Additional support questions for all new undergraduate students

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Introduction

There are 10 sections in this booklet, each section contains a set of questions designed to help you consolidate your understanding of the majority of the A Level Mathematics content. The final section contains extension and enrichment material, with thanks to Integral resources for providing some of this material. You can use your A Level textbooks and material to help you. You can also find additional support and enrichment material on the AMSP website (https://amsp.org.uk). You might also find the book *Bridging the Gap to University Mathematics by Hurst, Edward, Gould, Martin, Springer 2009* useful.

These questions have been designed specifically to help you consolidate your A Level Mathematics knowledge and understanding, with a particular focus on your core mathematics skills. If you are unable to do some of these questions please do not worry, but make sure you engage with the additional support we are offering.

You will find some of these questions more straight forward than others - we do not expect you to be able to do all of these! Some of these questions are more challenging than others, we will be running mini lectures in welcome week in order to help you with some of the content covered here.

Please note that we do not expect you to send us your solutions for marking/feedback, these questions are purely for your self-study and critical self-assessment. We will release the answers during welcome week.

Section A - Algebraic skills

1. Factorise the following polynomials:

   (a) $x^2 - 9$

   (b) $x^2 - 2x - 15$

   (c) $196 + 28x + x^2$

   (d) $8x^2 + 19x + 6$

   (e) $x^3 + x^2 - 17x + 15$

   (f) $2x^4 + 3x^3 + 30x^2 + 48x - 32$

2. Given that $x - 2$ is a factor of $3x^3 - 5x^2 + ax + 2$, find the value of a.

3. When $2x^4 + px^2 - 1$ is divided by $x + 2$, the remainder is 15. What is the value of p?

4. Solve $x - 5\sqrt{x} = -6$.

5. Rewrite the following as partial fractions:

   (a) $\dfrac{x+4}{1-x-2x^2}$

   (b) $\dfrac{4x^2-3x+2}{(x+1)(x-2)^2}$

   (c) $\dfrac{2x+3}{(x-1)(x^2+4)}$

   (d) $\dfrac{4x^2+5x+3}{(x-1)(2x+1)^2}$

6. Divide $\dfrac{x^2-3x-4}{x^2-25}$ by $\dfrac{x+1}{x-5}$.

7. Write down the first 4 terms in the binomial expansion of $\dfrac{1}{\sqrt{2-3x}}$.

8. Expand $\dfrac{9}{(1-x)(1+2x)^2}$ up to and including the term in x^2, stating the range of values for which your expansion is valid.

9. The first three terms of the expansion of $(8+kx)^n$ are $2-\frac{5x}{12}-\frac{25x^2}{288}$. Find k and n.

10. Solve the following inequalities

    (a) $|x-5|<7$

    (b) $|5-3x|\leq 10$

    (c) $|x+2|>|2x-5|$

11. Solve the simultaneous equations

$$4^{x-1} = 16^{2y} \qquad (1)$$
$$8^{x-3} = 2^{1+y} \qquad (2)$$

12. Find the coordinates of the points where the circle $x^2+y^2+6x-4y-4=0$ meets the line $y=3x+4$.

13. A circle has centre $(5,2)$ and passes through the point $(7,3)$. The line $y=2x-a$ is a tangent. Find the values of a.

14. The line $y=x+1$ does not intersect the circle $(x-1)^2+(y+2)^2=k$. Find the possible values of k.

15. If $\frac{x}{3}<3$ and $\frac{y}{2}<5$ which of the following are true statements:

    (a) $\frac{x}{y}<\frac{9}{10}$

    (b) $\frac{x}{y}>\frac{9}{10}$

    (c) $\frac{x}{y}$ can be greater than, equal to or less than $\frac{9}{10}$

    (d) there is not enough information.

16. Given that $x^2>9$ and $y<0$, does this mean that xy will always be positive? Explain your answer.

Section B - Curve sketching

1. Sketch the following curves, without using graphing software. Make sure you label all intersections with the axes, asymptotes and any stationary points:

   (a) $f(x) = \dfrac{3}{x}$

   (b) $y = \sqrt{4 - x}$

   (c) $y = 2x^2 - 3x + 1$

   (d) $y = x^5 - 3x$

2. Given that the curve $f(x) = \sqrt{x}$, describe the following transformations and give the equation of the transformed curve:

   (a) $f(x - 3)$

   (b) $f(x) + 2$

   (c) $4f(x)$

3. Given that the curve $f(x) = x^2 - 3x + 2$, describe the following transformations and give the equation of the transformed curve:

   (a) $f(x + 1)$

   (b) $-f(x)$

   (c) $f(2x)$

Section C - Differentiation

1. Differentiate the following functions with respect to x:

   (a) $\sqrt{2x - 5}$

   (b) $e^{4x} \cos 3x$

   (c) uv, given that u and v are functions of x.

   (d) $\dfrac{x^2 - \sin x}{\cos 2x}$

   (e) $\left(\dfrac{1}{x} + x\right)^{\frac{4}{3}}$

(f) $\dfrac{u}{v}$ given that u and v are functions of x.

(g) $\dfrac{e^{\frac{x}{2}}}{\sqrt{3-2x}}$

(h) $f(x)g(x)$

(i) $3x^2e^{1-5x}$

(j) $f(g(x))$

(k) $\dfrac{f(x)}{g(x)}$

2. Find the gradient of the curve $y = \dfrac{3}{2-5x}$ at the point $(0.2, 3)$.

3. Find the exact value of the stationary points on the curve $y = x\sqrt{2-3x}$ and determine the nature of them.

4. Find the exact equation of the tangent to the curve $y = \ln(1-2x)$ at the point when $x = -0.5$.

5. Given that $x = y^3 + 5\sqrt{y}$, find $\dfrac{dy}{dx}$.

6. Given that $y = \sin 2x$, show that $\dfrac{d^2y}{dx^2} = -4y$.

7. Find $\dfrac{dy}{dx}$ for the following:

   (a) $xy^3 = 5\ln y + 1$

   (b) $(2x-3)(y+1) = 2$

8. Find the equation of the normal to the curve $xy - 6y + 4x = 26$ at the point where $y = -2$.

9. Find the stationary points of the curve $xy + 38 = y^2 + 3x^2$ and determine their nature.

Section D - Integration

1. Evaluate the following, leaving your answers in exact form where appropriate:

   (a) $\int 2x\sqrt{4-3x^2}dx$

   (b) $\int_1^2 \frac{x}{\sqrt{2x-1}}dx$

   (c) $\int \cos^2 x dx$

   (d) $\int x^2 \sin(x^3)dx$

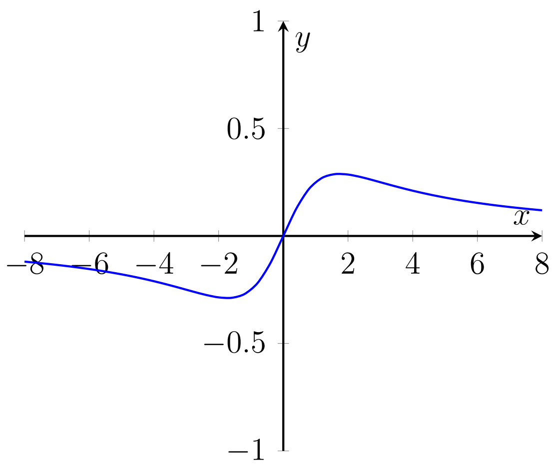
   (e) $\int_\pi^{3\pi} x \sin x dx$

   (f) $\int \ln x dx$

   (g) $\int \sin(-3x)\cos(5x)dx$

   (h) $\int_0^3 \frac{2x-1}{x^2-x+1}dx$

2. This diagram shows the curve $y = \dfrac{x}{x^2+3}$. The area enclosed by the curve, the x-axis and the lines $x = 7$ and $x = -7$ is $\ln a$. Find the value of a.



3. Find the volume of the solid generated by rotating $y - x = x^2 + 3$ through $360°$ about the x-axis from the lines $x = 1$ and $x = 5$.

4. Find the volume of the solid formed when $y = 5x^2$, from $y = 1.5$ to $y = 3$, is rotated through $180°$ about the y-axis. Leave your answer in exact form.

Section E - Trigonometry

1. Prove $\text{cosec}2\theta + \cot 2\theta \equiv \cot\theta$.

2. Solve $\cos 2\theta + 3\sin\theta = 2$ for the values of θ that satisfy $-2\pi \leq \theta \leq 2\pi$.

3. Find the minimum and maximum values of $y = 6\cos x + 8\sin x$ and find the corresponding values of x at which these occur.

4. Prove that $\cos^4 x - \sin^4 x \equiv \cos 2x$.

5. Express $2\cos^2 x - 3\sin^2 x$ in terms of $\cos 2x$.

6. Prove the following:

   (a) $\dfrac{\sin x}{1+\cos x} + \dfrac{1-\cos x}{\sin x} \equiv 2\tan\left(\dfrac{x}{2}\right)$

   (b) $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

7. Sketch, without using any graphing software, $y = \sqrt{3}\sin\theta + 5\cos\theta$.

8. Solve the following equations within the specified range:

   (a) $2\sin x - \text{cosec}x = 1$, $0° < x < 360°$

   (b) $\tan x + 4\cot x = 5$, $-180° < x < 180°$

Section F - Numerical methods

1. Show that the equation $e^x - 3x^2 = 0$ has three roots in the interval $[-1, 4]$. Use the Newton-Raphson method to find each of the roots correct to 2 decimal places.

2. Sketch the curves of $y = e^x$ and $y = \dfrac{4}{x}$. Use the Newton-Raphson method to find the value of x where the curve $y = \dfrac{4}{x} - e^x$ crosses the x axis, correct to 3 decimal places, taking a starting value of 2.

3. Use the Trapezium rule with strip widths of 0.125 to estimate the value of $\int_0^1 \frac{5}{1+x^2}dx$.

4. Use Simpson's rule, with 10 strips, to find an approximation to the area bounded by $y = xe^{-x^2}$, the x axis and $x = 1$. Give your answer correct to 5 significant figures.

Section G - Parametric equations

1. Find the Cartesian equations of each of these curves:

   (a) $x = 2t + 1$ and $y = 2t^2$

   (b) $x = 2\cos\theta$ and $y = 8\sin\theta$

   (c) $x = \dfrac{2+3t}{1-t}$ and $y = \dfrac{2-t}{1+t}$

2. Find the gradient of the curve with parametric equations $x = \sqrt{t+3}$ and $y = t^2$, at the point where $t = 2$.

3. Find the equation of the normal to the curve defined parametrically as $x = 3\cos\theta$ and $y = 2\sin\theta$ at the point $\left(\dfrac{3}{2}, \sqrt{3}\right)$.

4. Find, and classify, the stationary points on the curve with parametric equations $x = t^2$ and $y = t^3 - 3t$.

Section H - Differential equations

1. Find the general solutions to the following differential equations:

   (a) $\dfrac{dy}{dx} = \dfrac{3y}{1-5x}$.

   (b) $x\dfrac{dy}{dx} = y + 5$.

2. Find the particular solutions to the following differential equations:

   (a) $\dfrac{dy}{dx} = \dfrac{x^2}{y+3}$ at the point $(2, 1)$.

   (b) $x\dfrac{dy}{dx} = 2(3 - y)$ given that $y = 2$ when $x = 2$.

3. A spherical balloon, which is being inflated, has radius $r$cm at time t seconds. It takes 4 seconds to inflate the balloon to a radius of 18cm from its initial value of 1.5cm. The rate of increase of r is taken to be inversely proportional to r^2. Find the time it will take to inflate the balloon to a radius of 22cm.

4. The number of rabbits in a population, n, increases at a rate proportional to the number present. In 2017 there were 400 rabbits, in 2020 there were $9,650$ rabbits. What year will the number of rabbits exceed half a million?

5. Find the general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{1}{(1+x)(3+x)}$.

6. The gradient of a curve $y = f(x)$ is inversely proportional to the square root of x and the curve passes through $(0, 3)$ and $(4, 23)$. Find the equation of the curve.

Section I - Probability and Statistics

1. 100 households in a city in England were randomly selected and the weekly expenditure, £X per household on travel was recorded. The results are summarised below, where $\overline{x}$ denotes the sample mean.
   $\sum x = 9116 \sum(x - \overline{x})^2 = 68112$

   (a) Calculate the mean and standard deviation for the weekly expenditure on travel.

   (b) It is assumed that household expenditure on travel follows a normal distribution. Describe the process for doing a hypothesis test to find out whether the mean household expenditure on travel is greater than £90. (Important: Just describe the process, don't do the Hypothesis test. You may draw diagrams.)

2. The mean of a set of data is 30, whilst the mean of a second set of data is 25. One of the pieces of data from the first set is exchanged with one of the pieces of data from the second set. As a result, the mean of the first set of data decreases from 30 to 28, and the mean of the second set of data increases from 25 to 31. What is the mean of the set made by combining all the data?

3. X can be modelled using a Binomial distribution with $n = 6$ and $p = 0.1$ such that $X\ Bin(n, p)$. Find $P(X \geq 3)$.

4. $P(A) = \dfrac{1}{4}, P(B) = \dfrac{1}{15}, P(B|A) = \dfrac{1}{3}$. Calculate $P(A \cup B)$.

5. Given the following distribution,

$$P(N = n) = \begin{cases} \frac{k}{2}(\frac{1}{4})^{n-1} & \text{for } n = 1, 2 \\ k & \text{for } n = 3 \end{cases}$$

   Find $P(N \geq 2)$.

6. 60% of a company's data scientists are women and the remainder are men. This company let's each data scientist use either R or Python as a programming language. Each staff member chooses exactly one of the programming languages.

   $\frac{2}{5}$ of the male staff members chooses R;

   $\frac{2}{3}$ of the staff members who use R are women.

   What is the probability that a randomly selected staff member, is a woman who uses Python?

7. A farmer records the masses of a random sample of 50 potatoes she has grown. The farmer models the distribution of masses by $N(74.5, 15^2)$. Find the number of potatoes in the sample that this model would predict to have masses in the range: $70 \leq m \leq 80$.

Section J - Extension and enrichment

1. Solve the inequality $x^2 - 4|x| + 3 < 0$.

2. Given that $f(x) = \dfrac{x+5}{(x-1)(x+2)}$, find the first four terms of the binomial expansion of $f(x)$, stating the values of x for which the expansion is valid.

3. A fishtank is 30cm deep and is formed from a cuboid with horizontal dimension $x$cm by $y$cm. Find the formulae for the surface area

(assume the tank has no lid) and the volume of the tank. If the surface area is 6300cm^2 and the volume is 45000cm^3, find the size of the tank.

4. How many real solutions does the following equation have? $x^2 - 300x = 3000$.

5. Use the quotient rule to show that the derivative of $\tan x$ can be written as $1 + \tan^2 x$.

6. Given that $f(x) = (x - a)^2 g(x)$ show that $f'(a) = 0$.

7. By using the 'volume of revolution' approach, prove the formula for the volume of a cone.

8. Find $\int \sin^3 x dx$.

9. Find all the solutions of $32\cos^3 x - 48\cos^2 x + 22\cos x = 3$ in the range $0 \le x \le 360°$.

10. Find the points of intersection of the curves $y = \cos x$ and $y = 2 + \sqrt{3}\sin x$ for the range $0 < x < 9\pi$.