Bridging the Gap to Linear Mathematics

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Linear Mathematics (MATH1007) is a first year Core module. This document will help to bridge the gap from A level to University Mathematics. In these chapters, we will go over preliminary details in order to prepare you for some of the topics you will face this academic year. There are questions throughout this document with video links explaining the solutions.

Contents

1 Complex Numbers

1.1 Real and Imaginary numbers

We define $\sqrt{-1} = i$ where i is called an **imaginary number**. A complex number is written in the form $a + bi$ where a, b are real numbers (see equivalent ACF document for more details). In the equation $a + bi$, a is known as the 'real part' and b the 'imaginary part' (note that 'bi' is not considered as the imaginary part, but simply 'b' by itself).

Additionally, the set of complex numbers is denoted $\mathbb{C}$. All numbers are complex numbers! This is because real numbers (i.e. positives, negatives and zero) have imaginary part b = 0. For instance, the number 5 is complex because we can write it as 5+0i. It represents the biggest set of numbers and contains the reals $\mathbb{R}$.

1.2 Complex conjugation

The **complex conjugate** of a general complex number '$a + bi$' is written as '$a - bi$'. That is, the sign of the imaginary part is flipped from negative to positive and vice-versa.

1.3 Basic Operations

To **add** two complex numbers 'z' and 'w', simply take the real part (denoted Re) of z and add it to the real part of w. Then, take the imaginary part (denoted Im) of z and add it to the imaginary part of w (see below).
Example 1:

$$\text{For } z = \text{a + bi}, w = \text{c + di} : \text{z + w} = (a + c) + (b + d)i \tag{1}$$

$$(7 + 5i) + (2 + 8i) = 9 + 13i \tag{2}$$

To **subtract** two complex numbers, take the same approach as addition, but subtract!
Example 2:

$$(a + bi) - (c + di) = (a - c) + (b - d)i \tag{3}$$

$$(6 + 7i) - (2 + 3i) = 4 + 4i \tag{4}$$

To **multiply** two complex numbers, expand using the FOIL (first, outer, inner, last) method:
Example 3:

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 \tag{5}$$

By remembering $i^2 = -1$, we then find (5) = $(ac - bd) + (ad + bc)i$.

$$(3 + 2i)(4 + i) = 12 + 3i + 8i + 2i^2 = 10 + 11i \tag{6}$$

Division, however, is slightly more involved. We think of division as 'mulitplying by one over a complex number.' In essence, division by complex numbers is equivalent to multiplying the top and bottom by the complex conjugate of the denominator (see below):
Example 4:

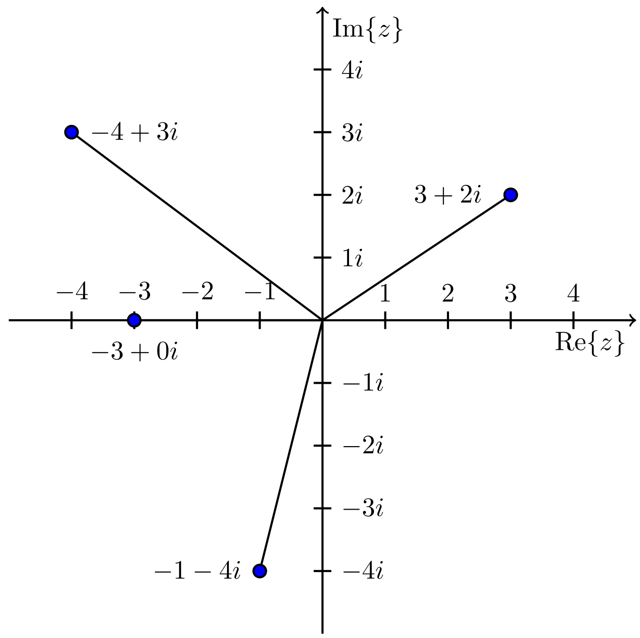
$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2} \tag{7}$$

Note that $c^2 + d^2$ is real. Here is an example of complex division:

$$\frac{5 + 4i}{3 + 2i} = \frac{5 + 4i}{3 + 2i} \cdot \frac{3 - 2i}{3 - 2i} = \frac{23}{13} + \frac{2}{13}i \tag{8}$$

1.4 Argand Diagram

An Argand diagram is a geometric plot of complex numbers. Complex numbers written in the form $x + yi$ are plotted as points in the form (x, y). The x axis is called the 'real axis' and the y axis is called the 'imaginary axis'.



Looking at the figure above, we have plotted the numbers $-4 + 3i$, $3 + 2i$, $-1 - 4i$ and $-3 + 0i$. Note that -3+0i simply has imaginary part '0', so it is purely real (it is **still** referred to as complex).

1.5 Polar form of complex numbers

Polar coordinates are written in the form $x = rcos\theta$, $y = rsin\theta$. See below to understand how these values are derived:

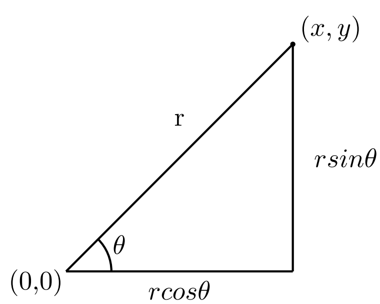


Figure 1: Polar Coordinates

We can then rewrite the comples form $z =$ '$x + yi$' as follows:

$$z = x + yi = rcos\theta + i(rsin\theta) = r(cos\theta + i\sin\theta) \tag{9}$$

r is the absolute value (modulus) of the complex number and θ is the argument of the complex number measured in *radians* (denoted arg).

$$r = |z| = \sqrt{x^2 + y^2} = \sqrt{(rcos\theta)^2 + (rsin\theta)^2} \tag{10}$$

1.5.1 Principal Argument

The **principal argument** of a complex number z, denoted Arg(z), is simply the argument but restricted as such:

$$-\pi < \text{Arg}(z) \leq \pi$$

Calculating the principal argument is usually relatively simple; if you find that your argument θ is outside of the above range, simply add/subtract 2π radians until you are within the range!

Just to make this clearer, arg(z) simply represents a general argument. We may write:

$$\text{arg}(z) = \text{Arg}(z) + 2\text{k}\pi \text{ ; where k is an integer!}$$

1.6 Eulers formula and De Moivre's

1.6.1 Euler's formula derivation

If we look at the Maclaurin expansion of e^x (if Maclaurin has not been covered, refer to the 'Bridging the gap to Calculus' PDF)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ... \tag{11}$$

Now, set $x = i\theta$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + ... \tag{12}$$

NOTE: $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$$e^{i\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...) \tag{13}$$

NOTE: The first bracket is the Maclaurin expansion of $cos\theta$ and the second is the Maclaurin expansion of $sin\theta$

$$e^{i\theta} = cos\theta + isin\theta \tag{14}$$

Now applying this to the general complex number

$$z = r(cos\theta + isin\theta) = re^{i\theta} \tag{15}$$

1.6.2 Introduction to De Moivre's

De Moivre's theorem provides a formula for computing powers of complex numbers. We gain some insight into the derivation of De Moivre's theorem by considering what happens when we multiply a complex number by itself:

$$z^2 = r^2(cos\theta + isin\theta)^2 = r^2(cos^2\theta + 2icos\theta sin\theta - sin^2\theta) = r^2(cos2\theta + isin2\theta) \tag{16}$$

In this example when the complex number is squared, it is the same as squaring the absolute value and multiplying the argument by 2. This can be generalised for higher integer values. If the complex number is raised to the nth power (z^n) this is the same as raising the absolute value to the nth power and multiplying the argument by n. More specifically:

$$z^n = r^n(cos\theta + isin\theta)^n = r^n(cos(n\theta) + isin(n\theta)) \tag{17}$$

This is quite obvious when we look at the alternative written form of a complex number

$$z = re^{i\theta} \tag{18}$$

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(cos(n\theta) + isin(n\theta)) \tag{19}$$

If confused, do not worry! The line above is explained in part by equation (14), but if you still don't understand it, your lecturers will offer more in depth derivations of these formulae!

Questions

1. $a = 5 - 3i$, $b = 2 + 7i$, $c = 4 - 6i$ and $d = 1 + 3i$.
   (a) $a + b$
   Link to solution
   (b) $c - d$
   Link to solution
   (c) $a \times c$
   Link to solution
   (d) $b \div d$
   Link to solution
2. Covert the cartesian form into polar form
   (a) 2+3i
   Link to solution
   (b) 6+8i
   Link to solution
   (c) 7+5i
   Link to solution
3. Convert the polar coordinate form to cartesian form
   (a) $z = \sqrt{3}(cos(\pi) + isin(\pi))$
   Link to solution
   (b) $z = \sqrt{5}(cos(\frac{\pi}{3}) + isin(\frac{\pi}{3}))$
   Link to solution
   (c) $\theta = \frac{\pi}{5}$, $\mid z \mid= 6$
   Link to solution
   (d) $\theta = \frac{\pi}{2}$, $\mid z \mid= 10$
   Link to solution
4. Sketch the points from question 3(a)-(d) on an argand diagram
   Link to solution
5. Using De Moivre's, simplify the below expressions
   (a) $(4(cos(2\theta) + isin(2\theta))^5$
   Link to solution
   (b) $(\sqrt{5}(cos(4\theta) + isin(4\theta))^2$
   Link to solution

(c) $(\sqrt{6}(cos(2\theta) + isin(2\theta))^{-3}$
Link to solution

(d) $(\sqrt{8}(cos(4\theta) + isin(4\theta))^{-7}$
Link to solution

2 Vectors

A **vector** is used to represent a quantity that has both **magnitude** and **direction**. If a quantity has only magnitude then it is known as a **scalar** (this will be briefly explained later in this section). Vectors come as either **row** vectors or **column** vectors (see below for general examples of both):

Row Vector: (a, b, c, d) is a (1 by 4) vector, i.e. 1 row and 4 columns.

Column Vector: $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a (3 by 1) vector, i.e. 3 rows and 1 column.

Note: A vector a is typically denoted $\mathbf{a}$ or $\vec{\mathbf{a}}$ and this is true throughout this section.

2.1 Distances, Unit Vectors/Norms

The distance between two points with position vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is given by

$$\|\mathbf{a} - \mathbf{b}\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

Since $\mathbf{a} - \mathbf{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$ produces another vector, say $\mathbf{c}$, $\|\mathbf{a} - \mathbf{b}\| = \|\mathbf{c}\|$. So, we can view the distance between two vectors as the length of the vector that joins them.

The distance between the origin $(0, 0, 0)$ and a point (a_1, a_2, a_3) is known as the **norm** (also known as **length** or **magnitude**) of a a vector.

The **norm** of a vector $\mathbf{a}$ is defined as

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Example 1:

For a vector $\mathbf{a} = \begin{pmatrix} -4 \\ 1 \\ -7 \end{pmatrix}$ the **norm** $\|\mathbf{a}\| = \sqrt{(4)^2 + (1)^2 + (-9)^2} = \sqrt{16 + 1 + 81} = \sqrt{98}$

2.1.1 Unit vectors

A **unit vectors** is a vector with norm = 1. See below for some examples.

General Case: The unit vector of $\mathbf{a}$, denoted $\hat{\mathbf{a}}$ (said as 'a hat'), is defined as

$$\hat{\mathbf{a}} = \frac{1}{\|\mathbf{a}\|} \times \mathbf{a}$$

Example 2:
$\mathbf{b}$ = (2,4,5) has unit vector $\hat{\mathbf{b}} = \frac{1}{\|\mathbf{b}\|} \times \mathbf{b} = \frac{1}{\sqrt{2^2+4^2+5^2}} \times (2,4,5)$.

We then have

$$\hat{\mathbf{b}} = \tfrac{1}{3\sqrt{5}}(2,4,5) = (\tfrac{2}{3\sqrt{5}}, \tfrac{4}{3\sqrt{5}}, \tfrac{5}{3\sqrt{5}})$$

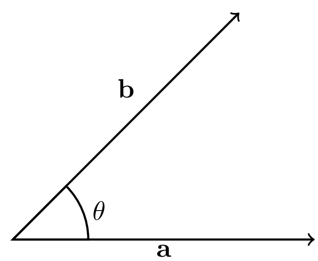
Example 3:
$\mathbf{c}$ = (0,3,4) has unit vector $\hat{\mathbf{c}} = \frac{1}{\|\mathbf{c}\|} \times \mathbf{c} = \frac{1}{\sqrt{0^2+3^2+4^2}} \times (0,3,4)$.

We then have

$$\hat{\mathbf{c}} = \tfrac{1}{5}(0,3,4) = (0, \tfrac{3}{5}, \tfrac{4}{5})$$

2.2 Dot (scalar) product

The **dot product** is an operation between two vectors (here in $\mathbb{R}^3$) where the operation produces a scalar (hence the name). Given two vectors **a** and **b**



the dot product between **a** and **b** is defined as follows:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

This is where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Example 4:

Calculate the dot product between vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix}$ where the angle between the vectors is 1.3453 *radians*.

Begin by calculating the modulus of both vectors:

$$\|\mathbf{a}\| = \sqrt{1^2 + 0^2 + 3^2} = \sqrt{10} \text{ and } \|\mathbf{b}\| = \sqrt{5^2 + 5^2 + 0^2} = \sqrt{50}$$

$$\implies \mathbf{a} \cdot \mathbf{b} = \sqrt{10}\sqrt{50}\cos(1.3453) \approx 5$$

Special cases:

- If $\theta = \pi/2$, the dot product is 0 (since $\cos(\pi/2) = 0$). In this case we say that **a** and **b** are orthogonal/perpendicular vectors.
- If $\theta = 0$, (if the angle between the two vectors is 0), we have $\cos(0) = 1$, meaning $\mathbf{a} \cdot \mathbf{b} = \mid \mathbf{a} \mid\mid \mathbf{b} \mid$.
- The dot product is **commutative**, that is $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

2.2.1 Alternative Dot Product definition

The more commonly used definition of the dot product is as follows

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

No angle is required with this method.

Example 5:

Calculate the dot product between vectors $\mathbf{a} = \begin{pmatrix} 5 \\ 8 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 9 \\ 0 \\ 4 \end{pmatrix}$.

Using $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$\mathbf{a} \cdot \mathbf{b} = 5 \times 9 + 8 \times 0 + 2 \times 4 = 53$$

Here we see that the angle is not required in order to calculate the dot product (this definition is more commonly used).

Example 6:

Calculate the dot product between the two vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ and find the angle between them.

Using $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$\mathbf{a} \cdot \mathbf{b} = 1 \times 3 + 6 \times 4 + 3 \times 2 = 33$$

In order to calculate the angle between the vectors we use $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$

$$\|\mathbf{a}\| = \sqrt{1^2 + 6^2 + 3^2} = \sqrt{46}, \ \|\mathbf{b}\| = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}$$

$$\implies 33 = \sqrt{46}\sqrt{29}\cos\theta \implies \theta = \cos^{-1}\left(\frac{33}{\sqrt{46}\sqrt{29}}\right) \approx 0.4429 \ radians$$

.

2.3 Cross (vector) product

We define the **cross product** as

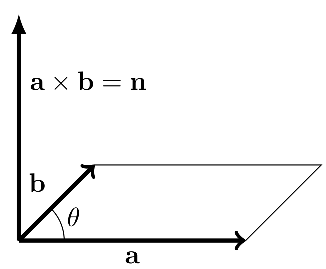
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (b_3 a_2 - b_2 a_3)\mathbf{i} - (b_3 a_1 - b_1 a_3)\mathbf{j} + (b_2 a_1 - b_1 a_2)\mathbf{k}$$

(See the **matrix determinants** section if this is unfamiliar). There are worked examples below if this is still confusing.

We may additionally define the **cross product** as follows:

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta \ \hat{\mathbf{n}}$$

The **cross product** actually generates another vector (hence the name); this vector is always perpendicular to both **a** and **b**. This can be seen in the diagram below.



We have labelled the new vector **n** to denote the term 'normal'. The **normal vector** is a vector perpendicular to a surface/plane (click here or keep reading as there will be more on this later).

We now list some special cases where θ takes certain values.

Special cases

- If $\theta = 0$, the cross product is 0 (since $\sin(0) = 0$). We then say that parallel vectors have a cross product of 0. An example of this is $\mathbf{a} \times \mathbf{a} = 0$.
- If $\theta = \pi/2$ then we get $\sin(\pi/2) = 1$. This means $\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \, \hat{\mathbf{n}}$.
- The cross product is not commutative. In fact, it is anti-commutative, i.e. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

We will also now show a property of the cross product to give you a rough idea of what is to come this year (do not worry if you do not yet fully grasp this; this is just to get you started)!

Example 7:

Let $\mathbf{a} = (2, -4, 4)$ and $\mathbf{b} = (4, 0, 3)$. Find $\mathbf{a} \times \mathbf{b}$.

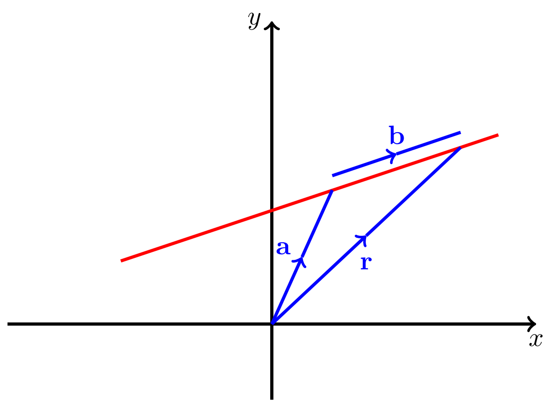
Applying the formula above, which will be derived in lectures at some point throughout your study, we find

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & 0 & 3 \end{vmatrix} = (3 \times (-4) - 0 \times 4)\mathbf{i} - (3 \times 2 - 4 \times 4)\mathbf{j} + (0 \times 2 - 4 \times (-4))\mathbf{k}$$

$\implies \mathbf{a} \times \mathbf{b} = (-12, 10, 16)$.

2.4 Lines in $\mathbb{R}^3$

2.4.1 Equation of lines in $\mathbb{R}^3$



The vector equation of a line is defined by

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Vector $\mathbf{a}$ is called the **position vector**, $\mathbf{b}$ is the **direction vector** and λ is a real constant. Changing the value of λ gives different points on the line.

Example 8:

Let $\mathbf{a} = \begin{pmatrix} 6 \\ 9 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$. Find the equation of the line that passes through both of these points.

Chose one of these points to be the position vector. ($\mathbf{a}=\begin{pmatrix} 6 \\ 9 \\ 2 \end{pmatrix}$ to be position vector). We then find the direction vector between the two points.

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

The equation of the line is given by

$$r = \begin{pmatrix} 6 \\ 9 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

Side note
The parametric equation of a line can be rearranged in order to give the **Cartesian equation** of a line.

$$\frac{x - a_1}{b_1} = \frac{x - a_2}{b_2} = \frac{x - a_3}{b_3} (= \lambda)$$

2.4.2 Intersecting lines

If we are given the equation of two straight lines, we can find the intersection of two straight lines. This is shown by example.

Example 9:

Let $L_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ and let $L_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Calculate where the lines cross each other

We equate x, y and z of L_1 and L_2 as follows:

$5 + \lambda = 2 + \mu \quad (1)$
$2 - 2\lambda = 2\mu \quad (2)$
$-1 - 3\lambda = 4 - \mu \quad (3)$

Note that the above resembles a set of simultaneous equations! We can solve these as follows:

Equation (1) + $\frac{1}{2}$ equation (2) $\implies 6 = 2 + 2\mu\mu = 2$
We then substitute $\mu = 2$ into the first equation to get $\lambda = -1$.

We now substitute in these values of λ and μ into L_1 and L_2.

to get $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$

So the two lines intersect at $(4, 4, 2)$.

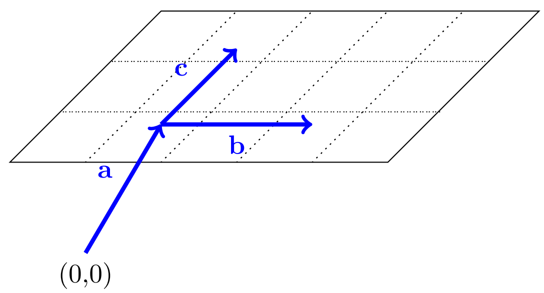
Sometimes lines in $\mathbb{R}^3$ lines do not intersect (for example if lines are parallel). Other questions may ask you to find if they intersect rather than ask where do they intersect.

2.5 Planes in $\mathbb{R}^3$

A **plane** is an infinite 2-dimensional surface in 3-dimensional space ($\mathbb{R}^3$). There are several ways to represent these mathematically and we will show you two different ways in this section!

2.5.1 Parametric Equation of a plane

The general **parametric** equation of a plane is:



$$r = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is the position vector and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ are direction vectors along the plane; that is, vectors that lie on the plane in 3-dimensional space. Also note that λ and μ are real constants and changing their values of gives different points in the plane. We now look at an example.

Example 10:

Given three vectors $\mathbf{a} = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -8 \\ 3 \\ -7 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}$ calculate the parametric equation of the plane containing all three points.

Looking at the general case above, We can assume that one of **a**,**b** or **c** is the position vector. By 'position vector', we mean a vector that takes you from the point (0,0) to a general point in the plane. Any choice is fine, but here we will take **a**.

We now need to find two direction vectors, as listed above. We can do this (generally) by finding the vectors that go from **a** to **b** and from **a** to **c**, calculated as follows:

$$\mathbf{b}-\mathbf{a}=\begin{pmatrix}-8\\3\\-7\end{pmatrix}-\begin{pmatrix}-5\\4\\1\end{pmatrix}=\begin{pmatrix}-3\\-1\\-8\end{pmatrix}$$

$$\mathbf{c}-\mathbf{a}=\begin{pmatrix}4\\6\\-3\end{pmatrix}-\begin{pmatrix}-5\\4\\1\end{pmatrix}=\begin{pmatrix}9\\2\\-4\end{pmatrix}$$

Then, substituting this into our parametric equation formula gives the equation of the plane to be:

$$r=\begin{pmatrix}-5\\4\\1\end{pmatrix}+\lambda\begin{pmatrix}-3\\-1\\-8\end{pmatrix}+\mu\begin{pmatrix}9\\2\\-4\end{pmatrix}$$

2.5.2 Cartesian Equation of a plane

The **Cartesian** equation of a plane is

$$ax+by+cz=d \tag{20}$$

where (a,b,c) represents the vector normal to the plane. The normal $\mathbf{n}$ is perpendicular to any vector in the plane.

Alternatively, the **Vector** equation of a plane is given by:

$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{a})=0.$$

Where '**a**' is a specific, known point on the plane and '**r**' is another general point on the plane. More precisely, $\mathbf{a}=(a,b,c)$ and $\mathbf{r}=(x,y,z)$.

Adding/subtracting two vectors that lie on a general plane 'P' produces another vector in P; for example, the addition/subtraction of any two vectors in the x-y plane produces another vector in the x-y plane! **Note** that this is true for every plane!

Hence, $(\mathbf{r}-\mathbf{a})$ remains on the plane. This makes it perpendicular to $\mathbf{n}$, since $\mathbf{n}$ is perpendicular to every point on the plane. By this, and recalling dot product properties, we know their dot product is 0.

If we now again look at

$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{a})=0,$$

we may expand out the bracket and add on each side, producing the following:

$$\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}.$$

The above line is equivalent to

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}$$

and if we expand the dot products, we find

$$\mathbf{a}\cdot\mathbf{n}=aa_1+ba_2+ca_3 \implies \mathbf{a}\cdot\mathbf{n}=d$$

We note that the right-hand-side $(aa_1+ba_2+ca_3)$ is a scalar; this is equal to d. We also note that this essentially gives the Cartesian equation of the plane (20). This is just a little manipulation to demonstrate that the **Cartesian** and **Vector** equations of planes are completely equivalent and interchangeable.

From section 2.3 we have found out how to calculate the normal of two vectors. This means if we have two direction vectors in the plane, we can use the cross product to find the normal vector. In order to find the value of d, substitute the coordinates of any point in the plane into the equation.

Example 11:

Two direction vectors in a plane are $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$. Find the cartesian equation of the plane.

Answer

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}. = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 5 \\ 1 & 7 & 3 \end{vmatrix} = (12-35)\mathbf{i} + (5-6)\mathbf{j} + (14-4)\mathbf{k} = -23\mathbf{i} - \mathbf{j} + 10\mathbf{k} = \begin{pmatrix} -23 \\ -1 \\ 10 \end{pmatrix}$$

This is the normal vector to the plane.

So we can substitute this normal vector into $ax + by + cz + d = 0$
$\therefore -23x - y + 10z + d = 0$.

We know that $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$ is a vector in the plane therefore we can substitue this into the equation.

$$-23(2) - (4) + 10(5) + d = 0 \implies d = 0$$

$$\implies -23x - y + 10z = 0$$

Questions

1. Let $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ -8 \\ 1 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 0 \\ -7 \\ 3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 9 \\ 5 \\ 2 \end{pmatrix}$.

   Calculate the following:

   (a) $\mathbf{c} \cdot \mathbf{b}$
   Link to Solution

   (b) $\mathbf{a} \cdot \mathbf{d}$

   (c) $\mathbf{d} \cdot \mathbf{a}$

   (d) what do you notice about the results in b) and c) and why?
   Link to solution to b) c) d)

   Calculate the following cross products:

   (e) $\mathbf{c} \times \mathbf{a}$
   Link to solution

   (f) $\mathbf{a} \times \mathbf{b}$

   (g) $\mathbf{d} \times \mathbf{b}$

   (h) $\mathbf{a} \times \mathbf{a}$
   Link to solution

   (i) what do you notice about h) and why?

2. Calculate the angle between $\mathbf{x} = \begin{pmatrix} 6 \\ 7 \\ 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$

   Link to Solution

3. Given points A=(1,1,1), B=(-1,1,0) and C=(2,0,3)

   (a) show that $\mathbf{n} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ is the normal to the plane containing A,B and C.
   Link to Solution

   (b) Find the Cartesian equation of the plane.
   Link to Solution

4. Do the lines $L_1 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $L_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ cross?

   Link to Solution

3 Matrix Algebra

3.1 Introduction

A **matrix** is a mathematical array that holds information, be it numbers, symbols, expressions etc. A general (n by m) matrix, denoted in bold capital letters **A**, is shown below:

$$\mathbf{A}_{n,m} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix} \tag{21}$$

Where each lower case 'a' represents an element, and the subscript represents its position in the matrix **A**. Below are several examples, each with their own explanation. This should help to bolster your understanding of the concept of a matrix.

Example 1:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{22}$$

The matrix **A** is (2 by 2) and contains the first four letters of the English alphabet.

Example 2:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Here, a new matrix **B** is (3 by 3) and contains a collection of 0s and 1s. This matrix is extremely important, and is called the identity matrix, denoted $\mathbb{I}$. It can be of any dimension (provided no. rows = no. columns) and it characterised by the leading diagonal having entries of 1, and all other entries being 0.

The two examples above are (n by n); that is, n rows by n columns. Note that despite this, a matrix can have any number of rows and columns, as long as they are integers of course. Below is one final example, but this time we have an (n by m) matrix.

Example 3:

$$\mathbf{C} = \begin{bmatrix} 4 & 8 & 3 & 1 \\ 1 & 3 & 2 & 4 \\ 5 & 2 & 0 & 7 \end{bmatrix} \tag{24}$$

Here, n = 3 and m = 4.

3.2 Matrix Addition and Subtraction

Adding and **subtracting** matrices is fairly straightforward. It is best displayed using an example.

Example 4:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & 5 \\ 7 & 6 & 1 \end{bmatrix} \tag{25}$$

Then,

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 1+4 & 0+2 & 1+1 \\ 0+0 & 3+2 & 1+5 \\ 2+7 & 1+6 & 0+1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 0 & 5 & 6 \\ 9 & 7 & 1 \end{bmatrix} \tag{26}$$

Subtraction is similar! In this case,

$$\mathbf{A}-\mathbf{B} = \begin{bmatrix} -3 & -2 & 0 \\ 0 & 1 & -4 \\ -5 & -5 & -1 \end{bmatrix} \tag{27}$$

3.3 Determinants

The **determinant** (denoted *det* of a matrix is a scalar value that is a function of the entries in the matrix. Note that, for a general matrix $\mathbf{M}$, the determinant is also denoted as $\|\mathbf{M}\|$.

3.3.1 2 by 2 Determinants

In order to calculate the determinant (of a (2 by 2) matrix, such as $\mathbf{E} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we use a simple formula:

$$det(\mathbf{E}) = ad - bc \tag{28}$$

3.3.2 3 by 3 Determinants

For matrix $\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, the determinant is calculated using the formula:

$$det(\mathbf{M}) = a * det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b * det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c * det \begin{bmatrix} d & e \\ g & h \end{bmatrix} \tag{29}$$

Your Linear Mathematics module will go into more detail as to why this formula arises, but in essence you expand along the top row and take the determinants of the (2 by 2) matrices across the bottom two rows (see formula above). Note that the smaller matrices are called 'minors', but this will be covered in MATH1007.

3.4 Matrix Multiplication

Multiplying and **dividing** matrices go hand-in-hand. Firstly, we will tackle multiplication.

Unlike the determinant, matrix multiplication is essentially the same for all possible pairs $\mathbf{A}$ and $\mathbf{B}$, except with more steps for higher dimension matrices.

A Condition and its Consequence (for general matrices $\mathbf{A}$ and $\mathbf{B}$):

For matrix multiplication to be possible, the number of columns in $\mathbf{A}$ must equal the number of rows in $\mathbf{B}$ (in this booklet, these matrices will be referred to as 'compatible').

As a consequence, the new matrix produced by multiplying two 'compatible' $\mathbf{A}$ and $\mathbf{B}$ will have the same number of rows as $\mathbf{A}$ and the same number of columns as $\mathbf{B}$.
In other words, let two matrices $\mathbf{L}$ and $\mathbf{P}$ be (n by m) and (i by j) respectively. If we wanted to perform the multiplication $\mathbf{LP}$, we need to check that m = i:
- If m $\neq$ i, $\mathbf{LP}$ is not a possible multiplication.
- If m = i, $\mathbf{LP}$ produces an (n by j) matrix.

Method
As for the method, matrix multiplication is performed by multiplying each row with each column (see example below):

Example 5:

$$\mathbf{A} = \begin{bmatrix} 4 & 6 & 2 \\ 3 & 9 & 7 \\ 6 & 1 & 6 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 7 & 1 \\ 4 & 2 & 1 \\ 3 & 6 & 6 \end{bmatrix} \tag{30}$$

Then, **AB** =

$$\begin{bmatrix} 4*3+6*4+2*3 & 4*7+6*2+2*6 & 4*1+6*1+2*6 \\ 3*3+9+4+7*3 & 3*7+9*2+7*6 & 3*1+9*1+7*6 \\ 6*3+1*4+6*3 & 6*7+1*2+6*6 & 6*1+1*1+6*6 \end{bmatrix} \tag{31}$$

i.e.

$$\mathbf{AB} = \begin{bmatrix} 42 & 52 & 22 \\ 66 & 81 & 54 \\ 40 & 80 & 43 \end{bmatrix} \tag{32}$$

3.4.1 Some Properties

1. **Non-Commutative**: $\mathbf{AB} \neq \mathbf{BA}$ (in general)
2. **Distributivity**: $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

More properties will be discussed in your Linear Mathematics module, but these two will provide you with an introduction.

3.5 Matrix Inverse

Calculating **matrix inverses** can be fairly involved. Here, you will be shown how to find the inverse of a (2 by 2) matrix. The method is as follows:

1. Calculate the determinant (recall (13) 'ad-bc')
2. Find the adjugate matrix
3. Multiply the adjugate matrix by 1/(ad-bc)

The **adjugate matrix** (denoted 'adj') is defined as the 'transpose of the cofactor matrix' (Linear and UG). More simply,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{33}$$

Matrix $\longrightarrow$ Adjugate

This is essentially a transformation of the entries in the original matrix. To shore up your understanding, an example of a (2 by 2) inversion is shown below:
Example 6:

$$\text{If } \mathbf{A} = \begin{bmatrix} 4 & 5 \\ 3 & 9 \end{bmatrix}, \text{ then } \mathbf{A}^{-1} = \frac{1}{det(\mathbf{A})} \begin{bmatrix} 9 & -5 \\ -3 & 4 \end{bmatrix}, \tag{34}$$

where det(A) = ad - bc, giving

$$\mathbf{A}^{-1} = \frac{1}{21} \begin{bmatrix} 9 & -5 \\ -3 & 4 \end{bmatrix} \tag{35}$$

This new matrix is the matrix inversion of **A** and has several properties, one of which will be discussed now. Similar to how multiplying a real number with its inverse always produces 1, multiplying a matrix with its inverse always produces the 'Identity matrix' as previously discussed.

i.e. $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbb{I}$

NOTE: A matrix is called 'non-invertible' if its determinant is equal to 0, since division by 0 is impossible/undefined.

3.6 Matrix Division

Now we have looked at the concept of a matrix inverse, we can assess how matrix division works! For matrices **A** and **B**, the division **A/B** is practically carried out by multiplying **A** with $\mathbf{B}^{-1}$.

In essence, $\mathbf{A}/\mathbf{B} = \mathbf{A}\mathbf{B}^{-1}$. We can now look at some examples!

Example 7: Calculate **A/B** where $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

We need to find the inverse of matrix **B**, which can be done using the method learnt in the previous section (see 2.5).

$$\mathbf{B}^{-1} = \frac{1}{det(\mathbf{B})} * \text{adj}(\mathbf{B})$$

$$\implies \mathbf{B}^{-1} = \frac{1}{2-12} \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -1/10 & 3/10 \\ 2/5 & -1/5 \end{bmatrix}$$

Then, we can carry out matrix multiplication as normal!

$$\mathbf{A}\mathbf{B}^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1/10 & 3/10 \\ 2/5 & -1/5 \end{bmatrix} = \begin{bmatrix} -3/10 + 2/5 & 9/10 - 1/5 \\ -2/10 + 8/5 & 6/10 - 4/5 \end{bmatrix} = \begin{bmatrix} 1/10 & 7/10 \\ 7/5 & -1/5 \end{bmatrix} \tag{36}$$

Example 8: Calculate **C/D** where $\mathbf{C} = \begin{bmatrix} 19 & 23 \\ 12 & 4 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 42 & 51 \\ 42 & 51 \end{bmatrix}$.

As before, we must find the inverse of **D**.

$$\implies \mathbf{D}^{-1} = \frac{1}{2142-2142} \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$$

This is undefined, since division by 0 is impossible. Hence, the inverse of **D** does not exist and division by **D** is not possible!

3.7 Eigenvalues and Eigenvectors

Note that for this subsection, **A** must be a square/(n by n) matrix. Also keep in mind that this can and will be applied to a (2 by 2) matrix as well; (3 by 3) is generally more difficult, so we thought we'd help you out!

Informal Definition: A non-zero column vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is an eigenvector of the matrix **A** with corresponding eigenvalue λ iff

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{37}$$

Solving this type of equation is difficult in that you cannot 'divide' by the vector **v** on both sides; this would leave a matrix being equal to a scalar, which is invalid. Instead, we must use an entirely new method.

Consider this: we need to 'convert' equation (37) into a solvable format. We can actually think about this equation in a different way. Recall the identity matrix $\mathbb{I}$, and that multiplying it with a matrix **A** or column vector **v** simply returns **A** or **v** respectively.

We then rewrite (37) as

$$\mathbf{A}\mathbf{v} = \lambda\mathbb{I}\mathbf{v}. \tag{38}$$

Both (37) and (38) mean the exact same thing, but (38) is a bit easier to manipulate. We can subtract the RHS on both sides and factor out **v**, leaving

$$(\mathbf{A} \text{ - } \lambda\mathbb{I})\ \mathbf{v} = \mathbf{0}$$

Here, **0** is the vector of entries 0 (the 'zero vector'). Since by assumption (see the **informal definition**) the **v** vector is non-zero, we can attempt to solve the equation for λ. We have (written out in full):

$$\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0} \Longrightarrow \begin{bmatrix} a_{1,1} - \lambda & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} - \lambda & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} - \lambda \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0} \tag{39}$$

We then take the determinant of the LHS! This temporarily eliminates **v** from our calculations; det(**v**) is non-zero, leaving det(**A** - $\lambda\mathbb{I}$) = det(**0**) = 0.

We have

$$(a_{1,1} - \lambda) * det \begin{bmatrix} a_{2,2} - \lambda & a_{2,3} \\ a_{3,2} & a_{3,3} - \lambda \end{bmatrix} - a_{1,2} * det \begin{bmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} - \lambda \end{bmatrix} + a_{1,3} * det \begin{bmatrix} a_{2,1} & a_{2,2} - \lambda \\ a_{3,1} & a_{3,2} \end{bmatrix} = 0$$

Note 0 is now a scalar!

After solving this, we obtain our eigenvalues λ_1, λ_2 and λ_3. We then substitute these back into our equation (**A** - $\lambda\mathbb{I}$) **v** = **0**, giving a set of simultaneous equations. These will allow us, by inspection, to find each eigenvalues corresponding eigenvector $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

We will now show a couples of examples, one (2 by 2) and the other (3 by 3), to shore up your understanding.

Example 9: Find the eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

We may write (**A** - $\lambda\mathbb{I}$) **v** = **0** as before; this is the step you will typically start on when solving these problems. We now have

$$\begin{bmatrix} -\lambda & 1 \\ -2 & -3 - \lambda \end{bmatrix} \mathbf{v} = \mathbf{0}$$

Now, we may assume: $\det \begin{bmatrix} -\lambda & 1 \\ -2 & -3 - \lambda \end{bmatrix} = 0$. Working through the determinant gives the equation

$$\lambda^2 + 3\lambda + 2 = 0 \tag{40}$$

We may use standard methods to solve this, leaving the eigenvalues λ_1 = -2 and λ_2 = -1. We may now substitute these two values back in.

- First, we assess λ_1 = -2: we have $\begin{bmatrix} -(-2) & 1 \\ -2 & -3 - (-2) \end{bmatrix} \mathbf{v}_1 = \mathbf{0}$.

The $\mathbf{v}_1$ here represents the eigenvector associated with eigenvalue -2. Then we may rewrite the above equation more explicitly as

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{41}$$

We can now multiply out the LHS, leaving

$$\begin{bmatrix} 2a + b \\ -2a - b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can now solve either the top line or the bottom line as a standard equation! It doesn't actually matter which one you choose, since they will output the same eigenvector! In this case we arbitrarily choose the top line.

We then have 2a+b $= 0 \implies$ b = -2a. a and b are arbitrary values: we are working out a vector that points in a direction, so naturally any choice of b and a satisfying b = -2a lies on that vector! Therefore, if we choose a = 1, we have b = -2.

We may then conclude that λ_1 = -2 has eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. A similar process can be carried out to find the eigenvector for λ_2 = -1, giving $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. In essence, for any k,t$\in \mathbb{R}$, λ_1 and λ_2 have eigenvectors k$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and t$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively. We have simply chosen the simplest one in this case, but any choice works: the most important part of your answer is the ratio between each entry in the eigenvector.

Example 10: For $\mathbf{B} = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$, carry out the process to find the correct eigenvalues and eigenvectors shown below:

$\lambda_1 = 3$ with $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1.5 \\ -0.5 \end{bmatrix}$, λ_2 = -5 with $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix}$ and finally $\lambda_3 = 6$ with $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 6 \\ 16 \end{bmatrix}$.

There will be a video detailing how to calculate this in case you get stuck, so don't be afraid to try the question yourself!

Questions

1. Find the **determinants** of the following matrices:

   (a) $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & -2 \end{bmatrix}$

   (b) $\mathbf{B} = \begin{bmatrix} 6 & 13 \\ 3 & 7 \end{bmatrix}$

   (c) $\mathbf{C} = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & 6 \\ 7 & 9 & 1 \end{bmatrix}$

   (d) $\mathbf{D} = \begin{bmatrix} 1 & -5 & 7 \\ 9 & 4 & 0 \\ -8 & -13 & 10 \end{bmatrix}$

   Video solution links):

   - 1(a) and 1(b)
   - 1(c) and 1(d)

2. Calculate, if possible, the following **multiplications**:

   (a) $\mathbf{AB}$ where $\mathbf{A} = \begin{bmatrix} 2 & 8 & 9 \\ 7 & 3 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 1 & 2 \\ 7 & 6 & 0 \\ 8 & 3 & 4 \end{bmatrix}$.

   (b) $\mathbf{BA}$ for the same matrices as in part (a).

   (c) $\mathbf{CD}$ where $\mathbf{C} = \begin{bmatrix} 7 & 6 & 3 \\ 2 & 4 & 9 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 2 & 9 & 1 \\ 6 & 5 & 3 \end{bmatrix}$

   (d) $\mathbf{EF}$ where $\mathbf{E} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 4 & 3 & 0 \end{bmatrix}$ and $\mathbf{F} = \begin{bmatrix} 1 & 7 & 4 \\ 1 & 1 & 3 \\ 3 & 4 & 3 \end{bmatrix}$.

   (e) $\mathbf{FE}$ for the same matrices as in part (d). Are they different? If so, why?

   Video solution link:

   - All of Question 2

3. Calculate, if possible, the **inverses** of the following matrices:

   (a) $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$

   (b) $\mathbf{B} = \begin{bmatrix} 15 & 42 \\ 71 & 143 \end{bmatrix}$

   (c) $\mathbf{C} = \begin{bmatrix} 8 & 10 \\ 4 & 5 \end{bmatrix}$

   Video solution link:

   - All of Question 3

4. Calculate, if possible, the following **divisions**:

(a) $\mathbf{A}/\mathbf{B}$ where $\mathbf{A} = \begin{bmatrix} 8 & 7 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 4 \\ 3 & 6 \end{bmatrix}$

(b) $\mathbf{C}/\mathbf{D}$ where $\mathbf{C} = \begin{bmatrix} 2 & 3 \\ 1 & 9 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 9 & 3 \\ 18 & 6 \end{bmatrix}$

Video solution link:

- All of Question 4

5. Find the **eigenvalues** and **eigenvectors** of the following matrices:

(a) $\mathbf{M} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

(b) The matrix **B** from **Example 10** (despite the answer already being provided, this serves as really good practice).

Video solution links:

- Question 5(a)
- Question 5(b)

4 Solutions to Questions:

4.1 Complex numbers

1. (a) $(5-3i)+(2+7i)=7+4i$

   (b) $(4-6i)-(1+3i)=3-9i$

   (c) $(5-3i)(4-6i)=20-30i-12i-18=2-42i$

   (d) $\frac{2+7i}{1+3i}=\frac{(2+7i)(1-3i)}{(1+3i)(1-3i)}=\frac{2-6i+7i+21}{1+9}=\frac{23+i}{10}$

2. (a) $\mid z\mid=\sqrt{2^2+3^2}=\sqrt{13}$
   $\theta=tan^{-1}(\frac{3}{2})=0.9827937232$
   $z=\sqrt{13}(cos(0.9827937232)+isin(0.9827937232)$

   (b) $\mid z\mid=\sqrt{6^2+8^2}=\sqrt{100}=10$
   $\theta=tan^{-1}(\frac{8}{6})=0.927295218$
   $z=10(cos(0.927295218)+isin(0.927295218)$

   (c) 7+5i $\mid z\mid=\sqrt{7^2+5^2}=\sqrt{74}$
   $\theta=tan^{-1}(\frac{5}{7})=0.620249486$
   $z=\sqrt{74}(cos(0.620249486)+isin(0.620249486)$

3. (a) $z=\sqrt{3}(-1+0)=-\sqrt{3}$

   (b) $z=\sqrt{5}(\frac{1}{2}+\frac{\sqrt{3}}{2}i)=\frac{\sqrt{5}}{2}+\frac{\sqrt{15}}{2}i$

   (c) $z=6(cos(\frac{\pi}{5})+isin(\frac{\pi}{5}))=4.854101966+3.526711514i$

   (d) $z=10(cos(\frac{\pi}{2}+isin(\frac{\pi}{2}))=0+10i$

4. Refer to videos

5. (a) $(4^5(cos(10\theta)+isin(10\theta))=1024(cos(10\theta)+isin(10\theta))$

   (b) $\sqrt{5}^2(cos(8\theta)+isin(8\theta))=5(cos(8\theta)+isin(8\theta)$

   (c) $6^{\frac{-3}{2}}(cos(-6\theta)+isin(-6\theta))=6^{\frac{-3}{2}}(cos(6\theta)-isin(6\theta))$

   (d) $8^{\frac{-7}{2}}(cos(-28\theta)+isin(-28\theta))=8^{\frac{-7}{2}}(cos(28\theta)-isin(28\theta))$

4.2 Vectors

1. (a) $\mathbf{c}\cdot\mathbf{b}=(0)(5)+(-8)(-7)+(3)(1)=59$

   (b) $\mathbf{a}\cdot\mathbf{d}=(2)(9)+(3)(5)+(-7)(2)=19$

   (c) $\mathbf{d}\cdot\mathbf{a}=(9)(2)+(5)(3)+(-7)(2)=19$

   (d) Answer b) and c) are the same, this is because the dot product is commutative.
   Calculate the following cross products:

   (e) $\mathbf{c}\times\mathbf{a}=\begin{pmatrix}40\\6\\14\end{pmatrix}$

   (f) $\mathbf{a}\times\mathbf{b}=\begin{pmatrix}-53\\-37\\-31\end{pmatrix}$

   (g) $\mathbf{d}\times\mathbf{b}=\begin{pmatrix}21\\1\\-97\end{pmatrix}$

(h) $\mathbf{a} \times \mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(i) result h) is the zero vector because $\theta = 0 \therefore sin(0) = 0$

2. Calculate the angle between $\mathbf{x} = \begin{pmatrix} 6 \\ 7 \\ 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$

   52.4°

3. Given points A=(1,1,1), B=(-1,1,0) and C=(2,0,3)

   (a) Find the $\vec{\mathbf{AB}}$ and $\vec{\mathbf{AC}}$

   $$\mathbf{AB} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix},$$
   $$\mathbf{AC} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

   These two vectors are in the plane with points A, B and C. The cross product of two vectors produces the vector normal to the two vectors. $\begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ hence shown

   (b) Find the equation of the plane.
   $-x + 3y + 2z - 4 = 0.$

4. Do the lines $L_1 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $L_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ cross?

   Lines do not cross.

4.3 Matrices

1. (a) $det(\mathbf{A})$ = -14

   (b) $det(\mathbf{B})$ = 3

   (c) $det(\mathbf{C})$ = -93

   (d) $det(\mathbf{D})$ = -105

2. (a) $\mathbf{AB} = \begin{bmatrix} 136 & 77 & 40 \\ 97 & 43 & 38 \end{bmatrix}$

   (b) $\mathbf{BA}$ = N/A

   (c) $\mathbf{CD}$ = N/A

   (d) $\mathbf{EF} = \begin{bmatrix} 8 & 21 & 20 \\ 4 & 11 & 7 \\ 7 & 31 & 25 \end{bmatrix}$

(e) $\mathbf{FE} = \begin{bmatrix} 25 & 15 & 8 \\ 15 & 12 & 2 \\ 22 & 18 & 7 \end{bmatrix}$

3. (a) $\mathbf{A}^{-1} = \begin{bmatrix} 4/7 & -1/7 \\ -5/7 & 3/7 \end{bmatrix}$

   (b) $\mathbf{B}^{-1} = \frac{-1}{837} \begin{bmatrix} 143 & -42 \\ -71 & 15 \end{bmatrix}$

   (c) $\mathbf{C}^{-1}$ = N/A

4. Calculate, if possible, the following **divisions**:

   (a) $\mathbf{A/B} = \begin{bmatrix} 1/3 & -2/9 \\ -1/6 & 5/18 \end{bmatrix}$

   (b) $\mathbf{C/D}$ = N/A

5. (a) $\lambda_1 = 2$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

   $\lambda_2 = 2 + \sqrt{2}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix}$

   $\lambda_3 = 2$ - $\sqrt{2}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ -1 \end{pmatrix}$

   (b) The matrix **B** from **Example 2** (despite the answer already being provided, this serves as really good practice).