Bridging the gap to Calculus

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Calculus (MATH1006) is a first year Core module. This document will help to bridge the gap from A level to University Mathematics. In these chapters, we will go over preliminary details in order to prepare you for some of the topics you will face this academic year. There are questions throughout this document with video links explaining the solutions.

Contents

1 Introduction definitions

This first chapter will offer some initial information, including standard definitions and where they apply. This information may not be new to you but will hopefully serve as a useful reminder.

- **Domain of a function**: The **domain** of a function represents the set of all inputs.
- **Range of a function**: The **range** is the set of outputs obtained when every element of the domain is acted on by the function.
- **One-to-Many**: This is where an input can be mapped to more than one output. E.g. $y = \sqrt{x}$ is one-to-many when defined across all real numbers, positive and negative (and zero). This is because $x = 4$ gives us both $y = 2$ and $y = -2$.
- **Many-to-Many**: This is where an input can be mapped to multiple outputs and outputs can be obtained from more than one input.

  A **function** specifically can be:
- **One-to-One**: This is where each input is mapped to a unique output. Note that these are the only kind of functions that can be inverted. (see page 6).
- **Many-to-One**: This is where different inputs can be mapped to the same output. E.g. $y = x^2$, since $x = 3$ and $x = -3$ both give $y = 9$.

  A **Function** is where each input only has one output; i.e. One-to-One or Many-to-One.
- **Independent Variables**: This is the input value for a function, it can be chosen freely and is therefore independent of other values. If the relationship is represented on a graph this value is on the x axis.
- **Dependent Variables**: This is the output value for a function, it is calculated from the input value and is therefore dependent on the input value. If the relationship is represented on a graph this value is on the y axis.

  E.g. Let $y = f(x)$. x is the independent variable and y is the dependent variable (since y depends on x).

2 Hyperbolic functions

2.1 Introduction

Hyperbolic functions are similar to trigonometric functions; they are derived from the exponential definitions and share many of the properties they hold. We define the following hyperbolic functions in terms of the exponential function as follows:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}. \tag{1}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}. \tag{2}$$

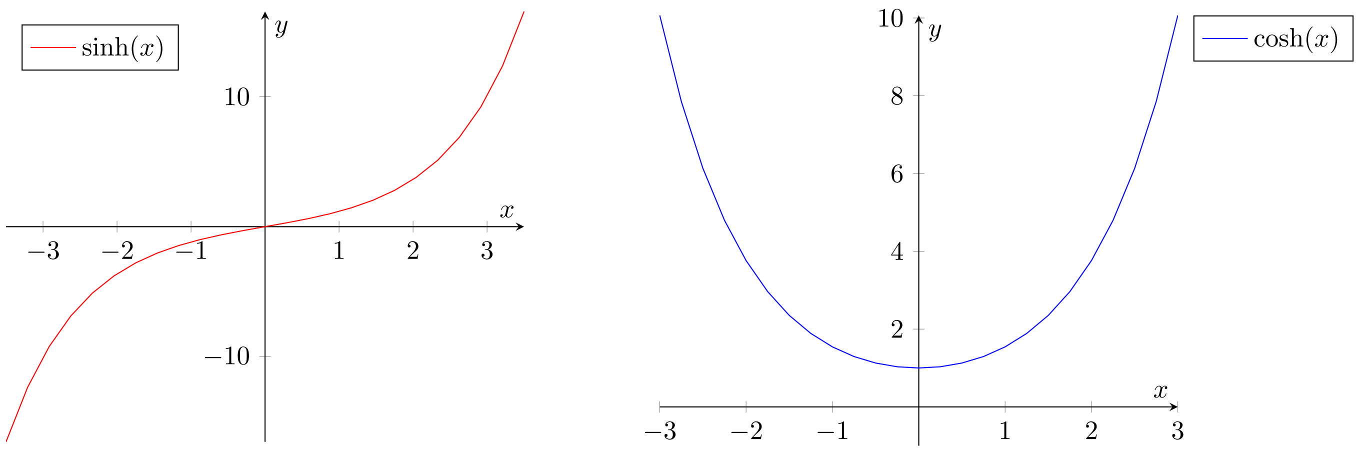


Figure 1: Graphs of $\sinh(x)$ and $\cosh(x)$

Above, we have a sketch of the hyperbolic sine and cosine (sinh and cosh respectively).

Odd and even functions are defined as follows:

Even functions: $f(x) = f(-x)$ (i.e. symmetric about the y-axis)
Odd functions: $f(x) = -f(-x)$ (i.e. has rotational symmetry about the origin)

Note that $y = \cosh(x)$ is an even function passing through (0,1) (like $y = \cos(x)$) and $y = \sinh(x)$ is an odd function passing through (0,0) (like $y = \sin(x)$).

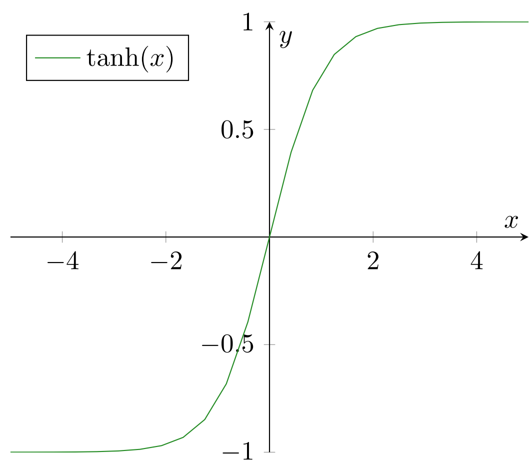
The following identity is useful when working with hyperbolic functions.

$$\cosh^2(x) - \sinh^2(x) = 1 \tag{3}$$

Notice how this is similar to the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$.

We now introduce the hyperbolic tangent function. The definition of $\tanh(x)$ can be derived from $\sinh(x)$ and $\cosh(x)$. (in a similar way that $\tan(x)$ can be derived from $\sin(x)$ and $\cos(x)$).

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



Some other useful hyperbolic functions are listed below:

$f(x)$	Exponential form
$\text{sech}(x) = \frac{1}{\cosh(x)}$	$\frac{2}{e^x+e^{-x}}$
$\text{cosech}(x) = \frac{1}{\sinh(x)}$	$\frac{2}{e^x-e^{-x}}$
$\coth(x) = \frac{1}{\tanh(x)}$	$\frac{e^x+e^{-x}}{e^x-e^{-x}}$

Below are the standard results from differentiating and integrating $\sinh(x)$ and $\cosh(x)$.

Derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh(x) = \cosh(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh(x) = \sinh(x)$$

Integrals:

$$\int \cosh(x)dx = \sinh(x) + C$$
$$\int \sinh(x)dx = \cosh(x) + C$$

It is easy to prove these standard integration and differentiation results from the exponential definitions.

Example 1: Find all real solutions to the equation $10\cosh(x) - 2\sinh(x) = 11$.

We start by substituting the exponential definitions for $\sinh(x)$ and $\cosh(x)$ from (1) and (2).

$$10(\tfrac{e^x+e^{-x}}{2}) - 2(\tfrac{e^x-e^{-x}}{2}) = 11 \implies 5(e^x + e^{-x}) - (e^x - e^{-x}) = 11$$

$$4e^x + 6e^{-x} - 11 = 0$$

We then multiply through by e^x to get:

$$4e^{2x} - 11e^x + 6 = 0$$

Set $y = e^x$ and factorise:

$$4y^2 - 11y + 6 = 0$$
$$(4y - 3)(y - 2) = 0$$
$$\implies y = 3/4 \text{ or } y = 2$$

Finally substitute $y = e^x$ and solve for x:

$$e^x = 3/4 \text{ or } e^x = 2$$
$$\implies x = \ln(\tfrac{3}{4}) \text{ or } x = \ln(2)$$

Questions:

1. Find all the real solutions of these equations.
   (a) $\cosh(x) + 2\sinh(x) = -1$
   Link to solution
   (b) $2\cosh(2x) + 10\sinh(2x) = 5$
2. Using the exponential definitions prove that:
   (a) $\cosh^2(x) - \sinh^2(x) = 1$
   Link to Solution
   (b) $\sinh(2x) = 2\sinh(x)\cosh(x)$
   Link to Solution

3. Differentiate each of the following with respect to x.
   (a) $y = 2\sinh(x)\cosh(2x)$
   Link to Solution
   (b) $(1+x)^3\cosh^3(3x)$
   Link to Solution
4. Integrate each of the following with respect to x.
   (a) $\cosh(2x) - 3\sinh(x)$
   Link to Solution
   (b) $\frac{\sinh(x)}{1+\cosh(x)}$
   Link to Solution

2.2 Inverse hyperbolic functions

Inverse hyperbolic functions are used in integration in similar ways to inverse trigonometric functions.

The cosh function is a many-to-one function, since more than one value of x can yield the same value of y. However if we restrict the domain of $\cosh(x)$ to $[1, \infty)$ then the function is one-to-one, meaning we can define the inverse $\text{arccosh}(x)$. Since the sinh and tanh functions are one-to-one there is no need for any restrictions when defining their inverse functions arcsinh and arctanh.

Function	Domain	Range
$\text{arcsinh}(x)$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\text{arccosh}(x)$	$[1, \infty)$	$[0, \infty)$
$\text{arctanh}(x)$	$(-1, 1)$	$(-\infty, \infty)$

To find the derivatives of inverse hyperbolic functions, we use implicit differentiation as follows:
Noting the condition that $a > 0$, we have

$$\begin{aligned} y = \text{arcsinh}(\frac{x}{a}) &\implies \sinh(y) = \frac{x}{a} \\ \frac{\mathrm{d}}{\mathrm{d}x}\sinh(y) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{a} \\ \cosh(y)\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{a} \qquad (4) \end{aligned}$$

Using the identity (3)

$$\cosh^2(x) - \sinh^2(x) = 1$$

we get:

$$\cosh(y) = \sqrt{1 + \sinh^2(y)} \qquad (5)$$

Thus by substitution of (5) into (4):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a\cosh(y)} = \frac{1}{a\sqrt{1+\sinh^2(y)}} = \frac{1}{\sqrt{a^2 + a^2\sinh^2(y)}} = \frac{1}{\sqrt{a^2+x^2}}.$$

The following standard results can be obtained in the same way, where $a > 0$.

$f(x)$	$f'(x)$
$\text{arcsinh}(\frac{x}{a})$	$\frac{1}{\sqrt{a^2+x^2}}$
$\text{arccosh}(\frac{x}{a})$	$\frac{1}{\sqrt{x^2-a^2}}$
$\text{arctanh}(\frac{x}{a})$	$\frac{a}{a^2-x^2}$

We also get the corresponding integration results for $a > 0$.

$f(x)$	$\int f(x)dx$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin(\frac{x}{a}) + C$, for $(\lvert x\rvert < a)$
$\frac{a}{a^2+x^2}$	$\arctan(\frac{x}{a}) + C$
$\frac{1}{\sqrt{a^2+x^2}}$	$\text{arcsinh}(\frac{x}{a}) + C$
$\frac{1}{\sqrt{x^2-a^2}}$	$\text{arccosh}(\frac{x}{a}) + C$, for $(x > a)$
$\frac{1}{a^2-x^2}$	$(\frac{1}{a})\,\text{arctanh}(\frac{x}{a}) + C$

Questions:

1. Find

   (a) $\int \frac{1}{\sqrt{4+x^2}}dx$
   Link to Solution

   (b) $\int \frac{1}{\sqrt{9-x^2}}dx$
   Link to Solution

   (c) $\int \frac{1}{\sqrt{x^2-4x+8}}dx$
   Link to Solution

   (d) $\int \frac{x^2}{\sqrt{x^6-1}}dx$ (HINT: Substitute $u = x^3$)
   Link to Solution

3 Differential equations

The **derivative** of a function is the rate by which the values of the function change as the independent variable changes. An ordinary differential equation is a relationship between a function and the function's derivatives. The **order** of a differential equation is the highest power derivative of the dependent variable.

Example 1: A first order differential equation

$$\frac{dy}{dx} = xy^2.$$

Example 2: A second order differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = x^2.$$

Below, is an example of a higher order differential equation:

$$\frac{d^{43}y}{dx^{43}} + 4\frac{d^{20}y}{dx^{20}} = 14x.$$

The order of the above differential equation is 43.

3.1 First order differential equations

In this section we will solve first order differential equations using different methods.

3.1.1 Separation of variables

Suppose a first order differential equation can be written in the form:

$$\frac{dy}{dx} = f(x)g(y), \tag{6}$$

then we can use the **separation of variables** method to solve the differential equation, where $f(x)$ depends only on x and $g(y)$ depends only on y.

If $g(y) \neq 0$ then we can **separate** the variables by dividing both sides by $g(y)$ and multiplying both sides by dx.

$$\int \frac{1}{g(y)} dy = \int f(x)dx \tag{7}$$

When you have rearranged to this form, you can integrate both sides.

Example 3: Consider the first order differential equation

$$\frac{dy}{dx} = 3xy.$$

Here $f(x) = 3x$ and $g(y) = y$.

If $y \neq 0$, we can divide by $g(y)$, obtaining

$$\int \frac{1}{y}\, dy = \int 3x\, dx \implies \ln(|y|) = \tfrac{3}{2}x^2 + c \implies |y| = e^{\frac{3}{2}x^2+c} = e^c e^{\frac{3}{2}x^2}$$
$$\implies y = \pm e^c e^{\frac{3}{2}x^2} \implies y = Ae^{\frac{3}{2}x^2}.$$

Where $A = \pm e^c$ is an arbitrary non-vanishing constant.

If $y = 0$, we cannot divide by $g(y)$, but a direct substitution in the original differential equation shows that the constant function $y(x) = 0$ is a solution.
Combining the outcomes of the $y \neq 0$ and $y = 0$ cases, the general solution can be conveniently written as

$$y = Ae^{\frac{3}{2}x^2} \ ,$$

where now A is an arbitrary constant: the solutions with $y \neq 0$ are recovered for $A \neq 0$ and the solution $y = 0$ is recovered for $A = 0$.

3.1.2 Integrating factor

Suppose a first order differential equations can be written in the form:

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{8}$$

Where $P(x)$ and $Q(x)$ are functions of one variable (x). In this case, we cannot use separation of variables. This is because we can't rearrange the equation into the form of (6).
Instead, we use the **integrating factor method**. This is carried out in the following steps:

1. Find the integrating factor, which is $e^{\int P(x)dx}$;
2. Multiply the equation by the integrating factor;
3. Notice the left hand side of the equation is the result of a product rule application (see examples below and videos if confused);
4. Integrate on both sides and solve for y.

Recall the formula for the product rule: $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$.

Also **note** that this method can only be carried out once the differential equation has been rearranged into the form of (8).

Example 4: Consider

$$\frac{dy}{dx} + \frac{y}{x} = \frac{2}{x^3}.$$

$P(x) = \frac{1}{x}$ by direct comparison with (8), so the integrating factor is:

$$e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x.$$

Following step 2, multiply the equation through by the integrating factor (x) to give:

$$x\frac{dy}{dx} + x\frac{y}{x} = x\frac{2}{x^3},$$

$$x\frac{dy}{dx} + y = \frac{2}{x^2}. \tag{9}$$

Notice that the left hand side of (9) looks like the result of a product rule application. We now recall the product rule formula. In this example, $u = x$ (hence $\frac{du}{dx} = 1$) and $v = y$ (hence $\frac{dv}{dx} = \frac{dy}{dx}$).

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x\frac{dy}{dx} + y = \frac{2}{x^2}$$

We integrate

$$\int \frac{\mathrm{d}}{\mathrm{d}x}(xy)\ dx = \int \frac{2}{x^2}\ dx$$

Notice that the left hand side is the integral of a derivative, so they cancel each other out. This gives

$$xy = \int \frac{2}{x^2}\ dx \implies xy = -\frac{2}{x} + C.$$

To give the general solution

$$y = -\frac{2}{x^2} + \frac{C}{x},$$

where $C \in \mathbb{R}$.

Example 5: Consider

$$x\frac{dy}{dx} + 2y = 10x^2.$$

First, we rearrange the differential equation into the form of (8).

$$x\frac{dy}{dx} + 2y = 10x^2 \implies \frac{dy}{dx} + \frac{2}{x}y = 10x.$$

Now it is in the correct form, we find the integrating factor:

$$P(x) = \frac{2}{x} \implies e^{\int \frac{2}{x} dx} = e^{2ln|x|} = x^2.$$

Multiply through by the integrating factor (x^2) to get:

$$x^2\frac{dy}{dx} + 2xy = 10x^3.$$

Looking at the equation we can see that the left hand side looks like the result of product rule application.

$$\frac{d}{dx}(x^2y) = 10x^3,$$

$$x^2y = \int 10x^3\ dx,$$

$$x^2y = \frac{10}{4}x^4 + C.$$

To give the general solution of

$$y = \frac{5}{2}x^2 + \frac{C}{x^2}.$$

3.2 Second Order Differential Equations

We now consider **second order** differential equations that can be written in the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \tag{10}$$

where a, b and c are real valued constants and $a \neq 0$.

3.2.1 Homogeneous equations

If $f(x) = 0$ then the second order differential equation is **homogeneous**.
We start by substituting our trial function $y = Ae^{mx}$ where A and m are undetermined constants and $A \neq 0$ because this gives the trivial solution $y(x) = 0$.

We find the first and second derivatives of our trial function to give us:

$$y = Ae^{mx} \implies \frac{dy}{dx} = Ame^{mx} \implies \frac{d^2y}{dx^2} = Am^2e^{mx}$$

We then substitute these results into (10).

$$Ae^{mx}(am^2 + bm + c) = 0 \implies am^2 + bm + c = 0$$

since $e^{mx} \neq 0$.

This is known as the **auxiliary equation**:

$$am^2 + bm + c = 0. \tag{11}$$

We can solve this like a normal quadratic to find the roots of the equation m_1 and m_2. Looking at the discriminant of the auxiliary equation, we can find which general case we are working with; this then allows us to find the general solution (see below for each case and examples).

<u>**General solution's for each case:**</u>

Case 1 - Real distinct roots: $b^2 - 4ac > 0$
In this case, the auxiliary equation (11) will have two real roots m_1 and m_2.
The general solution is in the form $y = Ae^{m_1x} + Be^{m_2x}$, where A, B $\in \mathbb{R}$.

Case 2 - Repeated roots: $b^2 - 4ac = 0$
In this case, the auxiliary equation will have one repeated root where $m_1 = m_2 = m$.
The general solution is in the form $y = (A + Bx)e^{mx}$, where A, B $\in \mathbb{R}$.

Case 3 - Complex roots:- $b^2 - 4ac < 0$
In this case, the auxiliary equation will have two complex roots m_1 and m_2.
The roots will be a complex conjugate pair in the form $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$ (See the Linear document for more on complex numbers).
The general solution is in the form $y = Ae^{m_1x} + Be^{m_2x}$. This can also be written as $y = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x)$, where A, B $\in \mathbb{R}$.

Example 7: Consider

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0.$$

From this we have the auxiliary equation below as $a = 1$, $b = 5$ and $c = 6$,

$$m^2 + 5m + 6 = 0,$$

$$a = 1, b = 5, c = 6 \implies b^2 - 4ac = 5^2 - 4(1)(6) = 1 > 0.$$

This is **case 1** as $b^2 - 4ac > 0$. The general solution for this case is:

$$y = Ae^{m_1x} + Be^{m_2x}.$$

Now solve the auxiliary equation by factorisation

$$m^2 + 5m + 6 = 0 \implies (m + 3)(m + 2) = 0.$$

This gives us roots

$$m_1 = -3 \text{ and } m_2 = -2,$$

to give the general solution

$$y = Ae^{-3x} + Be^{-2x}.$$

Example 8: Consider

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 9y = 0.$$

We can identify $a = 1$, $b = -4$ and $c = 9$ to give the auxiliary equation:

$$m^2 - 4m + 9 = 0.$$

$$a = 1, b = -4, c = 9 \implies b^2 - 4ac = (-4)^2 - 4(1)(9) = -20 < 0.$$

This is **case 3** as $b^2 - 4ac < 0$. The general solution for this case is:

$$y = Ae^{m_1x} + Be^{m_2x},$$

where we have the roots in the form $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$.

To solve the auxiliary equation, use the quadratic formula or completing the square (remembering these roots will be complex)

$$m^2 - 4m + 9 = 0 \implies (m - 2)^2 = -9 \implies m - 2 = \pm 3i \implies m = 2 \pm 3i$$

$$\implies m_1 = 2 + 3i,\ m_2 = 2 - 3i$$

To give the general solution

$$y = Ae^{(2+3i)x} + Be^{(2-3i)x}$$

Which can also be written in the form

$$y = e^{2x}(A\cos(3x) + B\sin(3x)).$$

3.2.2 Inhomogeneous Equations

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x). \tag{12}$$

If $f(x) \neq 0$ then the second order differential equation is **inhomogeneous**. As $f(x) \neq 0$ solving this differential equation is a bit more involved.
We can solve equations in the same form as (12) using the following steps:

1. Set the right hand side equal to 0 (we now have $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$).
2. Solve the now homogeneous equation using methods from section 3.2.1 to find the general solution. This is called the complementary function denoted y_{CF} (note that this solution alone does not solve the original inhomogeneous equation).
3. Use a trial function to find a solution to equation (12) that doesn't involve any constants of integration. This is called the particular integral denoted y_{PI} (use the table below).
4. The general solution to the second order inhomogeneous differential equation is denoted $y = y_{CF} + y_{PI}$.

Below is a table displaying which trial function to use for each possible $f(x)$ (where k, C, D and E are constants).

$f(x)$	Trial function
k	C
kx	$Cx + D$
kx^2	$Cx^2 + Dx + E$
$k\cos(x)$ or $k\sin(x)$	$C\sin(x) + D\cos(x)$
$k\cosh(x)$ or $k\sinh(x)$	$C\sinh(x) + D\cosh(x)$
e^{kx}	Ce^{kx}

Example 9: Consider

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 3x.$$

In this example we have already solved the homogeneous equation $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$ and hence found the complementary function ($y_{CF} = Ae^{-3x} + Be^{-2x}$). See **Example 7** .

We now need to calculate the **particular integral**. Looking at the right hand side of the inhomogeneous equation we have $f(x) = 3x$, therefore we use the trial function $Cx+D$ (as $f(x)$ is in the form kx from the table). We first work out the derivatives of our trial function; this is so we can substitute into our inhomogeneous equation and solve for our constants C and D.

$$y = Cx + D \implies \frac{dy}{dx} = C \implies \frac{d^2y}{dx^2} = 0.$$

We now need to substitute this into the inhomogeneous differential equation

$$0 + 5(C) + 6(Cx + D) = 3x \implies (6Cx) + (5C + 6D) = 3x.$$

By comparing coefficients, we know that ① $6Cx = 3x$ and ② $5C + 6D = 0$,

$$\therefore C = \frac{1}{2},$$

and substituting the value of a into ②

$$5\left(\tfrac{1}{2}\right) + 6D = 0 \implies D = -\frac{5}{12}.$$

The trial function is $y = Cx + D \implies y_{PI} = \frac{1}{2}x - \frac{5}{12}$. We know the general solution to the inhomogeneous equation is in the form $y = y_{CF} + y_{PI}$:

$$y = Ae^{-3x} + Be^{-2x} + \frac{1}{2}x - \frac{5}{12}.$$

Example 10: Consider

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-3x}.$$

This example is similar to example 7 but demonstrates this special case.

From **Example 7** we have that $y_{CF} = Ae^{-3x} + Be^{-2x}$ is the complementary function for the inhomogeneous equation.

The structure of the trial function should be Ce^{-3x} as $f(x) = e^{-3x}$. However, because this trial function, (Ce^{-3x}), is in the same form as one of the terms of our complementary function (Ae^{-3x}) we can't use this as it will fail - we encourage you to try show that this case does not work. Instead, we have to multiply this trial function by the independent variable x (now our trial function is Cxe^{-3x}). As Cxe^{-3x} is not in the same form as a term in the complementary function we can use this as our trial function.

In the case where this form is also in the complementary function we would multiple by x again and again until there was a form that isn't in the complementary function. Luckily this is not the case on this example so we can just use Cxe^{-3x}.

We work out the first and second order derivatives of our trial function. This is so we can substitute into the inhomogeneous equation and solve for our constant C.

$$y = Cxe^{-3x},$$

$$\frac{dy}{dx} = Ce^{-3x} - 3Cxe^{-3x},$$

$$\frac{d^2y}{dx^2} = -3Ce^{-3x} - 3Ce^{-3x} + 9Cxe^{-3x}.$$

Substitute into the inhomogeneous second order differential equation:

$$-3Ce^{-3x} - 3Ce^{-3x} + 9Cxe^{-3x} + 5(Ce^{-3x} - 3Cxe^{-3x}) + 6(Cxe^{-3x}) = e^{-3x},$$

$$-Ce^{-3x} = e^{-3x} \implies C = -1,$$

$$y_{PI} = -xe^{-3x} \text{ is the particular integral.}$$

Hence the general solution to **Example 10** is

$$y = Ae^{-3x} + Be^{-2x} - xe^{-3x}.$$

In the special case where the trial function is in the same form as a term in the complementary function, follow the below steps in order to find an alternative trial function:

1. Find the trial function
2. Check the trial function is not contained in the complementary function y_{CF}
   (a) If contained, we move to step 3
   (b) If not contained, continue with the method as normal, and find the particular integral
3. Multiply the trial function by the independent variable (typically x) and repeat step 2.

Questions

1. Using the method of separation of variables workout the general solution to the following first order ODE's:
   (a) $\frac{dy}{dx} = \frac{x}{y}$.
   Link to Solution
   (b) $\frac{dy}{dx} = 8xy$.
   Link to Solution
   (c) $3\frac{dy}{dx} = xy^4$.
   Link to Solution

2. Using the integrating factor method, workout the general solution to the following first order ODE's:
   (a) $\frac{dy}{dx} + 3xy = 5$
   Link to Solution
   (b) $\frac{dy}{dx} + 7x^2y = 6$
   Link to Solution
   (c) $3x\frac{dy}{dx} + 6x^2y = 24x$
   Link to Solution

3. Find the general solution to the following first order homogeneous ODE's:
   (a) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$
   Link to Solution
   (b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$
   Link to Solution
   (c) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$
   Link to Solution
   (d) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$
   Link to Solution

4. Find the general solution to the following second order inhomogeneous ODE's:
   (a) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 3x$
   Link to Solution
   (b) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 4e^{-3x}$
   HINT: Refer to the **special case** section and example on page 14
   Link to Solution

4 Taylor series and Maclaurin series

Polynomial functions are easy to work with. Therefore it is preferable to use them as approximations to more complicated functions. Let f be a function that is differentiable infinitely many times at a point a. The derivatives of the function f at a point a can be used to construct a polynomial approximation to the function.

The **Taylor series about a** for $\boldsymbol{f}$ is :

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^3(a)}{3!}(x-a)^3 + ... + \frac{f^n(a)}{n!}(x-a)^n + ... \qquad (13)$$

The point a is called the centre of the Taylor series.

When $a = 0$, the Taylor series becomes

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^3(0)}{3!}x^3 + ... + \frac{f^n(0)}{n!}x^n + ...$$

This is known as the **Maclaurin series** for f.

It is useful to know and recognise the following standard Maclaurin series:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ... \qquad (x \in \mathbb{R})$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - ... \qquad (x \in \mathbb{R})$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - ... \qquad (x \in \mathbb{R})$$

$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + ... \qquad (x \in \mathbb{R})$$

$$\cosh(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x + \frac{1}{8!}x^8 + ... \qquad (x \in \mathbb{R})$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + ... \qquad (x \in (-1,1))$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \qquad (x \in (-1,1))$$

Example 11: Find the first three terms in the Taylor expansion for $f(x) = x\sin(x)$ where $a = \frac{\pi}{2}$.

Calculate

$$f(x) = x\sin(x),$$
$$f'(x) = \sin(x) + x\cos(x),$$
$$f''(x) = \cos(x) + \cos(x) - x\sin(x) = 2\cos(x) - x\sin(x).$$

We now substitute $a = \frac{\pi}{2}$ into (13):

$$f(x) = x\sin(x) = \tfrac{\pi}{2}\sin(\tfrac{\pi}{2}) + \Big(\sin(\tfrac{\pi}{2}) + \tfrac{\pi}{2}\cos(\tfrac{\pi}{2})\Big)(x - \tfrac{\pi}{2}) + \tfrac{2\cos(\frac{\pi}{2}) - \frac{\pi}{2}\sin(\frac{\pi}{2})}{2!}(x - \tfrac{\pi}{2})^2 + ...$$

We now use the fact that $\cos(\frac{\pi}{2}) = 0$ and $\sin(\frac{\pi}{2}) = 1$ to get

$$f(x) = x\sin(x) = \tfrac{\pi}{2} + (x - \tfrac{\pi}{2}) + -\tfrac{\pi}{4}(x - \tfrac{\pi}{2})^2 + ...$$

Questions:

1. Using the Maclaurin expansion show that

   (a) $e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + ...$

   Link to Solution

2. Using the Standard Maclaurin expansions find the first three non zero terms of

   (a) $\sin^2(x)$
   Link to Solution

   (b) $\ln(\frac{1+x}{1-x})$
   Link to Solution

   (c) $e^x\cos(x)$
   Link to Solution

3. Using Taylor series find the first three terms of $\sin(x + \frac{\pi}{6})$ about $a = \frac{\pi}{2}$
   Link to Solution

5 Solutions to Questions:

5.1 Hyperbolic Functions

1. (a) $x = -\ln(3)$
   (b) Real solutions occur when $e^{2x} > 0$ so
   $2x = \ln(\frac{4}{3}) \implies x = \frac{1}{2}\ln(\frac{4}{3})$

2. (a) See video for solution.
   (b) See video for solution.

3. (a) $\frac{dy}{dx} = 4\sinh(x)\sinh(2x) + 2\cosh(x)\cosh(2x)$
   (b) $\frac{dy}{dx} = 3(1+x)^2\cosh^2(3x)[3(1+x)\sinh(3x) + \cosh(3x)]$

4. (a) $\frac{1}{2}\sinh(2x) - 3\cosh(x) + C$
   (b) $\ln(1 + \cosh(x)) + C$

5. (a) $\text{arcsinh}(\frac{x}{2}) + C$
   (b) $\arcsin(\frac{x}{3}) + C$
   (c) Complete the square first
   $\text{arcsinh}(\frac{x-2}{2}) + C$
   (d) $\frac{1}{3}\text{arccosh}(x^3) + C$

5.2 Differential equations

1. (a) $y = \sqrt{x^2 + 2c}$
   (b) $y = Ae^{4x^2}$ where $A = \pm e^c$
   (c) $y = \sqrt[3]{\frac{-2}{x^2+2c}}$

2. (a) $y = \frac{5}{3x} + ce^{\frac{-3}{2}x^2}$
   (b) $y = \frac{6}{7x^2} + ce^{-\frac{7}{3}x^3}$
   (c) $y = \frac{4}{x} + ce^{-x^2}$

3. (a) $y = Ae^{-3x} + Be^{-2x}$
   (b) $y = e^{-x}(A\text{cos}(\sqrt{3}x) + B\text{sin}(\sqrt{3}x))$
   (c) $y = e^{-2x}(A\text{cos}(x) + B\text{sin}(x))$
   (d) $y = (A + Bx)e^{-3x}$

4. (a) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 3x$
   You have already worked out the complementary function to this equation in 3c ($y_{CF} = e^{-2x}(A\text{cos}(x)+ B\text{sin}(x))$). Need to calculate the particular integral ($y_{PI} = \frac{3}{5}x - \frac{12}{25}$)
   $\implies$ y = $\text{e}^{-2x}(A\text{cos}(x) + B\text{sin}(x)) + \frac{3}{5}x - \frac{12}{25}$
   (b) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 4e^{-3x}$
   First need to calculate the **complementary function**
   $a = 1$, $b = 6$ and $c = 9$
   $(6)^2 - 4(1)(9) = 0 \implies$ **case 2**

Solve the auxiliary equation
$m^2 + 6m + 9 = 0 \implies (m+3)^2 = 0 \implies m = -3$
Looking at the general solution for this case $y = (A + Bx)e^{mx}$
$y = (A + Bx)e^{-3x}$. Now need to calculate the particular integral. $f(x) = 4e^{-3x}$ mean the form of the particular integral should be $y = Ce^{-3x}$. However, this form is in the complementary function () therefore multiple by x so the form is now Cxe^{-3x}. However this form is also in the complementary function (). This means we need to multiple by x again Cx^2e^{-3x}. This is not in the complementary function so we can use this. $y = Cx^2e^{-3x}$, $\frac{dy}{dx} = 2Cxe^{-3x} - 3Cx^2e^{-3x}$ and $\frac{d^2y}{dx^2} = 2Ce^{-3x} - 6Cxe^{-3x} + 9Cx^2e^{-3x} - 6Cxe^{-3x}$
$2Ce^{-3x} - 6Cxe^{-3x} + 9Cx^2e^{-3x} - 6Cxe^{-3x} + 6(2Cxe^{-3x} - 3Cx^2e^{-3x}) + 9(Cx^2e^{-3x}) = 4e^{-3x}$
$2Ce^{-3x} = 4e^{-3x} \implies 2C = 4 \implies C = 2$ The particular integral is $y = 2x^2e^{-3}$
The general solution $y = (A + Bx)e^{-3x} + 2x^2e^{-3x}$

5.3 Taylor and Maclaurin series

1. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ...$
2. (a) $x^2 - 3x^4 + \frac{x^6}{36} + ...$
   (b) $2x + \frac{2x^3}{3} + \frac{2x^5}{5} + ...$
   (c) $1 + x - \frac{2}{3}x^3 + ...$
3. $\frac{\sqrt{3}}{2} - \frac{1}{2}(x - \frac{\pi}{2}) - \frac{\sqrt{3}}{4}(x - \frac{\sqrt{3}}{4}(x - \frac{\pi}{2})^2 + ...$