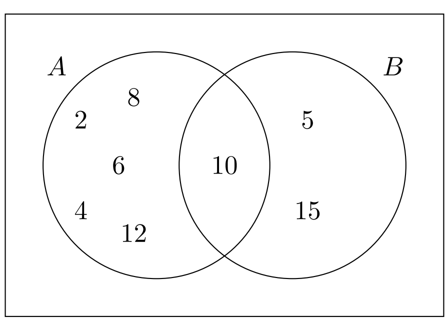
Now what you've seen a few examples, we will provide you with a comprehensive list of well-defined sets and how to denote them!

Symbol	Set represented by symbol	Example Elements
$\mathbb{Z}$	All integers, positive and negative.	...,-2,-1,0,1,2,...
$\mathbb{Q}$	All rational numbers; all numbers with an integer factorisation	1, 5/4, 524/71, etc.
$\mathbb{R}$	All real numbers! Includes $\mathbb{Q}$ (rationals) and $\mathbb{Q}'$ (irrationals)	$\sqrt{2}$, 3, 59/9, 3+$\sqrt{5}$, etc.
$\mathbb{N}$	The 'natural numbers' (positive integers)	1,2,3,4,...

Your ACF module will go into more depth regarding the properties of these sets, but hopefully this small table prepares you for the type of sets you will need to remember.

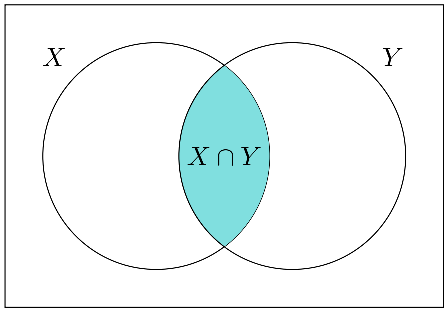
1.3 Venn Diagrams

Sets can also be displayed using **Venn Diagrams**; these are a pictorial representation of sets and their elements. The below diagram shows the sets A = {even numbers} and B = {multiples of 5} and how they may overlap. Note that the rectangle simply represents the '**universal set**' U, which is the set containing all objects or elements. Additionally note that in the below example, only a few elements are listed for simplicity.



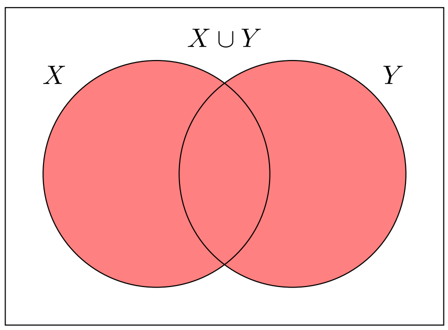
As you can see, sets can have overlapping elements! In this case, 10 is both an even number and a multiple of 5. If we look more generally at these diagrams, we can introduce some concepts (below, X and Y are general sets).

1.3.1 Intersection



The blue shaded region X ∩ Y represents the **intersect** of sets X and Y. These are all the elements that reside in both X and Y simultaneously and we can think of the intersect as being elements in X AND Y. This gives rise to another concept; the **union**.

1.3.2 Union

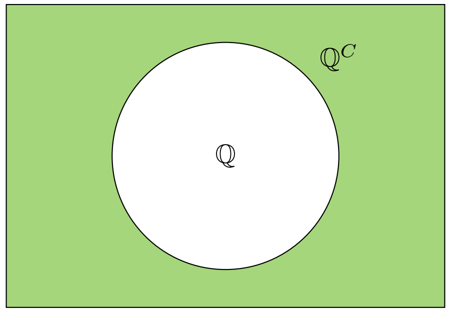


The union of X and Y, denoted $X \cup Y$, represents every element included in each set, including overlapping elements! In the case of A and B in the example above, $A \cup B = \{2, 4, 6, 8, 10, 12, 5, 15, ...\}$. Note that 10 only appears once. The union of general sets X and Y can be thought of as X OR Y OR BOTH - this phrasing describes how the union covers all possible elements in both sets.

1.3.3 Complement

We now introduce the **complement** of a set. The complement of a general set is defined as follows: The complement X^C of a set X is an entirely new set, containing all the elements not in X.

For example, when looking at $\mathbb{Q} = \{\text{rational numbers}\}$, we have that $\mathbb{Q}^C = \{\text{irrational numbers}\}$. We can display this on a venn diagram as follows:



As you can see, $\mathbb{Q}^C$ represents every element that is not included in $\mathbb{Q}$. Below we have listed a few elements in each section of the venn diagram above, just to aid your understanding of what this image means.

- **White area** $\mathbb{Q}$: 1/2, 19/256, 5, 0 etc.
- **Green area** $\mathbb{Q}^C$: $\sqrt{2}$, π, $\sqrt{3}$, 'e' etc.

We may also generally note that writing $(\mathrm{X}^C)^C = \mathrm{X}$.

Questions

1. Write out the following sentences in notation form (the first is an example to show you what you need to do):

   (a) For 'the set A of natural numbers divisible by 3', we may write $\{x \in \mathbb{N} : x = 3m, \text{ where } m \in \mathbb{N}\}$.

   (b) 'B is the set of real numbers less than 5'.

   (c) 'C is the set of all natural numbers with no divisors other than 1 and itself'.

   (d) 'D is the set of all natural numbers not between 80 and 120'.

   Video solution link:

   - All of Question 1 (excluding 1(a))

2. List 5 elements of the following sets:

   (a) A is the set of all square numbers.

   (b) $B = \{(x, y) \in \mathbb{R}^2 : x = 2y\}$

   (c) $\mathbb{C}$ (if stuck, see 'Bridging the Gap to Linear mathematics' document or watch solution video).

   Video solution link:

   - All of Question 2

3. For the following venn diagram, find

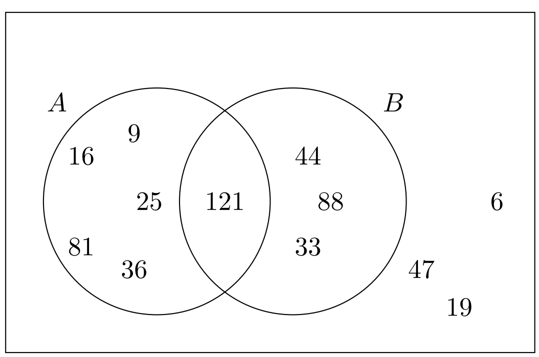
   (a) The elements in A

   (b) The elements in A, but not B

   (c) $A \cap B$

   (d) $A \cup B^C$

   (e) $B \cap A^C$



   Video solution link:

   - All of Question 3

2 Mathematical Reasoning/Logic

2.1 Brief Introduction

This subsection will be somewhat short, but should provide a basis of understanding. Logic is an extremely important part of mathematics and mathematical notation. One of the main purposes of logic is to prove certain theorems, using several methods (these methods will be specified in each section). Now, we study the concept of reasoning.

2.2 Reasoning

The expression A $\implies$ B reads 'A implies B'. For events A and B, this means 'if A, then B'. See below for an example.

Example 1

$$\text{Event A is `}x = 1\text{', Event B is `}x > 0\text{'} \tag{5}$$

Which of these statement implies the other? It may seem obvious, but this process is very important groundwork and will allow you to tackle harder examples in the future.

Clearly, A $\implies$ B: x being 1 means that x is greater than zero necessarily.
However, B $\not\Longleftarrow$ A: x being greater than zero does not necessarily imply $x = 1$ (x could then be any other positive number, for instance). Even though $x > 0$ permits $x = 1$, it does not imply it explicitly.

Here is another example.

Example 2
Let x and y be positive integers.

$$\text{Statement: } x + y = 2 \Longrightarrow x = y = 1 \tag{6}$$

This statement is true! Since x and y are both positive integers (or alternatively, are elements of the Natural numbers $\mathbb{N}$), the only solution must be that $x = y = 1$.

If you look closely, you will also notice that $x + y = 2 \Longleftarrow x = y = 1$ (backwards implication).

Two-way implication means we have an 'if and only if' (denoted 'iff') statement. This means that one event can only occur if the other does (and vice versa). We may write

$$x + y = 2 \iff x = y = 1 \tag{7}$$

Remember the condition that x and y are natural numbers! Without this condition, the above iff statement would not be true.

You will encounter more examples in the worked example videos associated with this document and in your ACF module content. We will now go over 'statements' themselves in a bit more detail.

2.3 Quantifiers

But before that, we will briefly discuss **quantifiers**. These mathematical symbols hold a lot of information, and some will be listed below!

1. $\forall$ = 'for all'

2. $\exists$ = 'there exists'

For instance, we may write $\forall x \in \mathbb{Z}$, $\exists \in \mathbb{N}$, $n > 1$ such that n divides x. This reads: 'for all integers x, there exists a natural number n greater than 1 such that n divides x.' This is obviously true, since every integer is divisible by the absolute value of itself (-9 is divisible by 9 for instance). Quantifiers serve as a method of abbreviation, as longer mathematical expressions are quite tedious to write out using words.

2.4 Statements

The notation ¬A reads 'not A'. Mathematically, this symbol ¬ is used to negate statements or quantifiers. For context, $\neg(0 \in \mathbb{N})$ and $0 \notin \mathbb{N}$ mean exactly the same thing. We can use this notation to define the **contrapositive** of a statement:

Definition

$$\text{A} \implies \text{B has contrapositive } \neg\text{B} \implies \neg\text{A} \tag{8}$$

Finding the contrapositive essentially boils down to negating 'everything' you see, including the quantifiers. Just to help you understand this process, we have included a little table on how to negate certain quantifiers and mathematical symbols!

Symbol	Negation
$\forall$	$\exists$
$\exists$	$\forall$
$\in$	$\notin$

It also maintains validity, meaning that if a statement is true, so is its contrapositive. Similarly, if a statement is false, then so is the contrapositive. We now look at a few specific examples, just so you can get to grips with this concept.

Some Examples

1. 'If it is raining, then the grass is wet'. Then, the contrapositive reads 'if the grass is not wet, then it is not raining'.

2. '$x \in \mathbb{N} \implies x^2 \geq x$' has contrapositive '$x^2 < x \implies x \notin \mathbb{N}$'.

3. 'If $x = 0$, then $\sin(x) = 0$'. The contrapositive reads 'if $\sin(x) \neq 0$, then $x \neq 0$.

A method we sometimes use is 'disproof by **counterexample**.' In essence, we can prove a statement to be false by providing one case that does not work. There are questions below for you to gain some practice in this, but it tends to be fairly straightforward!

Questions:

1. Show by counterexample that the following implications are false:

   (a) If n is prime, then $3n - 4$ is prime.

   (b) If $x \in \mathbb{R}$, then $x^2 \leq 0$.

   Video solution link:

   - All of Question 1

2. Show that each of the following statements are true, by finding the contrapositive:
   (a) If n^2 is odd, then n is odd.
   (b) If $m + n$ is even, then m and n are both even or m and n are both odd.

   Video solution link:

   - All of Question 2

3 Methods of Proof

Proving a result is one of the most fundamental things any mathematician can do. It provides us with clarity as to whether a theory, statement or proposition holds under known mathematical restrictions. Although there are many methods of proof, several frequently arise as the easiest to carry out. In this section, we will cover proof by contradiction and proof by induction.

3.1 Contradiction

Proof by contradiction is a really useful tool. It essentially works like this:

1. Statement to prove
2. Assume the opposite is true
3. Use pre-known methods to show this assumption contradicts itself
4. This then proves the original statement

It might seem confusing as to why proving something is not true shows the opposite is true, but it is actually rigorous mathematically (provided your 'opposite' assumption is the right one).

Example 1: Prove that $\sqrt{2}$ is irrational.

- **Step 1**: We want to prove that $\sqrt{2}$ is irrational.

- **Step 2**: Assume that $\sqrt{2}$ is **rational**.

We may then write $\sqrt{2} = \frac{a}{b}$, where a and b are integers (we may assume positive, since we are looking at $+\sqrt{2}$ here). We may assume that $\frac{a}{b}$ is irreducible; that is, a and b do not have any common divisors.

Just briefly, an example of an irreducible fraction is $\frac{19}{21}$, since 19 and 21 have no common divisors.

- **Step 3**: Show that this assumption does not make sense.

Squaring both sides, we get $2 = \frac{a^2}{b^2} \implies 2b^2 = a^2$. We now notice that if a^2 is equal to a multiple of 2, then we can write write a = 2c, since if a square is even then so is its root. Substituting this in gives the following:

$$2b^2 = 4c^2 \implies b^2 = 2c^2. \tag{9}$$

Clearly, from equation (9), b^2 is also a multiple of 2, implying b is also. However, this would then mean that $\sqrt{2} = \frac{a}{b}$ is not in its simplest form, since a and b are both reducible by 2. We now run into the contradiction!

We may then write that $\sqrt{2}$ must be irrational and conclude our proof. We will show one more example of a proof by contradiction below. We typically write ■ to denote the completion of a proof (this will be used from here on).

Example 2: For a,b$\in \mathbb{Z}$, prove that $a^2 - 4b - 3 = 0$ has no solutions.

- **Step 1**: We want to prove that $a^2 - 4b - 3 = 0$ has no integer solutions.

- **Step 2**: Assume that $a^2 - 4b - 3 = 0$ has integer solutions.
Then, $a^2 = 4b + 3 = 0$, where the RHS is odd, therefore the LHS is odd as well!

- **Step 3**: Show that this assumption does not make sense.

Since a^2 is odd, so is a (by properties of square roots). We may then write $a = 2n{+}1$, where $n \in \mathbb{Z}$. Substituting this in gives the following:

$$(2n+1)^2 = 4b + 3 \implies 4n^2 + 4n + 1 = 4b + 3 \implies 4(n^2 + n - b) = 2 \implies n^2 + n - b = \frac{2}{4} = \frac{1}{2} \tag{10}$$

Now, we notice a contradiction! Both b and n are integers, meaning the LHS of the equation above must also be an integer. However, the RHS is a fraction!
We then have a contradiction! Therefore, the opposite of the assumption must be true! i.e. $a^2 - 4b - 3 = 0$ has no integer solutions, and we an end our proof. ■

You will encounter this type of proof frequently this year, so make sure you memorise this method by doing the example questions provided by your lecturers.

3.2 Induction

Induction is another useful tool for proof. In principle, induction works by proving some initial case, then by proving generally that if one case is true, so is its consecutive case. These will come to be known as the anchor/base step and the induction step.

In terms of a step by step process, we describe proof by induction as follows: For a collection of statements P(n), where $n \in \mathbb{N}$,:

1. Prove P(1) (or P(2) etc. depending on what integer is the first for which the statement is supposed to be true). This is the **Anchor Step**.
2. Assume P(k) is true for some $k \in \mathbb{N}$. This is the **Induction Hypothesis**
3. Prove P(k) $\implies$ P(k+1) using P(k).

This method, though simple, is mathematically rigorous.

Example 3: Prove, by induction, that $11^n - 6$ is divisible by 5 for all $n \in \mathbb{N}$.

Here, P(n) is the statement that '$11^n - 6$ is divisible by 5'.

- **Step 1**: Prove P(1) is true.

P(1) is the statement that '11^1 - 6 is divisible by 5', i.e. that 5 is divisible by 5 (which is obviously true).

- **Step 2**: Assume P(k) is true for some k$\in \mathbb{N}$.

- **Step 3**: Prove P(k) $\implies$ P(k+1).

In Step 2 we assumed P(k) to be true; that is, 11^n - 6 is divisible by 5. Now, we analyse P(k+1) and see whether we can gather any information about it from our assumption.

Our assumption essentially implies that 11^j - 6 = 5m (for some integer m). This implies that 11^k = 5m + 6. Clearly, P(k+1) is written 11^{k+1} - 6. Expanding this gives the following string of equations:

$$11^{k+1} - 6 = (11 * 11^k) - 6 = 11(5m+6) - 6 = 55m + 66 - 6 = 55m + 60 = 5(11m+12). \quad (11)$$

Since 5(11m+12) is clearly divisible by 5, we have proven that P(k) $\implies$ P(k+1) for any k$\in \mathbb{N}$. Hence, we end our proof. ■

Example 4: Use induction to prove that $\sum_{i=1}^{k}$3i = 3+6+9+12+...+3(k-1)+3k = $\frac{3k(k+1)}{2}$.

Note that this new notation $\sum$ represents a repeated sum. Essentially, $\sum_{m=1}^{n} t_m = t_1 + t_2 + ... + t_n$.

Let P(k) be the statement that $\sum_{i=1}^{k}$3i = $\frac{3k(k+1)}{2}$.

- **Step 1**: Prove P(1) is true.
Statement P(1) reads $\sum_{i=1}^{1}$3i = $\frac{3(1+1)}{2}$,and we can see that the LHS is $\sum_{i=1}^{1}$3i = 3(1) = 3. This is true as the LHS is 3 and the RHS is $\frac{3(1+1)}{2}$, which also equals 3.

- **Step 2**: Assume that the statement P(k) is true. That means $\sum_{i=1}^{k}$3i = $\frac{3k(k+1)}{2}$.

- **Step 3**: Prove P(k) $\implies$ P(k+1).

We know $\sum_{i=1}^{k}$3i = $\frac{3k(k+1)}{2}$ is true from step 2.

We now look at P(k+1) =: $\sum_{i=1}^{k+1}$3i = 3+6+9+...+3k+3(k+1). We notice that this is equivalent to

$$\sum_{i=1}^{k} 3i + 3(k+1) = \frac{3k(k+1)}{2} + 3(k+1) = \frac{3k(k+1) + 6(k+1)}{2} = \frac{3(k+1)(k+2)}{2} = \frac{3(k+1)(k+1+1)}{2}. \quad (12)$$

You can see that here we use the induction assumption, i.e. Step 2. By the principle of mathematical induction, we have shown the statement P(n) to be true for all n. ■

Induction involves a fair amount of intuition and 'spotting things', which can be difficult for some to grasp early on. However, as you go through examples this intuition will be built and solidified.
Contradiction and Induction are just two of the several methods of proof you will encounter, but hopefully this works as an introduction!

Questions:

1. Prove by contradiction that for all integers n, if n^3+5 is odd, then n is even. **Video solution link:**
   - Question 1
2. Prove by induction that $2^n > 4n$ for $n \geq 5$. **Video solution link:**

- Question 2

Although there aren't many questions here, they give an idea of what kind of questions you will face this year! Many more examples will be shown and thoroughly explained to you, so don't worry if this is confusing.

4 Inequalities

Most of you will have encountered at least basic inequalities before, so this section will focus on slightly harder topics to prepare you for the year ahead.

4.1 Basic Properties

Here, we will list several standard rules and properties obeyed by inequalities.

1. $a < b$ means 'a is less than b', and $a > b$ means 'a is more than b'. For example, $5 < 6$ and $5 > 2$.
2. $a \leq b$ means 'a is less than or equal to b', and $a \geq b$ means 'a is greater than or equal to b'. For example, $15 \leq 23$ and $4 \geq 4$. Note that a number 'n' is considered to be less/greater than or equal to itself.
3. $c < d$ and $e < f \not\Longrightarrow ce < df$. To show this, we use the example $-1 < 3$ and $-5 < -3$. (-1)(-5) = 5 and (3)(-3) = -9, but 5 $\not\leq$ -9. In fact, $5 \geq -9$. This is just to illustrate how inequalities behave differently to normal equations.
4. Inequalities do not exist for complex numbers! There is actually no notion of 'size' within the Argand plane, only lengths! (see 'Bridging the Gap to Linear Mathematics' for more).

More facts will be listed in MATH1007, but the above represents a few of the simpler ones to get you started!

4.2 Solving Inequalities

Solving inequality equations is usually fairly straightforward, unless a modulus sign |...| is included. You will learn how to tackle those later on, but here you will be given an overview of how solving these types of equations will work.

Example 1: Find x when $7x + 9 \leq 72$.

$$7x + 9 \leq 72 \Longrightarrow 7x \leq 63 \Longrightarrow x \leq 9. \tag{13}$$

Through standard processes, we have reached our answer. It shows that any choice of $x \leq 9$ satisfies the original equation. For instance, choosing $x = 5$ gives 7(5)+9 = 44, which is indeed ≤ 72.

Example 2: Find x when $x + 1 > 5x$ - 2.

$$x + 1 > 5x - 2 \Longrightarrow x + 3 > 5x \Longrightarrow 3 > 4x \Longrightarrow \frac{3}{4} > x. \tag{14}$$

We arrive at our answer! Notice that this means any value of x less than 3/4 will satisfy the original inequality!

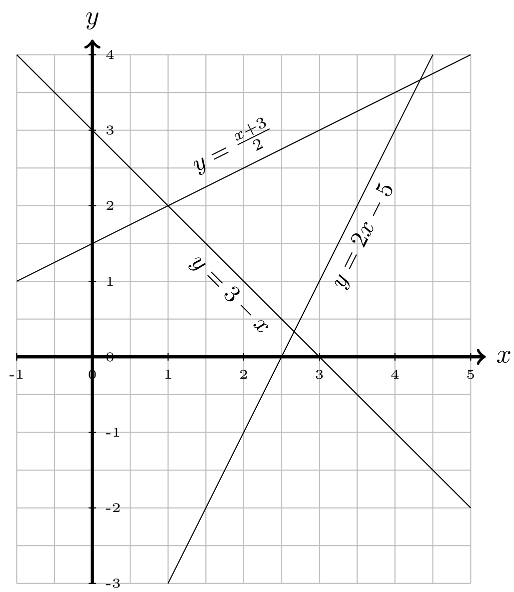
Example 3: Sketch the region that satisfies the inequalities $2y$-$x \leq 3$, $2x$-$y \leq 5$ and $x + y \geq 3$.
First, we temporarily ignore the inequality signs and rearrange the equations to obtain 3 lines of the form $y = ax + b$. We arrive at the following trio:

$$y = \frac{x+3}{2}, y = 2x - 5, y = 3 - x \tag{15}$$

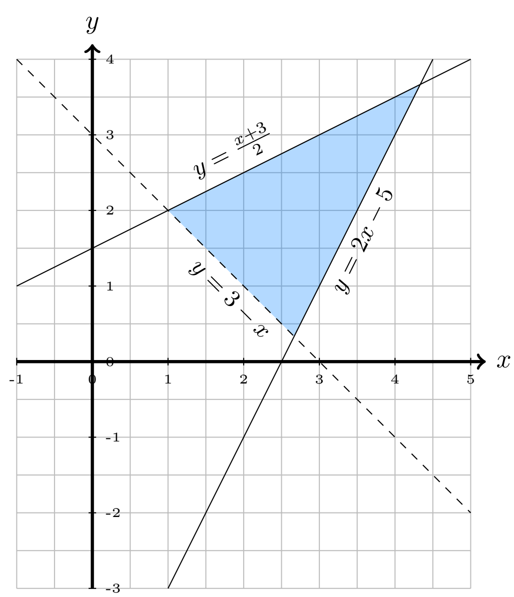
Keep in mind that this is what these lines look like with inequality symbols included:

$$y \leq \frac{x+3}{2},\ y \geq 2x-5 \text{ and } y > 3-x.$$

Plotting (15) on a graph gives the following:



If we then reconsider our inequality signs, we can map out a shaded area, giving:



If you assess the inequalities we calculated, it is clear why the blue shaded area is the overlapping section (try this sketch yourself!). In this triangular area, any choice of x and y will satisfy all 3 inequalities - try some out, it works!

The dotted line is a convection and represents and inequality without an 'or equal to'. In summary, when sketching inequalities:

1. $<$ and $>$ are represented by dashed lines '- - - - - -'.

2. $\leq$ and $\geq$ are represented by solid lines '————'.

Questions

1. Sketch the following inequalities: (see associated answer video for the sketch and explanations).

   (a) $7x - 14y \geq 0$

   (b) $x^2 + 3x - 4 \leq 0$

   (c) $-x^2 + 2x \leq 0$

   Video solution link:

   - All of Question 1

2. Solve the following inequalities:

   (a) $3x \leq 1$

   (b) $x^2 + 5x + 6 \leq 0$

   (c) $x^2 + 5x + 6 \geq 0$

   Video solution link:

   - All of Question 2

5 Solutions to Questions

5.1 Sets and Set Notation

1. (a) (already shown)

   (b) B = $\{x \in \mathbb{R} : x < 5\}$

   (c) B = {prime numbers}

   (d) D = $\{y \in \mathbb{N} : y \leq 80 \ \& \ y \geq 120\}$

2. (these are just example answers so yours might still be correct even if not shown here!)

   (a) 1, 4, 9, 16, 25, ...

   (b) (2,1), (4,2), (6,3), (19,9.5), (0,0), ...

   (c) i, 2 + 5i, 3, 16 - 45i, 0, ...

3. (a) A = { 9, 16, 25, 36, 81, 121 }

   (b) A = { 9, 16, 25, 36, 81 }

   (c) A $\cap$ B = {121}

   (d) A $\cup$ B^C = { 9, 16, 25, 36, 81, 121, 6, 19, 47 }

   (e) B $\cap$ A^C = { 33, 44, 88 }

5.2 Mathematical Reasoning/Logic

1. (a) Case n = 13: 3(13) - 4 = 35 $\notin$ {prime numbers}
   (b) Case $x = 0$: $0^2 \ngeq 0$

2. (a) Contrapositive: If n is even, then n^2 is even.
   Let n be even $\Longrightarrow$ n = 2k for some k $\in \mathbb{Z}$. Then, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even!
   Since the contrapositive is true, so is the original statement! ■

   (b) Contrapositive: If one of m and n is even and the other is odd, then m + n is odd.
   Let n be odd and m be even, i.e. m = 2k and n = 2p + 1 for some k, p $\in \mathbb{Z}$.
   Then m + n = 2k + 2p + 1 = 2(k + p) + 1, which is odd!
   Since the contrapositive is true, so is the original statement! ■

5.3 Methods of Proof

These answers are best shown in the answer videos since they are fairly involved questions.

1. Question 1

2. Question 2

5.4 Inequalities

1. See answer video: Question 1
2. (a) $x \leq \frac{1}{3}$
   (b) -3 $\leq x \leq$ -2
   (c) $x \leq$ -3 & -2 $\leq x$